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UNSTEADY PRESSURE MOTION OF VISCOUS - PLASTIC FLUID BETWEEN TWO INFINITE PLANES

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ABSTRACT. In this paper we consider the problem on unsteady one - dimensional pressure flow of Svedov-Bingham's fluid between two infinite planes.

We show that this problem can be solved completely by using Sliozkin - Targ's method with the approximation

$$\varphi(t) = \frac{2}{3}\frac{\partial u}{\partial t}\bigg|_{y=y_0}$$

We have some notices on the possibility of the application of this method for solving a class on unsteady flow of viscous-plastic fluid in pipe-lines without any supplementary assumption.

1. The motion equation and its conditions

Consider one-dimensional unsteady pressure flow of viscous-plastic fluid between two horizontal infinite planes.

The system of cartesian coordinates was given as in Fig. 1, where coordinate plane xOy is between two infinite planes. Since considered flow is symmetric, so we need only consider its upper half. Through the forthcoming, we shall adapt the traditionnal terminologies and notations [1, 2]

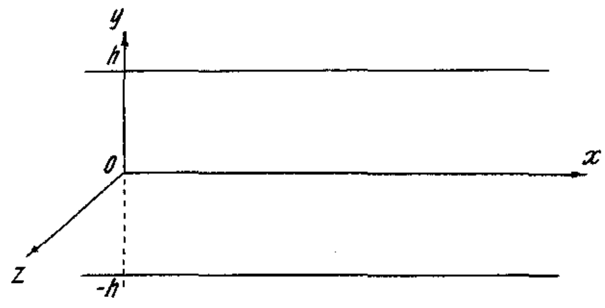

Fig. 1

From the system of Henky-Iliusin's motion equation of continuity and symmetry of flow [1, 2] we obtain the following motion equation:

$$\rho\frac{\partial u}{\partial t} = \eta\frac{\partial^2 u}{\partial y^2} + \frac{\Delta p}{\ell} \quad y_0 \le y \le h \tag{1.1}$$

$$u_0(t) = u(y_0, t) \quad y \le y_0 \tag{1.2}$$

where

$u = u(y,t)$ is velocity of flow in the viscous-plastic zone

$u_0 = u_0(t) = u(y_0,t)$ - velocity of flow in the zone of elastic core.

$y_0 = y_0(t)$ - semi-width of elastic core (1.3)

$\frac{\Delta p}{\ell}$ - pressure gradient,

η - structural viscosity,

ρ - density of considered viscous-plastic fluid

With boundary conditions:

$$u(h,t) = 0 \tag{1.4}$$

$$\frac{\partial u(y_0,t)}{\partial y} = 0 \tag{1.5}$$

$$\frac{\partial u(y_0,t)}{\partial t} = \frac{\Delta p}{\rho\ell} - \frac{\tau_0}{\rho y_0} \tag{1.6}$$

(The last formula expresses the motion equation of elastic core) and initial conditions

$$u(y,0) = 0 \tag{1.7}$$

$$y_0(0) = h$$

The size of elastic core $y_0(t)$ has to satisfy the condition

$$\lim_{t\to\infty} y_0(t) = y_0(\infty) = \frac{\ell\tau_0}{\Delta p} \tag{1.8}$$

2. Sliozkin-Targ's approximation and corollary of first average value theorem

The problems on unsteady flows of viscous-plastic fluid belong to the class of problems with mobile boundary; it is difficult to find their complete solution even by approximate method. In [1] A. A. Abbasov gave approximate solution of considered problem by using assumption

$$\varphi(t) = \frac{1}{h-y_0}\int\limits_{y_0}^{h}\frac{\partial u(y,t)}{\partial t}dy = \frac{1}{h-y_0}\int\limits_{y_0}^{h}\frac{\partial u(y_0,t)}{\partial t}dy = \frac{\Delta p}{\rho \ell} - \frac{\tau_0}{\rho y_0} \tag{2.1}$$

In this paper we solve this problem by Sliozkin-Targ's method by using approximation

$$\varphi(t) = \frac{1}{h-y_0}\int\limits_{y_0}^{h}\frac{\partial u(y,t)}{\partial t}dy \tag{2.2}$$

without any supplementary assumption.

Consider the function [4]

$$f(x) = f_2(x) - f_1(x) \tag{2.3}$$

where $y = f_1(x)$ and $y = f_2(x)$ are two parabolas with the common symmetric axis. Without loss of generality we may assume that

$$f_1(x) = bx^2 \quad \text{is a parabola } (P_1) \tag{2.4}$$

$$f_2(x) = ax^2 + c \quad \text{is a parabola } (P_2) \tag{2.5}$$

where c is the distance between vertices of two parabolas, $x = 0$ is common symmetric axis and $a \neq b \neq 0$.

Corollary. Assume that two parabolas (P_1) and (P_2) are crossed at the point $M_0(x_0, y_0)$. (It is clear that $x_0 \neq 0$): Then there exists $\xi \in [0, x_0]$ such that

$$f(\varsigma) = \frac{1}{x_0}\int\limits_{0}^{x_0} f(x)dx \tag{2.6}$$

and we always have

$$f(\varsigma) = \frac{2}{3}c = \frac{2}{3}[f_2(0) - f_1(0)] = \frac{2}{3}f(0) \tag{2.7}$$

3. Results

Substituting $\frac{\partial u}{\partial t}$ in (1.1) by its average value in the viscous-plastic zone $y_0 \le y \le h$ (Sliozkin-Targ approximation (2.2)). We get the approximation equation:

$$\rho\varphi(t) = \eta\frac{\partial^2 u}{\partial y^2} + \frac{\Delta p}{\ell} \tag{3.1}$$

The solution of equation (3.1) satisfying (1.4), (1.5) is

$$u(y,t) = \frac{\Delta p}{2\ell\eta}(h^2 - y^2) - \frac{\rho\varphi}{2\eta}(h^2 - y^2) - \Big(\frac{\Delta p}{\ell\eta}y_0 - \frac{\rho\varphi}{\eta}y_0\Big)(h - y) \tag{3.2}$$

or

$$u(y,t) = \Big(\frac{\Delta p}{2\ell\eta} - \frac{\rho\varphi}{2\eta}\Big)\left[(h - y_0)^2 - (h - y_0)\right]; \quad y_0 \le y \le h.$$

The velocity profile (3.2) expresses a semi-parabola with vertex at the $y = y_0$.

Applying the obtained corollary above with playing the role of $f(x)$ and note that $c = \frac{\partial u}{\partial t}\big|_{y=0}$ we have

$$\varphi(t) = \frac{1}{h - y_0}\int_{y_0}^{h}\frac{\partial u}{\partial t}dy = \frac{2}{3}\frac{\partial u}{\partial t}\Big|_{y=y_0} = \frac{2}{3}\frac{\partial u(y_0,t)}{\partial t} \tag{3.3}$$

Substituting (1.6) into (3.3) yields

$$\varphi(t) = \frac{2}{3}\Big(\frac{\Delta p}{\rho\ell} - \frac{\tau_0}{\rho y_0}\Big) \tag{3.4}$$

and velocity profile in the cross-section will be:

$$u(y,t) = \Big(\frac{\Delta p}{6\ell\rho} + \frac{\tau_0}{3\eta y_0}\Big)(h^2 - y^2) - \Big(\frac{\Delta p}{3\ell\eta}y_0 + \frac{2\tau_0}{3\eta}\Big)(h - y); \quad y_0 \le y \le h \tag{3.5}$$

$$u_0(t) = u(y_0,t) = \Big(\frac{\Delta p}{6\ell\rho} + \frac{\tau_0}{3\eta y_0}\Big)(h^2 - y_0^2) - \Big(\frac{\Delta p}{3\ell\eta}y_0 + \frac{2\tau_0}{3\eta}\Big)(h - y_0); \quad y \le y_C \tag{3.6}$$

The discharge of flow is determined as follows:

$$Q(t) = 2y_0u_0(t) + 2\int_{y_0}^{h} u(y,t)dy = \Big(\frac{\Delta p}{3\ell\eta} + \frac{2\tau_0}{3\eta y_0}\Big)\Big(\frac{2h^3}{3} - h^2y_0 + \frac{y_0^3}{3}\Big) \tag{3.7}$$

The size of elastic core in formulas (3.5), (3.6) and (3.7) $y_0 = y_0(t)$ is not determines yet. Substituting (3.5) or (3.6) into (1.6) we get differential equation for determining elastic core

$$\Big[-\frac{\tau_0}{3\eta}\frac{h^2}{y_0^2}+\frac{\tau_0}{3\eta}-\frac{\Delta p}{3\ell\eta}h+\frac{\Delta p}{3\ell\eta}y_0\Big]y_0' = \frac{\Delta p}{\rho\ell}-\frac{\tau_0}{\rho y_0}\,. \tag{3.8}$$

The solution of this equation satisfying condition (1.7) will be

$$\frac{1}{2}(h^2-y_0^2)+\Big(\frac{2\ell}{\Delta p}\tau_0-h\Big)(h-y_0)+h^2\ln\frac{h}{y}+$$
$$\Big[2\Big(\frac{\ell}{\Delta p}\tau_0\Big)^2-\frac{\ell}{\Delta p}\tau_0 h-h^2\Big]\ln\frac{\frac{4p}{\ell}h-\tau_0}{\frac{4p}{\ell}y_0-\tau_0}=-\frac{3\eta}{\rho}t$$

Denote by $y_0(\infty) = \ell\tau_0/\Delta p$ a semi-width of elastic core of respective steady flow, we get

$$\frac{1}{2}(h^2-y_0^2)+[2y_0(\infty)-h](h-y_0)+h^2\ln\frac{h}{y_0}+$$
$$[2y_0^2(\infty)-hy_0(\infty)-h^2]\ln\frac{h-y_0(\infty)}{y_0-y_0(\infty)}=-\frac{3\eta}{\rho}t \tag{3.9}$$

Drawing by computer, basing on numerical date for (3.9) and (3.5), (3.6) gives plots of developing elastic core in Fig. 2 and velocity profile in Fig. 3 [5]

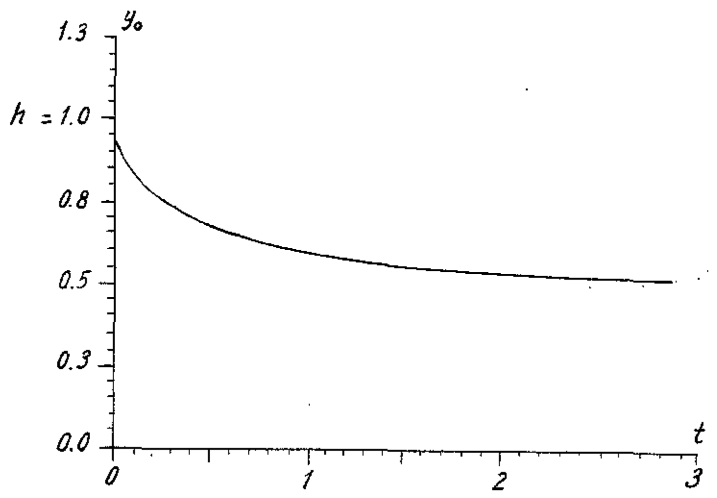

Fig. 2

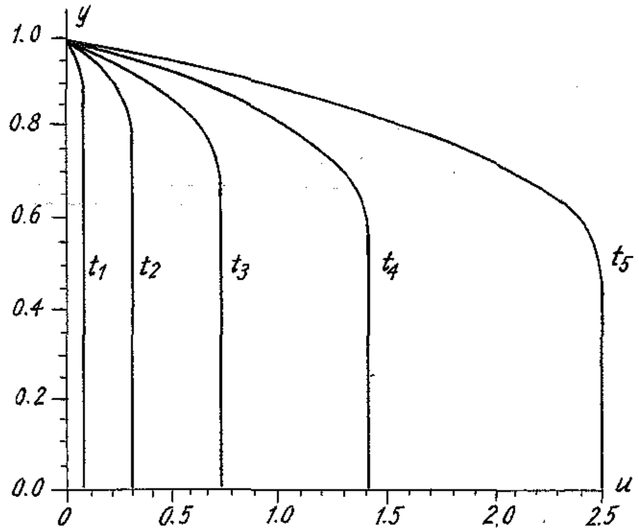

Fig. 3

$t_1 = 0.010$, $t_2 = 0.067$, $t_3 = 0.227$, $t_4 = 0.511$, $t_5 = 2.237$

4. Concluding remarks

From the obtained results (3.5), (3.6), (3.7) and (3.9) we can get the well-known formulae for the respective steady flow [1, 4]. Note that, the solution (3.9) and (3.5) will satisfy initial condition (1.7) and condition (1.8) when $t \to \infty$.

Note that we can always write one condition at the mobile boundary expressing the motion equation of elastic core in form $\left.\frac{\partial u}{\partial t}\right|_{y=y_0} = f(y_0)$. So combining Sliozkin-Targ's method and properties of velocity profile we can solve some other problems on unsteady flow of viscous-plastic fluid with higher approximations.

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CHUYỂN ĐỘNG CÓ ÁP KHÔNG DỪNG CỦA CHẤT LỎNG NHỚT DẺO GIỮA HAI BẢN PHẲNG SONG SONG

Trong bài báo này chúng tôi xét bài toán dòng chảy có áp không dừng của chất lỏng Svedov-Bingham giữa hai bản phẳng song song.

Chúng tôi chỉ ra rằng bài toán có thể giải trọn vẹn bằng phương pháp Sliozkin-Targ với xấp xỉ $\varphi(t) = \frac{2}{3}\frac{\partial u}{\partial t}\Big|_{y=y_0}$.

Chúng tôi có một vài lưu ý về khả năng áp dụng phương pháp này để giải lớp các bài toán dòng chảy không dừng của chất lỏng nhớt dẻo trong ống dẫn mà không cần giả thiết phụ nào khác.