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ONE METHOD TO DETERMINE THE SOLUTION VALUES AT THE BOUNDARY FOR THE VERTICAL TWO-DIMENSIONAL EQUATION SYSTEM

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ABSTRACT. In this paper, the boundary conditions for modified Navier-Stokes equations system are presented, and the complementary equations on the boundaries are established.

Keywords: Partial differential equation, the equation of mathematical physics, linear algebra.

1. Navier-Stokes equation system

The Navier-Stokes equation system describing vertical two-dimensional unsteady flow for viscous incompressible fluid has the following form (see [1]-[5])

$$
\begin{aligned}
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial x}&=\nu\Delta u,\\
\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial z}&=\nu\Delta w,\\
\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}&=0,
\end{aligned}
\tag{1.1}
$$

where x, z are coordinate axes, (u,w) is the velocity vector, p - the pressure, ρ - the density, ν - the kinematic viscosity, $\Delta=\dfrac{\partial}{\partial x^2}+\dfrac{\partial}{\partial z^2}$.

It is well known that, the equation system (1.1) with the initial condition

$$U_1(x,z,0)=U_1^0(x,z),$$

where

$$U_1=\begin{pmatrix}u\\ w\end{pmatrix}$$

and the boundary conditions on the boundary ∂G of the region in consideration G:

$$U_1(x,z,t)\Big|_{\partial G} = 0$$

has the unique solution in the space of generalised functions (see [1], [2]).

The determination of pressure p from equation (1.1) is very difficult. To avoid it, the artificial compression component is added to the continuity equation of (1.1). Then we obtain the following equation system (see [3], [4]). (Let us take $\rho = 1$)

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{\partial p}{\partial x} &= \nu\Delta u, \\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial z} + w\frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} &= \nu\Delta w, \\ \frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0. \end{aligned} \tag{1.2}$$

In actual fact, it is difficult to give all 2 boundary conditions at any part of the boundary.

To overcome this insufficiency of given boundary conditions, at the neighborhood of the part of boundary, where all the 2 boundary conditions can not be given, we consider the following modified Navier-Stokes equation system for determining the solution values on this part of boundary.

2. Modified Navier-Stokes equation system (1.2)

Let the right hand side of the equation system (1.2) be known. For example, their values were taken at the previous time step, or the fluid is inviscous ($\nu = 0$). Then we obtain the following equation system

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{\partial p}{\partial x} &= \phi_1, \\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} &= \phi_2, \\ \frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0, \end{aligned} \tag{2.1}$$

where ϕ_1, ϕ_2 are known functions.

The system (2.1) may be written in the vectorial form:

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial z} = \Phi, \tag{2.2}$$

where

$$U = \begin{pmatrix} u \\ w \\ p \end{pmatrix}, \quad \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} u & 0 & 1 \\ 0 & u & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} w & 0 & 0 \\ 0 & w & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The system (2.1) is quasilinear hyperbolic equation system. Really, the matrix $A_* = s_1 A + s_2 B$, where s_1, s_2 are real numbers and $s_1^2 + s_2^2 \neq 0$, has three different real eigen values:

$$\lambda_1^* = s_1 u + s_2 w = w_*,$$
$$\lambda_2^* = \frac{w_*}{2} + \frac{\sqrt{\Delta_*}}{2},$$
$$\lambda_3^* = \frac{w_*}{2} - \frac{\sqrt{\Delta_*}}{2}, \quad \text{where } \Delta_* = w_*^2 + 4(s_1^2 + s_2^2).$$

It is well know that (see [6], [7]) the boundary problem of linear symmetric hyperbolic equation system

$$D\frac{\partial V}{\partial t} + \overline{A}\frac{\partial V}{\partial x} + \overline{B}\frac{\partial V}{\partial z} = F, \tag{2.3}$$

where D, $\overline{A}$, $\overline{B}$ are symmetric matrices and $D > 0$, has a solution if, besides the initial conditions the number of given boundary conditions is equal to one of the negative eigen-values λ_j of the matrix $\overline{A}_n = n_x\overline{A} + n_z\overline{B}$, where $(n_x,\ n_z)$ is the external normal vector of the boundary ∂G.

In addition, the equation system (2.3) has the unique solution continuously depending on the initial conditions and on the right-hand side if the boundary conditions are dissipative.

The boundary condition is said to be dissipative if any vector V satisfying this condition has to satisfy the following inequality

$$\iint_S \left([n_t D + n_x\overline{A} + n_z\overline{B}]V, V\right) dS \geq 0, \tag{2.4}$$

where $S = \partial G \times [0, T]$, (n_t, n_x, n_z) is the external normal vector of the surface S.

In the case, when the boundary ∂G is fixed, then $n_t = 0$ and inequality (2.4) may be replaced by the following inequality

$$(\overline{A}_n V, V) = \left([n_x\overline{A} + n_z\overline{B}]V, V\right) \geq 0 \tag{2.5}$$

Let us take in the equation system (2.3)

$$V = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \; D = E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; \overline{A} = \begin{pmatrix} \overline{u} & 0 & 1 \\ 0 & \overline{u} & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; \overline{B} = \begin{pmatrix} \overline{w} & 0 & 0 \\ 0 & \overline{w} & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{2.3a}$$

where $\overline{u}$ and $\overline{w}$ are known values.

It is easy to verify that, the eigen values of the matrix $\overline{A}_n = n_x\overline{A} + n_y\overline{B}$ are

$$\lambda_1 = n_x\overline{u} + n_z\overline{w} = \overline{w}_n, \quad \lambda_2 = \frac{\overline{w}_n + \sqrt{\Delta}}{2}, \quad \lambda_3 = \frac{\overline{w}_n - \sqrt{\Delta}}{2}, \tag{2.6}$$

where $\Delta = \overline{w}_n^2 + 4$.

From (2.6) one deduces the number of boundary conditions which is necessary to be given on the boundary for equation system (2.3), (2.3a)

a - At the inflow boundary ($\overline{w}_n < 0$) it is necessary to give two conditions

b - At the outflow boundary ($\overline{w}_n > 0$) only one condition is required

c - At the solid boundary ($\overline{w}_n = 0$) one condition is needed.

Now we find the dissipative boundary conditions for the equation system (2.3), (2.3a) by the same method as in [10].

Putting

$$V = PW, \tag{2.7}$$

where $W = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \quad P = \begin{pmatrix} n_x & n_z & 0 \\ n_z & -n_x & 0 \\ 0 & 0 & 1 \end{pmatrix},$

we have

$$P^* = P^{-1} = P,$$

where P^* is the transposed matrix of P and

$$(\overline{A}_n V, V) = (\overline{A}_n PW, PW) = (P^*\overline{A}_n PW, W) = (\widetilde{A}_n V, V), \tag{2.8}$$

where

$$\widetilde{A}_n = P^*\overline{A}_n P = \begin{pmatrix} \overline{w}_n & 0 & 1 \\ 0 & \overline{w}_n & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The eigen-values of the matrix $\widetilde{A}_n$ are

$$\lambda_1 = \overline{w}_n, \quad \lambda_2 = \frac{\overline{w}_n + \sqrt{\Delta}}{2}, \quad \lambda_3 = \frac{\overline{w}_n - \sqrt{\Delta}}{2}, \tag{2.9}$$

ıere $\Delta = \overline{w}_n^2 + 4$ and their corresponding eigen vectors are

$$Z_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} \dfrac{\overline{w}_n + \sqrt{\Delta}}{a} \\ 0 \\ 2/a \end{pmatrix}, \quad Z_3 = \begin{pmatrix} \dfrac{\overline{w}_n - \sqrt{\Delta}}{b} \\ 0 \\ 2/b \end{pmatrix},$$

ıere $a = \sqrt{(\overline{w}_n + \sqrt{\Delta})^2 + 4}, \quad b = \sqrt{(\overline{w}_n - \sqrt{\Delta})^2 + 4}.$

Denoting

$$Z = (Z_1, Z_2, Z_3) = \begin{pmatrix} 0 & \dfrac{\overline{w}_n + \sqrt{\Delta}}{a} & \dfrac{\overline{w}_n - \sqrt{\Delta}}{b} \\ 1 & 0 & 0 \\ 0 & 2/a & 2/b \end{pmatrix},$$

is well known that for symmetric matrix $\widetilde{A}_n$ we have

$$Z^* \widetilde{A}_n Z = \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \tag{2.10}$$

here Z^* is the transport matrix of Z.

Putting

$$W = ZT, \tag{2.11}$$

here $T = (t_1, \ t_2, \ t_3)^T$, from (2.8), (2.10) and (2.11) one deduces

$$\begin{aligned}(\overline{A}_n V, V) &= (\widetilde{A}_n W, W) = (\widetilde{A}_n ZT, ZT) = (Z^* \widetilde{A}_n ZT, T) \\ &= (\Lambda T, T) = \lambda_1 t_1^2 + \lambda_2 t_2^2 + \lambda_3 t_3^2,\end{aligned} \tag{2.12}$$

here

$$\begin{cases} t_1 &= w_2, \\ t_2 &= \dfrac{a}{2\sqrt{\Delta}} w_1 + \dfrac{a(\sqrt{\Delta} - \overline{w}_n)}{4\sqrt{\Delta}} w_3, \\ t_3 &= -\dfrac{b}{2\sqrt{\Delta}} w_1 + \dfrac{b(\sqrt{\Delta} + \overline{w}_n)}{4\sqrt{\Delta}} w_3, \end{cases} \qquad \begin{cases} w_1 &= \dfrac{\overline{w}_n + \sqrt{\Delta}}{a} t_2 + \dfrac{\overline{w}_n - \sqrt{\Delta}}{b} t_3, \\ w_2 &= t_1, \\ w_3 &= \dfrac{2}{a} t_2 + \dfrac{2}{b} t_3 . \end{cases} \tag{2.13}$$

Using (2.9) and (2.12) we seek the boundary conditions so that any vector V ıtisfying this condition verifies the following inequality

$$\overline{w}_n t_1^2 + \frac{\overline{w}_n + \sqrt{\Delta}}{2} t_2^2 + \frac{\overline{w}_n - \sqrt{\Delta}}{2} t_3^2 \geq 0. \tag{2.14}$$

These boundary conditions are:

1. At the inflow boundary ($\overline{w}_n < 0$)

   a. The condition pair

$$\left\{\begin{array}{l} w_2 = 0, \\ -\quad w_1 + \dfrac{\overline{w}_n + \sqrt{\Delta}}{2} w_3 = 0 \end{array}\right. \tag{2.15}$$

is dissipative because of (2.13) $\left\{\begin{array}{ll} t_1 & = 0, \\ t_3 & = 0 \end{array}\right.$ and the inequality (2.14) is satisfied.

   b. The condition pair

$$\left\{\begin{array}{l} w_2 \pm \dfrac{\alpha a}{2\sqrt{\Delta}}\left(w_1 + \dfrac{\sqrt{\Delta} - \overline{w}_n}{2} w_3\right) = 0, \\ (-b \pm a\beta) w_1 + \left(\dfrac{b(\overline{w}_n + \sqrt{\Delta})}{2} \pm a\beta \dfrac{\sqrt{\Delta} - \overline{w}_n}{2}\right) w_3 = 0 \end{array}\right.$$

is also dissipative if α and β satisfy the inequality

$$\overline{w}_n \alpha^2 + \frac{\overline{w}_n + \sqrt{\Delta}}{2} + \frac{\overline{w}_n - \sqrt{\Delta}}{2} \beta^2 \geq 0.$$

2. At the outflow boundary ($\overline{w}_n > 0$).

Each of the following two boundary conditions is dissipative

a. $$-w_1 + \frac{\sqrt{\Delta} + \overline{w}_n}{2} w_3 = 0 \tag{2.16}$$

because of (2.13): $t_3 = 0$ and (2.14) is satisfied.

b. $$w_3 = 0 \tag{2.17}$$

because of (2.13): $t_2 = -\frac{a}{b} t_3$, $\frac{a}{b} > 1$ so (2.14) is satisfied.

Now we consider the transformation

$$V = U - \Psi, \tag{2.18}$$

where

$$V = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad U = \begin{pmatrix} u \\ w \\ p \end{pmatrix}, \quad \Psi = P\Psi_1 = \begin{pmatrix} \varphi_1 \\ \eta_1 \\ \varsigma_1 \end{pmatrix}$$

and φ_1, η_1, ς_1 are functions of variables (x, z, t). These functions have continuous first-order derivatives and have the following properties. On any part (a_i, b_i) of the boundary ∂G:

- If the boundary condition is $w_n = \varphi$ then $\varphi_1\big|_{(a_i,b_i)} = \varphi$.
- If the boundary condition is $w_s = \eta$ then $\eta_1\big|_{(a_i,b_i)} = \eta$.
- If the boundary condition is $p = \varsigma$ then $\varsigma_1\big|_{(a_i,b_i)} = \varsigma$. (2.19)
- If the boundary condition is $p - w_n = \chi$ then $(\varsigma_1 - \varphi_1)\big|_{(a_i,b_i)} = \chi$.
- If the boundary condition is $\dfrac{\overline{w}_n + \sqrt{\Delta}}{2}p - w_n = f$ then

$$\left(\frac{\overline{w}_n + \sqrt{\Delta}}{2}\varsigma_1 - \varphi_1\right)\Big|_{(a_i,b_i)} = f.$$

Putting (2.18) into the linearized equation system (2.2) with the coefficient matrixes $\overline{A}$, $\overline{B}$, where $\overline{u}$, $\overline{w}$ are known values of the previous time step t_k, we obtain the equation system (2.3), (2.3a)

$$\frac{\partial V}{\partial t} + \overline{A}\frac{\partial V}{\partial x} + \overline{B}\frac{\partial V}{\partial z} = F, \tag{2.3}$$

where

$$F = \Phi - \frac{\partial \Psi}{\partial t} - \overline{A}\frac{\partial \Psi}{\partial x} - \overline{B}\frac{\partial \Psi}{\partial z}\ .$$

The linearized equation system (2.2) and the equation system (2.3) are equivalent. If the linearized system (2.2) has a unique solution continuously depending on the initial condition and on the right-hand side, the system (2.3) has also a unique solution that continuously depends on the initial condition and on the right-hand side, and vice versa.

The solution is now said to satisfy the c-property if it continuously depends on the initial condition and on the right-hand side.

From this we can show the boundary conditions under which the linearized boundary value problem (2.2) has unique solution satisfying the c-property.

In fact, from (2.7) and (2.18) one deduces:

$$W = P^{-1}V = PV = P(U - \Psi) = PU - \Psi_1,$$

where

$$W = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \quad PU = \begin{pmatrix} n_x & n_z & 0 \\ n_z & -n_x & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ w \\ p \end{pmatrix} = \begin{pmatrix} w_n \\ w_s \\ p \end{pmatrix}, \quad \Psi_1 = \begin{pmatrix} \varphi_1 \\ \eta_1 \\ \varsigma_1 \end{pmatrix}.$$

Therefore

$$\begin{cases} w_1 &= w_n - \varphi_1, \\ w_2 &= w_s - \eta_1, \\ w_3 &= p - \varsigma_1. \end{cases} \tag{2.20}$$

1. At the inflow boundary ($\overline{w}_n < 0$)

The condition pair $\begin{cases} w_s = \eta, \\ -\quad w_n + \dfrac{\sqrt{\Delta} + \overline{w}_n}{2} p = f \end{cases}$ ensure that the linearized boundary problem (2.2) has a unique solution satisfying the c-property. Indeed, from (2.19), (2.20) one deduces (2.15) which is dissipative for the equation system (2.3).

2. At the outflow boundary ($\overline{w}_n > 0$)

Each of the following two boundary conditions ensures that the linearized boundary problem (2.2) has a unique solution satisfying the c-property

a. $$-w_n + \frac{\overline{w}_n + \sqrt{\Delta}}{2} p = f ,$$

b. $$p = \varsigma ,$$

because the corresponding boundary condition (2.16) and (2.17) for the system (2.3) are dissipative.

We consider the simple case, where the flow is very slow and the convective components $\left(u\dfrac{\partial u}{\partial x} + w\dfrac{\partial u}{\partial z}\right)$, $\left(u\dfrac{\partial w}{\partial x} + w\dfrac{\partial w}{\partial z}\right)$ are very small in comparison with the other ones. Omitting the high order of the small components or bringing them to the right-hand side and considering them as known, using their values at the previous time step t_k, we obtain the equation system

$$\frac{\partial U}{\partial t} + A_M \frac{\partial U}{\partial x} + B_M \frac{\partial U}{\partial z} = \Psi, \tag{2.21}$$

where

$$U = \begin{pmatrix} u \\ w \\ p \end{pmatrix}, \quad A_M = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad B_M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Equation system (2.21) is also a quasilinear equation of hyperbolic type. The matrix $\overset{M}{A_n} = n_x A_M + n_z B_M$ has the following three eigen-values: $\overset{M}{\lambda_1} = 0$, $\overset{M}{\lambda_2} = 1$, $\overset{M}{\lambda_3} = -1$. Therefore at any of the boundary type for the linearized boundary problem (2.21) it is necessary to give only one boundary condition.

By an argument analogous to that used for the equation system (2.2) we get boundary conditions for the linearized equation system (2.21), which has a unique solution satisfying the c-property. In fact, each of the following three boundary conditions ensures the uniqueness of a solution satisfying the c-property

$$\begin{aligned}
&\text{a.} \quad p - w_n = \chi, \\
&\text{b.} \quad w_n = \varphi, \\
&\text{c.} \quad p = \varsigma.
\end{aligned}$$

3. Complementary equations on the boundary

In order to determine the three unknown function u, w, p on the boundary ∂G, where only one or two boundary conditions are given, it is necessary to construct some complementary equations, which combining with the given boundary conditions give a close equation system for determining the functions u, w, p on the boundary.

Let σ_i and $\vec{r}_i$ $(i = 1,2,3)$ are left eigen values and corresponding left eigen vectors of the matrix A of the equation system (2.2) we have

$$\vec{r}_i A = \sigma_i \vec{r}_i,$$

$$\vec{r}_i\Big(\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x}\Big) = \vec{r}_i\Big(\frac{\partial U}{\partial t} + \sigma_i\frac{\partial U}{\partial x}\Big) = \vec{r}_i\Big(\frac{dU}{dt}\Big)_{x_i}.$$

Denoting

$$\Omega = \begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \dfrac{\sqrt{\Delta_A} - u}{2} \\ 1 & 0 & -\dfrac{\sqrt{\Delta_A} + u}{2} \end{pmatrix}, \quad \Delta_A = u^2 + 4,$$

from (2.2) it yields

$$\begin{aligned}
B\frac{\partial U}{\partial z} &= \Phi - \Big(\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x}\Big) = \Phi - \Omega^{-1}\Omega\Big(\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x}\Big) \\
&= \Phi - \Omega^{-1}\Big\|\vec{r}_i\Big(\frac{dU}{dt}\Big)_{x_i}\Big\|,
\end{aligned} \tag{3.1}$$

where

$$\Big(\frac{dU}{dt}\Big)_{x_i} = \frac{\partial U}{\partial t} + \sigma_i\frac{\partial U}{\partial x}, \quad \Big\|\vec{r}_i\Big(\frac{dU}{dt}\Big)_{x_i}\Big\| = \begin{pmatrix} \vec{r}_1\Big(\dfrac{dU}{dt}\Big)_{x_1} \\ \vec{r}_2\Big(\dfrac{dU}{dt}\Big)_{x_2} \\ \vec{r}_3\Big(\dfrac{dU}{dt}\Big)_{x_3} \end{pmatrix}.$$

By an analogous argument one deduces:

$$A\frac{\partial U}{\partial x} = \Phi - Q^{-1}\left\|\vec{\nu}_j\left(\frac{dU}{dt}\right)_{z_j}\right\|, \tag{3.2}$$

where μ_j and $\vec{\nu}_j$ $(j = 1,2,3)$ are left eigen values and their corresponding left eigen vectors of the matrix B in (2.2)

$$\left(\frac{dU}{dt}\right)_{z_j} = \frac{\partial U}{\partial t} + \mu_j\frac{\partial U}{\partial z}, \quad \left\|\vec{\nu}_j\left(\frac{dU}{dt}\right)_{z_j}\right\| = \begin{pmatrix} \vec{\nu}_1\left(\frac{dU}{dt}\right)_{z_1} \\ \vec{\nu}_2\left(\frac{dU}{dt}\right)_{z_2} \\ \vec{\nu}_3\left(\frac{dU}{dt}\right)_{z_3} \end{pmatrix},$$

$$Q = \begin{pmatrix} \vec{\nu}_1 \\ \vec{\nu}_2 \\ \vec{\nu}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{\sqrt{\Delta_B} - w}{2} \\ 0 & 1 & -\frac{\sqrt{\Delta_B} + w}{2} \end{pmatrix}, \quad \Delta_B = w^2 + 4.$$

Putting (3.1), (3.2) into (2.2) we get (see [8], [9])

$$\frac{\partial U}{\partial t} + \Phi - \Omega^{-1}\left\|\vec{\tau}_i\left(\frac{dU}{dt}\right)_{x_i}\right\| - Q^{-1}\left\|\vec{\nu}_j\left(\frac{dU}{dt}\right)_{y_j}\right\| = 0, \tag{3.3}$$

where

$$\Omega^{-1} = \begin{pmatrix} 0 & \frac{\sqrt{\Delta_A} + u}{2\sqrt{\Delta_A}} & \frac{\sqrt{\Delta_A} - u}{2\sqrt{\Delta_A}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{\Delta_A}} & \frac{-1}{\sqrt{\Delta_A}} \end{pmatrix}, \quad Q^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{\Delta_B} + w}{2\sqrt{\Delta_B}} & \frac{\sqrt{\Delta_B} - w}{2\sqrt{\Delta_B}} \\ 0 & \frac{1}{\sqrt{\Delta_B}} & \frac{-1}{\sqrt{\Delta_B}} \end{pmatrix}.$$

Equation system (3.3) may be rewritten in the evident form

$$\begin{aligned} \frac{\partial u}{\partial t} - \left(\frac{du}{dt}\right)_{x_1} - \frac{\alpha_1}{2\sqrt{\Delta_A}}\left(\frac{du}{dt}\right)_{x_2} - \frac{1}{\sqrt{\Delta_A}}\left(\frac{dp}{dt}\right)_{x_2} - \frac{\alpha_2}{\sqrt{\Delta_A}}\left(\frac{du}{dt}\right)_{x_3} \\ + \frac{1}{\sqrt{\Delta_A}}\left(\frac{dp}{dt}\right)_{x_3} + \phi_1 = 0, \end{aligned} \tag{3.4}$$

$$\begin{aligned} \frac{\partial w}{\partial t} - \left(\frac{dw}{dt}\right)_{z_1} - \frac{\beta_1}{2\sqrt{\Delta_B}}\left(\frac{dw}{dt}\right)_{z_2} - \frac{1}{\sqrt{\Delta_B}}\left(\frac{dp}{dt}\right)_{z_2} - \frac{\beta_2}{\sqrt{\Delta_B}}\left(\frac{dw}{dt}\right)_{z_3} \\ + \frac{1}{\sqrt{\Delta_B}}\left(\frac{dp}{dt}\right)_{z_3} + \phi_2 = 0, \end{aligned} \tag{3.5}$$

$$
\begin{aligned}
&\frac{\partial p}{\partial t}-\frac{1}{\sqrt{\Delta_A}}\left(\frac{du}{dt}\right)_{x_2}-\frac{\alpha_2}{\sqrt{\Delta_A}}\left(\frac{dp}{dt}\right)_{x_2}+\frac{1}{\sqrt{\Delta_A}}\left(\frac{du}{dt}\right)_{x_3}-\frac{\alpha_1}{\sqrt{\Delta_A}}\left(\frac{dp}{dt}\right)_{x_3} \\
&\quad-\frac{1}{\sqrt{\Delta_B}}\left(\frac{dw}{dt}\right)_{z_2}-\frac{\beta_2}{\sqrt{\Delta_B}}\left(\frac{dp}{dt}\right)_{z_2}+\frac{1}{\sqrt{\Delta_B}}\left(\frac{dw}{dt}\right)_{z_3}-\frac{\beta_1}{\sqrt{\Delta_B}}\left(\frac{dp}{dt}\right)_{z_3}=0,
\end{aligned}
\tag{3.6}
$$

where $\alpha_1=\dfrac{\sqrt{\Delta_A}+u}{2}$, $\alpha_2=\dfrac{\sqrt{\Delta_A}-u}{2}$, $\beta_1=\dfrac{\sqrt{\Delta_B}+w}{2}$, $\beta_2=\dfrac{\sqrt{\Delta_B}-w}{2}$.

From equation system (3.4)-(3.6) it is easy to deduce the complementary equations on the boundary.

1. At the left boundary

   a. For the inflow ($w_n=-u<0$)

$$
\begin{aligned}
&\frac{\partial p}{\partial t}-\frac{1}{\alpha_1}\frac{\partial u}{\partial t}+\frac{1}{\alpha_1}\left(\frac{du}{dt}\right)_{z_1}+\frac{1}{\alpha_1}\left(\frac{du}{dt}\right)_{x_3}-\frac{1+\alpha_1^2}{\alpha_1\sqrt{\Delta_A}}\left(\frac{dp}{dt}\right)_{x_3}-\frac{1}{\sqrt{\Delta_B}}\left(\frac{dw}{dt}\right)_{z_2} \\
&\quad-\frac{\beta_2}{\sqrt{\Delta_B}}\left(\frac{dp}{dt}\right)_{z_2}+\frac{1}{\sqrt{\Delta_B}}\left(\frac{dw}{dt}\right)_{z_3}-\frac{\beta_1}{\sqrt{\Delta_B}}\left(\frac{dp}{dt}\right)_{z_3}-\frac{1}{\alpha_1}\phi_1=0.
\end{aligned}
\tag{3.7}
$$

   b. For the outflow ($w_n=-u>0$)

   - Equation (3.5),
   - Equation (3.7).

2. At the right boundary

   a. For the inflow ($w_n=u<0$)

$$
\begin{aligned}
&\frac{\partial p}{\partial t}+\frac{1}{\alpha_2}\frac{\partial u}{\partial t}-\frac{1}{\alpha_2}\left(\frac{du}{dt}\right)_{z_1}-\frac{1}{\alpha_2}\left(\frac{du}{dt}\right)_{x_2}-\frac{1+\alpha_2^2}{\alpha_2\sqrt{\Delta_A}}\left(\frac{dp}{dt}\right)_{x_2}-\frac{1}{\sqrt{\Delta_B}}\left(\frac{dw}{dt}\right)_{z_2} \\
&\quad-\frac{\beta_2}{\sqrt{\Delta_B}}\left(\frac{dp}{dt}\right)_{z_2}+\frac{1}{\sqrt{\Delta_B}}\left(\frac{dw}{dt}\right)_{z_3}-\frac{\beta_1}{\sqrt{\Delta_B}}\left(\frac{dp}{dt}\right)_{z_3}+\frac{1}{\alpha_2}\phi_1=0.
\end{aligned}
\tag{3.8}
$$

   b. For the outflow ($w_n=u>0$)

   - Equation (3.5) ,
   - Equation (3.8) .

By an analogous argument we can obtain the characteristic form of the equation system (2.21)

$$\frac{\partial u}{\partial t} - \left(\frac{du}{dt}\right)_{x_1} - \frac{1}{2}\left(\frac{du}{dt}\right)_{x_2} - \frac{1}{2}\left(\frac{dp}{dt}\right)_{x_2} - \frac{1}{2}\left(\frac{du}{dt}\right)_{x_3} + \frac{1}{2}\left(\frac{dp}{dt}\right)_{x_3} + \phi_1 = 0, \tag{3.9}$$

$$\frac{\partial w}{\partial t} - \left(\frac{dw}{dt}\right)_{x_1} - \frac{1}{2}\left(\frac{dw}{dt}\right)_{x_2} - \frac{1}{2}\left(\frac{dp}{dt}\right)_{x_2} - \frac{1}{2}\left(\frac{dw}{dt}\right)_{x_3} + \frac{1}{2}\left(\frac{dp}{dt}\right)_{x_3} + \phi_2 = 0, \tag{3.10}$$

$$\frac{\partial p}{\partial t} - \frac{1}{2}\left(\frac{du}{dt}\right)_{x_2} - \frac{1}{2}\left(\frac{dp}{dt}\right)_{x_2} + \frac{1}{2}\left(\frac{du}{dt}\right)_{x_3} - \frac{1}{2}\left(\frac{dp}{dt}\right)_{x_3} - \frac{1}{2}\left(\frac{dw}{dt}\right)_{x_2}$$
$$- \frac{1}{2}\left(\frac{dp}{dt}\right)_{x_2} + \frac{1}{2}\left(\frac{dw}{dt}\right)_{x_3} - \frac{1}{2}\left(\frac{dp}{dt}\right)_{x_3} = 0.$$

From these equations we obtain the following complementary equations on the boundary

1. At the left boundary

- Equation (3.10),
- $$\frac{\partial p}{\partial t} - \frac{\partial u}{\partial t} + \left(\frac{du}{dt}\right)_{x_1} + \left(\frac{du}{dt}\right)_{x_3} - \left(\frac{dp}{dt}\right)_{x_3}$$
$$- \frac{1}{2}\left(\frac{dw}{dt}\right)_{x_2} - \frac{1}{2}\left(\frac{dp}{dt}\right)_{x_2} + \frac{1}{2}\left(\frac{dw}{dt}\right)_{x_3} - \frac{1}{2}\left(\frac{dp}{dt}\right)_{x_3} - \phi_1 = 0.$$

2. At the right boundary

- Equation (3.10),
- $$\frac{\partial p}{\partial t} + \frac{\partial u}{\partial t} - \left(\frac{du}{dt}\right)_{x_1} - \left(\frac{du}{dt}\right)_{x_2} - \left(\frac{dp}{dt}\right)_{x_2}$$
$$- \frac{1}{2}\left(\frac{dw}{dt}\right)_{x_2} - \frac{1}{2}\left(\frac{dp}{dt}\right)_{x_2} + \frac{1}{2}\left(\frac{dw}{dt}\right)_{x_3} - \frac{1}{2}\left(\frac{dp}{dt}\right)_{x_3} + \phi_1 = 0.$$

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MỘT PHƯƠNG PHÁP XÁC ĐỊNH GIÁ TRỊ NGHIỆM TẠI BIÊN ĐỐI VỚI HỆ PHƯƠNG TRÌNH HAI CHIỀU ĐỨNG

Để giải hệ phương trình Navier-Stokes hai chiều đứng, cần phải cho tại biên hai điều kiện biên. Tuy nhiên việc cho đủ cả hai điều kiện biên trên mọi đoạn biên là rất khó khăn.

Bài báo trình bày một phương pháp xác định các giá trị nghiệm tại các đoạn biên, mà ở đó không thể cho đủ cả hai điều kiện biên, nhờ việc giải một hệ phương trình biến dạng của hệ phương trình Navier-Stokes, trong lân cận đoạn biên đó.