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INTERACTION BETWEEN NONLINEAR PARAMETRIC AND FORCED OSCILLATIONS

NGUYEN VAN DAO, NGUYEN VAN DINH, TRAN KIM CHI

The interaction of nonlinear oscillations is an important and interesting problem, which has attracted the attention of many researchers. Minorsky N. [5] has stated "Perhaps the whole theory of nonlinear oscillations could be formed on the basis of interaction".

The interaction between the forced and "linear" parametric oscillations when the coefficient of the harmonic function of time is linear relative to the position has been studied in [1, 4]. In this paper this kind of interaction is considered for "nonlinear" parametric oscillation with cubic nonlinearity of the modulation depth. The asymptotic method of nonlinear mechanics [1] is used. Our attention is focused on the stationary oscillations and their stability. Different resonance curves are obtained.

1. Equation of motion and approximate solution

Let us consider a nonlinear system governed by the differential equation

$$\ddot{x} + \omega^2 x = \varepsilon\Big[\Delta x - h\dot{x} - \gamma x^3 + 2px^3\cos 2\omega t + rcos(\omega t - \delta)\Big], \qquad (1.1)$$

where $\varepsilon > 0$ is the small parameter; $h \geq 0$ is the damping coefficient; $\gamma > 0$, $p > 0$, $r > 0$, $\omega > 0$ are the constant parameters; $\varepsilon\Delta = \omega^2 - 1$ is the detuning parameter, where the natural frequency is equal to unity; and $\delta \geq 0$ is the phase shift between two excitations. The frequency of the forced excitation is nearly equal to the own frequency ω, and the frequency of the nonlinear parametric excitation is nearly twice as large. So, both excitations are in fundamental resonance. They will interact one to another.

Introducing new variables a and ψ instead of x and $\dot{x}$ as follows,

$$x = a\cos\theta. \quad \dot{x} = -a\omega\sin\theta, \quad \theta = \omega t + \psi, \qquad (1.2)$$

we have a system of two equations which is fully equivalent to (1.1)

$$\frac{da}{dt} = -\frac{\varepsilon}{\omega}F\sin\theta, \quad a\frac{d\psi}{dt} = -\frac{\varepsilon}{\omega}F\cos\theta, \qquad (1.3)$$

where

$$F = \Delta x - h\dot{x} - \gamma x^3 + 2px^3 \cos 2\omega t + r\cos(\omega t - \delta).$$

The equations (1.3) belong to the standard form, for which the asymptotic method is applied [1]. Thus, in the first approximation we can replace the right hand sides of (1.3) by their averaged values in time. We have the following averaged equations:

$$\frac{da}{dt} = -\frac{\varepsilon}{4\omega} f_0, \quad a\frac{d\psi}{dt} = -\frac{\varepsilon}{2\omega} g_0, \tag{1.4}$$

where

$$\begin{aligned} f_0 &= 2h\omega a + pa^3 \sin 2\psi + 2r\sin(\psi + \delta), \\ g_0 &= aE + pa^3 \cos 2\psi + r\cos(\psi + \delta), \\ E &= \Delta - \frac{3}{4}\gamma a^2. \end{aligned} \tag{1.5}$$

The stationary solution (a_0, ψ_0) of the equations (1.4) are determined by equations $\frac{da}{dt} = 0$, $\frac{d\psi}{dt} = 0$ or

$$\begin{aligned} \overline{f}_0 &= 2h\omega a_0 + pa_0^3 \sin 2\psi_0 + 2r\sin(\psi_0 + \delta) = 0, \\ \overline{g}_0 &= a_0 E_0 + pa_0^3 \cos 2\psi_0 + r\cos(\psi_0 + \delta) = 0, \\ E_0 &= \Delta - \frac{3}{4}\gamma a_0^2, \end{aligned} \tag{1.6}$$

or equivalently

$$\begin{aligned} f_1 &= \overline{f}_0 \cos\psi_0 - \overline{g}_0 \sin\psi_0 \\ &= \frac{3}{2} r\sin\delta + \frac{r}{2}\sin(2\psi_0 + \delta) + 2h\omega a_0 \cos\psi_0 + (pa_0^2 - E_0) a_0 \sin\psi_0 = 0, \\ g_1 &= \overline{f}_0 \sin\psi_0 + \overline{g}_0 \cos\psi_0 = \\ &= \frac{3}{2} r\cos\delta - \frac{r}{2}\cos(2\psi_0 + \delta) + 2h\omega a_0 \sin\psi_0 + (pa_0^2 + E_0) a_0 \cos\psi_0 = 0. \end{aligned} \tag{1.7}$$

Below, for simplicity, we consider only the case $\delta = 0$. To eliminate $\cos 2\psi_0$ and $\sin 2\psi_0$ from (1.6) and (1.7), we use the combinations

$$\begin{aligned} f &= \frac{r}{2}\overline{f}_0 - pa_0^3 f_1 = rh\omega a_0 - \left[pa_0^4(pa_0^2 - E_0) - r^2\right]\sin\psi_0 \\ &\quad - 2ph\omega a_0^4 \cos\psi_0 = 0, \\ g &= \frac{r}{2}\overline{g}_0 + pa_0^3 g_1 = \frac{r}{2} a_0 E_0 + \frac{3}{2} rpa_0^3 + 2ph\omega a_0^4 \sin\psi_0 \\ &\quad + \left[\frac{r^2}{2} + pa_0^4(pa_0^2 + E_0)\right]\cos\psi_0 = 0. \end{aligned} \tag{1.8}$$

The condition for equivalence of (1.6) and (1.8) is $r^2 \neq 4p^2a_0^6$. As usual, equations (1.8) are considered as two linear algebraic equations relative to two unknowns u and v : $u = \sin\psi_0$; $v = \cos\psi_0$. The elimination of the phase ψ_0 can be done by using the relationship $u^2 + v^2 = 1$. Two cases must be identified:

1. The "ordinary" case when the determinant D of the coefficients of u and v in (1.8) is different from zero, where

$$D = \begin{vmatrix} 2ph\omega a_0^4 & \dfrac{r^2}{2} + pa_0^4(pa_0^2 + E_0) \\ pa_0^4(pa_0^2 - E_0) - r^2 & 2ph\omega a_0^4 \end{vmatrix},$$

$$D = 4p^2h^2\omega^2a_0^8 + \left[r^2 - pa_0^4(pa_0^2 - E_0)\right]\left[\frac{r^2}{2} + pa_0^4(pa_0^2 + E_0)\right]. \tag{1.9}$$

2. The "critical" case when $D = 0$

2. Resonance curves in system without damping

Supposing that $h = 0$, the equations (1.6) become

$$\begin{aligned} &(pa_0^3\cos\psi_0 + r)\sin\psi_0 = 0, \\ &a_0(E_0 - pa_0^2) + r\cos\psi_0 + 2pa_0^3\cos^2\psi_0 = 0. \end{aligned} \tag{2.1}$$

From the equations (2.1) it follows

a) $\psi_0 = 0$ which corresponds to the resonance curve $C_1^{(1)}$:

$$E_0 = -pa_0^2 - \frac{r}{a_0} \quad \text{or} \quad \Delta = \left(\frac{3\gamma}{4} - p\right)a_0^2 - \frac{r}{a_0}, \quad a_0 \neq 0. \tag{2.2}$$

b) $\psi_0 = \pi$ which corresponds to the resonance curve $C_1^{(2)}$:

$$\Delta = \left(\frac{3}{4}\gamma - p\right)a_0^2 + \frac{r}{a_0}, \quad a_0 \neq 0. \tag{2.3}$$

c) $\psi_0 = \pm\arccos\left(\dfrac{-r}{pa_0^3}\right)$ which corresponds to the resonance curve C_2:

$$E_0 = pa_0^2 - \frac{r^2}{pa_0^4} \quad \text{or} \quad \Delta = \left(\frac{3}{4}\gamma + p\right)a_0^2 - \frac{r^2}{pa_0^4} \tag{2.4}$$

with limitation: $r^2 \leq p^2a^6$.

The curves $C_1^{(1)}$, $C_1^{(2)}$ and C_2 are presented in Fig. 1, where the curve C_2 is only the upper part of the curve (2.4) ended at point $I(r^2 = p^2a_0^6)$. The parameters for Fig. 1 are chosen so that $4p > \gamma$.

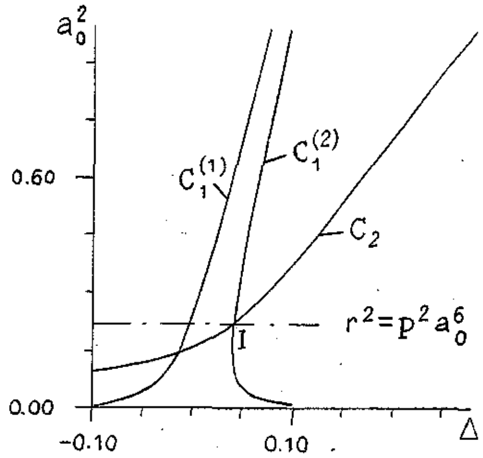

Fig. 1

3. Resonance curves in the system with damping

Solving the system (1.8) relative to $\sin\psi_0$ and $\cos\psi_0$ we obtain

$$
\begin{aligned}
&\sin\psi_0 = \frac{D_1}{D}, \quad \cos\psi_0 = \frac{D_2}{D}, \\
&D_1 = -\frac{1}{2} rh\omega a_0 \left[r^2 + 4pa_0^4(E_0 + 2pa_0^2)\right], \\
&D_2 = \frac{ra_0}{2}\left\{4ph^2\omega^2 a_0^4 + (E_0 + 3pa_0^2)\left[pa_0^4(pa_0^2 - E_0) - r^2\right]\right\}, \\
&D = 4p^2h^2\omega^2 a_0^8 + \left[r^2 - pa_0^4(pa_0^2 - E_0)\right]\left[\frac{r^2}{2} + pa_0^4(pa_0^2 + E_0)\right], \\
&\quad r \neq 2pa_0^3.
\end{aligned}
\tag{3.1}
$$

Eliminating the phase ψ_0 from (3.1), we obtain the following equation of the resonance curve H_1:

$$W(\Delta, a_0^2) = 0, \tag{3.2}$$

where

$$W = D_1^2 + D_2^2 - D^2. \tag{3.3}$$

After simple, but rather long calculations, we can write (3.3) in the form

$$W = \left(\frac{r^2}{4} - p^2a_0^6\right)\overline{W}, \tag{3.4}$$

where

$$\overline{W} = \alpha_0 E_0^4 + \alpha_1 E_0^3 + \alpha_2 E_0^2 + \alpha_3, \tag{3.5}$$

$$\alpha_0 = p^2 a_0^{10}, \quad \alpha_1 = 2pr^2 a_0^6, \quad E_0 = \Delta - \frac{3\gamma}{4} a_0^2,$$
$$\begin{aligned} \alpha_2 &= -2p^4 a_0^{14} + 8p^2 h^2 \omega^2 a_0^{10} + p^2 r^2 a_0^8 + r^4 a_0^2, \\ \alpha_3 &= p^6 a_0^{18} - 8p^4 h^2 \omega^2 a_0^{14} - 3r^2 p^4 a_0^{12} + 16p^2 h^4 \omega^4 a_0^{10} \\ &\quad - 20p^2 r^2 h^2 \omega^2 a_0^8 + 3r^4 p^2 a_0^6 + r^4 h^2 \omega^2 a_0^2 - r^6. \end{aligned} \tag{3.6}$$

It is easy to verify that the determinant D is different from zero along the resonance curve (3.2). Since $r^2 \neq 4p^2 a_0^6$ (equivalence condition of (1.6) and (1.8)) the equation (3.2) is equivalent to

$$\overline{W}(\Delta, a_0^2) = 0. \tag{3.7}$$

The resonance curves have three branches and are presented in Figs 2-3 for the parameters $r = 0.01$, $p = 0.1$, $\gamma = 0.25$, and $\omega^2 = 1.1$. With increasing h, the upper branch 1 moves up and the two lower branches 2 and 3 are tied and then separated, as branches 4 and 5, see Fig. 2 for $h = 0.01$ and Fig. 3 for $h = 0.027$.

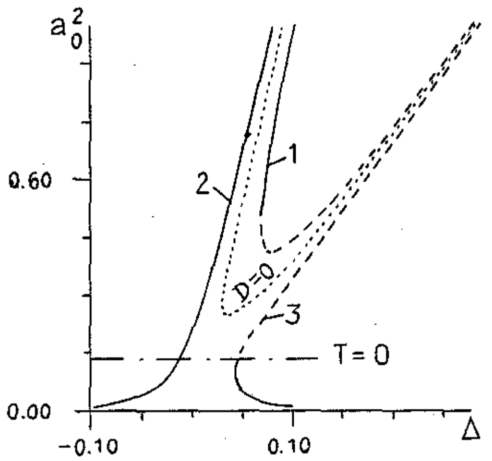

Fig. 2

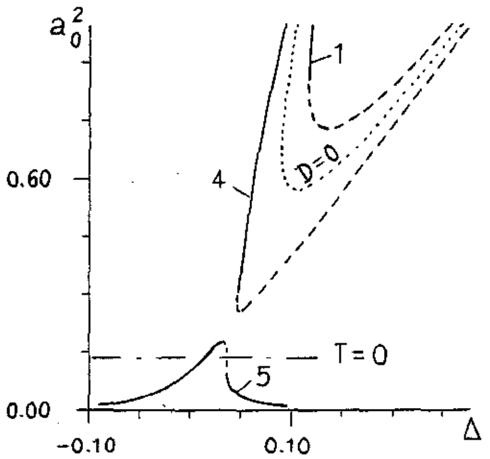

Fig. 3

4. Stability of stationary oscillations

Setting in (1.4) $a = a_0 + \delta a$, $\psi = \psi_0 + \delta\psi$ and neglecting the terms with higher than one degree relative to δa, $\delta\psi$ we have the following equations in variation:

$$\begin{aligned} \frac{d\delta a}{dt} &= -\frac{\varepsilon}{4\omega}\left[\left(\frac{\partial f_0}{\partial a}\right)_0 \delta a + \left(\frac{\partial f_0}{\partial \psi}\right)_0 \delta\psi\right], \\ a_0 \frac{d\delta\psi}{dt} &= -\frac{\varepsilon}{2\omega}\left[\left(\frac{\partial g_0}{\partial a}\right)_0 \delta a + \left(\frac{\partial g_0}{\partial \psi}\right)_0 \delta\psi\right], \end{aligned} \tag{4.1}$$

where the symbol $(\ )_0$ denotes that $a = a_0$, $\psi = \psi_0$. The characteristic equation of this system of equations is

$$\begin{vmatrix} -\frac{\varepsilon}{4\omega}\left(\frac{\partial f_0}{\partial a}\right)_0 - \lambda & -\frac{\varepsilon}{4\omega}\left(\frac{\partial f_0}{\partial \psi}\right)_0 \\ -\frac{\varepsilon}{2\omega}\left(\frac{\partial g_0}{\partial a}\right)_0 & -\frac{\varepsilon}{2\omega}\left(\frac{\partial g_0}{\partial \psi}\right)_0 - a_0\lambda \end{vmatrix} = 0. \tag{4.2}$$

The first stability condition will be

$$S_1 = a_0 \frac{\partial \overline{f}_0}{\partial a_0} + 2\frac{\partial \overline{g}_0}{\partial \psi_0} = 4h\omega a_0 > 0. \tag{4.3}$$

The second stability condition is

$$S_2 = \frac{\partial \overline{f}_0}{\partial a_0}\frac{\partial \overline{g}_0}{\partial \psi_0} - \frac{\partial \overline{f}_0}{\partial \psi_0}\frac{\partial \overline{g}_0}{\partial a_0} > 0. \tag{4.4}$$

From equations (1.8)

$$f = \left(\frac{r}{2} - pa_0^3\cos\psi_0\right)\overline{f}_0 + (pa_0^3\sin\psi_0)\overline{g}_0,$$
$$g = \left(pa_0^3\sin\psi_0\right)\overline{f}_0 + \left(pa_0^3\cos\psi_0 + \frac{r}{2}\right)\overline{g}_0,$$

and from $\overline{f}_0 = 0$, $\overline{g}_0 = 0$, it follows:

$$\frac{\partial f}{\partial a_0}\frac{\partial g}{\partial \psi_0} - \frac{\partial f}{\partial \psi_0}\frac{\partial g}{\partial a_0} = \left(\frac{r^2}{4} - p^2a_0^6\right)S_2. \tag{4.5}$$

The second stability condition (4.4) is equivalent to

$$S_2 = \frac{1}{T}\left(\frac{\partial f}{\partial a_0}\frac{\partial g}{\partial \psi_0} - \frac{\partial f}{\partial \psi_0}\frac{\partial g}{\partial a_0}\right) = \frac{1}{2DT}\frac{\partial W}{\partial a_0} > 0, \quad T = \frac{r^2}{4} - p^2a_0^6. \tag{4.6}$$

According to (3.4) we have $\frac{\partial W}{\partial a_0} = T\frac{\partial \overline{W}}{\partial a_0}$ along the resonance curve $\overline{W} = 0$. Therefore the condition (4.6) takes the form:

$$S_2 = \frac{a_0}{D}\frac{\partial \overline{W}}{\partial a_0^2} > 0. \tag{4.7}$$

It is noted that since $\overline{W}\big|_{a_0=0} = -r^6 < 0$, $D\big|_{a_0=0} = \frac{r^2}{2} > 0$ one can easily identify the regions of the (a_0, Δ) - plane where the functions $\overline{W}$ and D are positive

(+) and negative (−) and therefore know the stability branches of the resonance curves. In Figures 2 - 3 these branches are presented by heavy lines, while the instability branches are shown by dotted lines.

5. Conclusion

The interaction between cubic nonlinear parametric and forced oscillations in a system governed by the differential equation (1.1) has been investigated by the asymptotic method of nonlinear mechanics. The typical amplitude curves of stationary oscillations are presented in Figs 1-3. The amplitude curves in Fig 1-2 are similar to that of the interaction between linear parametric and forced oscillations (see [1], Figs 94 and 98, page 275). The amplitude curves in Fig. 3 characterize the nonlinear system under consideration. For small values of a_0 the forced component is dominated and the corresponding parts of resonance curves are similar to those of forced oscillation. For large values of a_0 the influence of the parametric component is clear, and as the result of the interaction between two oscillations, the resonance curve has the form of an upward parabola.

The stability of the stationary oscillations obtained is studied by using the variational equations. The stability criterion in the form (4.7) is convenient for geometric interpretation. The jump phenomenon takes place on some branches of the resonance curve.

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TƯƠNG TÁC GIỮA DAO ĐỘNG CƯỠNG BỨC VÀ THÔNG SỐ PHI TUYẾN

Sự tương tác của các dao động phi tuyến là một bài toán hay, quan trọng và đã thu hút sự chú ý của nhiều nhà nghiên cứu. Minorsky N. đã phát biểu rằng: "Toàn bộ lý thuyết dao động phi tuyến có thể được hình thành dựa trên cơ sở của sự tương tác".

Sự tương tác giữa dao động cưỡng bức và dao động thông số "tuyến tính", khi hệ số của hàm điều hòa của thời gian là tuyến tính đối với thông số định vị đã được nghiên cứu trong các tài liệu [1] và [4]. Trong bài báo này xét sự tương tác giữa dao động thông số phi tuyến bậc ba với dao động cưỡng bức. Phương pháp tiệm cận của cơ học phi tuyến [1] đã được sử dụng để nghiên cứu các dao động dừng và sự ổn định của chúng.

Các đường biên - tần điển hình của dao động dừng được biểu diễn trên hình 1-3. Các đường cong trên hình 1-2 có dạng tương tự như trường hợp tương tác giữa dao động cưỡng bức và thông số "tuyến tính" (xem [1], hình 94 và 98 trang 275). Các đường cộng hưởng trên hình 3 rất đặc trưng cho hệ phi tuyến khảo sát. Với các giá trị a_0 nhỏ thành phần cưỡng bức đóng vai trò áp đảo và phần đường cộng hưởng tương ứng có dạng tương tự như trong trường hợp dao động cưỡng bức thuần túy. Với những giá trị lớn của a_0, ảnh hưởng của thành phần thông số khá rõ. Kết quả của sự tương tác giữa hai dao động kể trên là đường cộng hưởng có dạng parabôn.

Sự ổn định của các dao động dừng được nghiên cứu bằng cách sử dụng phương pháp biến phân. Tiêu chuẩn ổn định dưới dạng (4.7) rất thuận lợi cho việc phân định các nhánh ổn định.

Hiện tượng nhảy biên độ cũng xuất hiện trên một số nhánh của đường cộng hưởng.