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INTERACTION BETWEEN THE FORCED AND PARAMETRIC EXCITATIONS WITH DIFFERENT DEGREES OF SMALLNESS

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ABSTRACT. The nonlinear system under consideration in this paper has a specification which can be stated as an interaction between the first order of smallness nonresonance parametric excitation and the second order of smallness resonance forced excitation. In the first approximation these excitations have no effect. However, they do interact one with another in the second approximation.

The equations for the amplitude and phase of oscillation are found by means of the asymptotic method. The stationary oscillations and their stability are of special interest.

1. The equation of motion and asymptotic solutions

Let us consider a nonlinear system governed by the differential equation

$$\ddot{x} + \omega^2 x = \varepsilon p x \cos\omega t + \varepsilon^2\big[\Delta x - 2h\dot{x} - \beta x^3 + r\cos(\omega t - \eta)\big], \tag{1.1}$$

where

$$\omega^2 = 1 + \varepsilon^2\Delta, \tag{1.2}$$

ε is a small dimensionless parameter, 1 is natural frequency, Δ is detuning parameter, p, h, β , r, η, ω are constants and overdots denote differentiation with respect to time t.

We look for the solution of the equation (1.1) in the form:

$$x = a\cos\theta + \varepsilon u_1(a,\psi,\theta) + \varepsilon^2 u_2(a,\psi,\theta) + \dots, \tag{1.3}$$

where $\theta = \omega t + \psi$, $u_i(a,\psi,\theta)$ are periodic functions with period 2π with respect to both angular variables ψ and θ, and a and ψ are functions of time which will be determined from the equations:

$$\begin{aligned}\frac{da}{dt} &= \varepsilon A_1(a,\psi) + \varepsilon^2 A_2(a,\psi) + \dots,\\ \frac{d\psi}{dt} &= \varepsilon B_1(a,\psi) + \varepsilon^2 B_2(a,\psi) + \dots.\end{aligned} \tag{1.4}$$

these equations $A_i(a,\psi)$, $B_i(a,\psi)$ are periodic functions of the angular variable with period 2π.

Substituting the expressions (1.3) and (1.4) into the equation (1.1) and comring the coefficient of ε^1 we obtain

$$-2\omega A_1 \sin\theta - 2\omega a B_1 \cos\theta + \omega^2\Big(\frac{\partial^2 u_1}{\partial\theta^2} + u_1\Big) = ap\cos(\theta-\psi)\cos\theta. \tag{1.5}$$

›mparing the harmonics in (1.5) gives:

$$\begin{aligned} A_1 &= B_1 = 0,\\ u_1 &= \frac{pa}{2\omega^2}\Big[\cos\psi - \frac{1}{3}\cos(2\theta-\psi)\Big]. \end{aligned} \tag{1.6}$$

Comparing the coefficients of ε^2 in (1.1) we have

$$\begin{aligned} &-2\omega A_2 \sin\theta - 2\omega a B_2 \cos\theta + \omega^2\Big(\frac{\partial^2 u_2}{\partial\theta^2} + u_2\Big) = pu_1\cos\omega t + \Delta a\cos\theta\\ &+ 2h\omega a\sin\theta - \beta a^3\cos^3\theta + r\cos[\theta-(\psi+\eta)]. \end{aligned} \tag{1.7}$$

ıuating the coefficients of the first harmonics $\sin\theta$ and $\cos\theta$ in (1.7) we obtain

$$\begin{aligned} A_2(a,\psi) &= -ha - \frac{p^2 a}{8\omega^3}\sin 2\psi - \frac{r}{2\omega}\sin(\psi+\eta),\\ B_2(a,\psi) &= -\frac{\Delta}{2\omega} - \frac{p^2}{12\omega^3} + \frac{3\beta}{8\omega}a^2 - \frac{p}{8\omega^3}\cos 2\psi - \frac{r}{2\omega a}\cos(\psi+\eta), \end{aligned} \tag{1.8}$$

$^2 \approx 1$.

Thus, in the second approximation one has:

$$x = a\cos\theta + \frac{\varepsilon p}{2\omega^2}\Big[\cos\psi - \frac{1}{3}\cos(2\theta-\psi)\Big], \quad \theta = \omega t + \psi, \tag{1.9}$$

here a and ψ satisfy the following differential equations:

$$\begin{aligned} \frac{da}{dt} &= -\frac{\varepsilon^2}{2\omega}\Big[2h\omega a + \frac{p^2 a}{4}\sin 2\psi + r\sin(\psi+\eta)\Big],\\ \frac{d\psi}{dt} &= -\frac{\varepsilon^2}{2\omega}\Big[\Delta + \frac{p^2}{6} - \frac{3\beta}{4}a^2 + \frac{p^2}{4}\cos 2\psi + \frac{r}{a}\cos(\psi+\eta)\Big], \end{aligned} \tag{1.10}$$

$\neq 0$.

2. Stationary solutions

Denoting

$$C = \Delta + \frac{p^2}{6} - \frac{3}{4}\beta a_0^2, \quad D = \frac{p^2}{4}, \quad H = 2h\omega, \tag{2.1}$$

we have the following equations for stationary values a_0, ψ_0 satisfying the relations:

$$\frac{da}{dt} = \frac{d\psi}{dt} = 0; \quad f = g = 0, \tag{2.2}$$

where

$$\begin{aligned} f &= Ha_0 + Da_0 \sin 2\psi_0 + r\sin(\psi_0 + \eta), \\ g &= Ca_0 + Da_0 \cos 2\psi_0 + r\cos(\psi_0 + \eta). \end{aligned} \tag{2.3}$$

We transform equations (2.2) into two equivalent ones:

$$\begin{aligned} f\cos\psi_0 - g\sin\psi_0 &= (D-C)a_0\sin\psi_0 + Ha_0\cos\psi_0 + r\sin\eta = 0, \\ f\sin\psi_0 + g\cos\psi_0 &= Ha_0\sin\psi_0 + (D+C)a_0\cos\psi_0 + r\cos\eta = 0. \end{aligned} \tag{2.4}$$

The condition for reality of $\sin\psi_0$ and $\cos\psi_0$ is [2, 3]:

$$a_0^2\left[(D-C)^2 + H^2\right] \geq r^2\sin^2\eta, \tag{2.5}$$

$$a_0^2\left[(D+C)^2 + H^2\right] \geq r^2\cos^2\eta. \tag{2.6}$$

a) Supposing that

$$M = D^2 - (H^2 + C^2) \neq 0, \tag{2.7}$$

we have from equations (2.4):

$$\begin{aligned} \sin\psi_0 &= \frac{r\left[H\cos\eta - (D+C)\sin\eta\right]}{\left[D^2 - (H^2+C^2)\right]a_0}, \\ \cos\psi_0 &= \frac{r\left[H\sin\eta - (D-C)\cos\eta\right]}{\left[D^2 - (H^2+C^2)\right]a_0}. \end{aligned} \tag{2.8}$$

Eliminating ψ_0 we obtain:

$$W(a_0, \omega) = 0, \tag{2.9}$$

where

$$W = a_0^2\left[D^2 - (H^2+C^2)\right]^2 - r^2\left[H^2 + D^2 + C^2 - 2DC\cos 2\eta - 2HD\sin 2\eta\right]. \tag{2.10}$$

b) If

$$M = D^2 - (H^2 + C^2) = 0, \tag{2.11}$$

i.e. if the resonance curve takes the form

$$C = \pm\sqrt{D^2 - H^2}\,,$$

or

$$\frac{3}{4}\beta a_0^2 = \delta \pm \sqrt{\left(\frac{p^2}{4}\right)^2 - 4h^2\omega^2}\,, \quad \delta = \Delta + \frac{p^2}{6}\,, \tag{2.12}$$

then by (2.8) one should have

$$N_1 = H\cos\eta - (D+C)\sin\eta = 0, \quad N_2 = H\sin\eta - (D-C)\cos\eta = 0,$$

or equivalently,

$$N_1\cos\eta + N_2\sin\eta = 0, \quad N_1\sin\eta - N_2\cos\eta = 0.$$

These relations give:

$$H = D\sin 2\eta, \quad C = D\cos 2\eta.$$

Substituting these values into (2.5) and (2.6) we obtain the following restriction to the amplitude a_0:

$$a_0^2 \geq \frac{r^2}{4D^2}\,. \tag{2.13}$$

Note. As it will be seen later, the curve (2.11) serves as the boundary of the stability zone.

3. System without friction

Now, let us consider a special case when $h = 0$ and the equations (2.4) have the form:

$$\begin{aligned}(D-C)a_0\sin\psi_0 &= -r\sin\eta,\\ (D+C)a_0\cos\psi_0 &= -r\cos\eta.\end{aligned} \tag{3.1}$$

a) If $D - C \neq 0$ and $D + C \neq 0$, then the resonance curve C_1 is determined by the equation of type (2.10) with $H = 0$:

$$W(\omega^2, a_0^2) = 0, \tag{3.2}$$

where

$$W(\omega^2, a_0^2) = a_0^2(D^2 - C^2)^2 - r^2(D^2 + C^2 - 2DC\cos 2\eta). \tag{3.3}$$

In a particular case, when $\eta = 0, \pi$ the resonance curve C_1 degenerates into

1) The curve C_1^1 : $D = C$ (double)
2) The curve C_1^3 : $a_0^2(D+C)^2 - r^2 = 0$.

When $\eta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ the resonance curve C_1 degenerates into

3) The curve C_1^4: $D = -C$ (double)
4) The curve C_1^6: $a_0^2(D-C)^2 - r^2 = 0$.

b) If $D - C = 0$ (the resonance curve C_2), then from (3.1) we have

$$0.a_0 \sin\psi_0 = -r\sin\eta \Rightarrow \sin\eta = 0 \Rightarrow \eta = 0, \pi,$$

$$2Da_0\cos\psi_0 = -r\cos\eta = \pm r \Rightarrow \psi_0 = \arccos\Big(\pm\frac{r}{2Da_0}\Big) \Rightarrow a_0^2 \geq \frac{r^2}{4D^2}\cdot$$

c) If $D + C = 0$ (the resonance curve C_3), then from (3.1) we have

$$0.a_0\cos\psi_0 = -r\cos\eta \Rightarrow \cos\eta = 0 \Rightarrow \eta = \frac{\pi}{2}, \frac{3\pi}{2},$$

$$2Da_0\sin\psi_0 = -r\sin\eta = \pm r \Rightarrow \psi_0 = \arcsin\Big(\pm\frac{r}{2Da_0}\Big) \Rightarrow a_0^2 \geq \frac{r^2}{4D^2}\cdot$$

4. Stability of stationary oscillations

With the notation (2.1) the equations (1.10) can be written in the form:

$$\begin{aligned} \frac{da}{dt} &= -\frac{\varepsilon^2}{2\omega}\big[Ha + Da\sin 2\psi + r\sin(\psi+\eta)\big], \\ a\frac{d\psi}{dt} &= -\frac{\varepsilon^2}{2\omega}\big[Ca + Da\cos 2\psi + r\cos(\psi+\eta)\big]. \end{aligned} \tag{4.1}$$

To study the stability of stationary oscillations with amplitude a_0 and phase ψ_0 determined from equations (2.2) or (2.4) we introduce the variations:

$$\widetilde{a} = a - a_0, \quad \widetilde{\psi} = \psi - \psi_0.$$

Substituting these values into (4.1) we obtain

$$\begin{aligned} \frac{d\widetilde{a}}{dt} &= -\frac{\varepsilon^2}{2\omega}\Big\{(H + D\sin 2\psi_0)\widetilde{a} + \big[2Da_0\cos 2\psi_0 + r\cos(\psi_0+\eta)\big]\widetilde{\psi}\Big\}, \quad (4.2) \\ a_0\frac{d\widetilde{\psi}}{dt} &= -\frac{\varepsilon^2}{2\omega}\Big\{(C + C'a_0 + D\cos 2\psi_0)\widetilde{a} - \big[2Da_0\sin 2\psi_0 + r\sin(\psi_0+\eta)\big]\widetilde{\psi}\Big\}, \end{aligned}$$

where $C' = -\frac{3}{2}\beta a_0$.

The characteristic equation for last two equations is

$$a_0\lambda^2 + \frac{\varepsilon^2}{2\omega}H_*\lambda - \frac{\varepsilon^4}{4\omega^2}S = 0, \tag{4.3}$$

where λ is characteristic numbers,

$$H_* = a_0\big[H - D\sin 2\psi_0 - r\sin(\psi_0 + \eta)\big] = 4h\omega a_0 > 0, \tag{4.4}$$

$$\begin{aligned} S = {} & (H + D\sin 2\psi_0)\big[2Da_0\sin 2\psi_0 + r\sin(\psi_0 + \eta)\big] \\ & + (C + C'a_0 + D\cos 2\psi_0)\big[2Da_0\cos 2\psi_0 + r\cos(\psi_0 + \eta)\big]. \end{aligned} \tag{4.5}$$

The expression for S can be written as

$$S = a_0(D^2 - H^2 - C^2 - a_0CC') + a_0^2C'D\cos 2\psi_0. \tag{4.6}$$

From (2.2) and (2.3) it follows:

$$Da_0\cos 2\psi_0 = -Ca_0 - r(\cos\psi_0\cos\eta - \sin\psi_0\sin\eta).$$

Substituting here the expressions $\cos\psi_0$ and $\sin\psi_0$ from (2.8) we obtain

$$Da_0\cos 2\psi_0 = -Ca_0 - \frac{r^2}{a_0(D^2 - H^2 - C^2)}(C - D\cos 2\eta).$$

Thus, we have

$$\begin{aligned} 2(D^2 - H^2 - C^2)S = {} & 2a_0(D^2 - H^2 - C^2)^2 - 4a_0^2CC'(D^2 - H^2 - C^2) \\ & - 2r^2CC' + 2r^2DC'\cos 2\eta = \frac{\partial W}{\partial a_0}, \end{aligned}$$

$$S = \frac{1}{2(D^2 - H^2 - C^2)}\cdot\frac{\partial W}{\partial a_0}, \quad (D^2 - H^2 - C^2) \neq 0. \tag{4.7}$$

Thus, the stability condition of the stationary solutions a_0 and ψ_0 takes the form

$$M\frac{\partial W}{\partial a_0} > 0, \tag{4.8}$$

$$M = H^2 + C^2 - D^2. \tag{4.9}$$

The resonance curve ($W = 0$) divides the plane (a_0, ω) into regions, in each of which the expression W has a definite sign (+ or −). If moving up along the straight line parallel to the axis a_0, we pass from a region $W < 0$ to a region $W > 0$, then at the point of intersection between the straight line and the resonance curve the derivative $\partial W/\partial a_0$ is positive. So, this point corresponds to a stable state of oscillation if $M > 0$ and to an unstable one if $M < 0$. On the contrary, if we pass from a region $W > 0$ to a region $W < 0$, then the point of intersection corresponding to a stable state of oscillation if $M < 0$ and to an unstable one if $M > 0$.

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TƯƠNG TÁC GIỮA CÁC KÍCH ĐỘNG THÔNG SỐ VÀ CƯỠNG BỨC
CÓ BẬC BÉ KHÁC NHAU

Sự tương tác giữa kích động thông số không cộng hưởng có độ bé bậc một với kích động cưỡng bức cộng hưởng có độ bé bậc hai đã được khảo sát. Ở xấp xỉ thứ nhất các kích động này không gây ra hiệu quả. Song chúng tương tác lẫn nhau trong xấp xỉ thứ hai. Các dao động dừng và sự ổn định của chúng đã được đặc biệt quan tâm nghiên cứu.