Vietnam Journal of Mechanics, NCST of Vietnam Vol. **21**, 1999, No 4 (194 - 202)

IDENTIFYING THE RESONANCE CURVE OF A SYSTEM SUBJECTED TO LINEAR AND QUADRATIC PARAMETRIC EXCITATIONS
(case without damping)

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Continuing our study in [3], in the present paper we examine the system without damping $h = 0$. In this particular case, the original equations and the associated ones are very simple and can easily be solved. The "exact" original resonance curve C_0 and the "exact" associated one C will be given. Thus, we are able to compare two resonance curves and to "estimate" the indirect method proposed. Although the system without damping seems to be trivial, the "structure" of its resonance curve is not so. On the other hand, the "difference" between C_0 and C is "greater": in the non equivalence line, there are not only strange representative points (which are ordinary points) but also strange dephases (at critical representative points). Consequently, the indirect method presented in [3] must be modified and developed.

§1. System under consideration - The direct method

In the case without damping, the system under consideration is described by the differential equation

$$\ddot{x} + \omega^2 x = \varepsilon\Big\{\Delta x - \gamma x^3 + 2px\cos 2\omega t + 2qx^2\cos\omega t\Big\} \tag{1.1}$$

The asymptotic method [1] leads to the averaged differential equations

$$\begin{cases} \dot{a} = \dfrac{-\varepsilon a}{2\omega} f_0 = \dfrac{-\varepsilon a}{2\omega}\Big\{p\sin 2\theta + \dfrac{1}{2}qa\sin\theta\Big\}, \\ a\dot{\theta} = \dfrac{-\varepsilon a}{2\omega} g_0 = \dfrac{-\varepsilon a}{2\omega}\Big\{\Big(\Delta - \dfrac{3\gamma}{4}a^2\Big) + p\cos 2\theta + \dfrac{3}{2}qa\cos\theta\Big\}. \end{cases} \tag{1.2}$$

and stationary oscillations are determined by the "original" equations:

$$\begin{cases} f_0 = p\sin 2\theta + \dfrac{1}{2}qa\sin\theta = 0, \\ g_0 = \Big(\Delta - \dfrac{3\gamma}{4}a^2\Big) + p\cos 2\theta + \dfrac{3}{2}qa\cos\theta = 0. \end{cases} \tag{1.3}$$

Solving (1.3) we obtain

$$\theta = \theta_1 = 0, \qquad \text{along} \quad \Delta = \frac{3\gamma}{4}a^2 - p - \frac{3}{2}qa, \tag{1.4}$$

$$\theta = \theta_2 = \pi, \qquad \text{along} \quad \Delta = \frac{3\gamma}{4}a^2 - p + \frac{3}{2}qa, \tag{1.5}$$

$$\theta = \pm\theta_3 = \pm\arccos\Big(\frac{-qa}{4p}\Big), \text{ along} \quad \Delta = \frac{3\gamma}{4}a^2 + p + \frac{q^2a^2}{4p}, \tag{1.6}$$

$$\text{under restriction} \qquad a^2 \le 4a_*^2 = \frac{16p^2}{q^2}\cdot \tag{1.7}$$

Thus, in the (semi-upper) plane $R(\Delta; a^2 > 0)$, the "original" resonance curve C_0 consists of three branches

- the left-half "parabola" P_1 : (1.4) with $\theta_1 = 0$,

- the right-half "parabola" P_2 : (1.5) with $\theta = \pi$,

- the segment J_1J_2 of the line (1.6), respectively bounded below and above by $J_1(\Delta = p; a^2 = 0)$ and $J_2\big(\Delta = \frac{3\gamma}{4}4a_*^2 + 5p; a^2 = 4a_*^2\big)$ with two dephases $\pm\theta_3 = \pm\arccos\Big(\frac{-qa}{4p}\Big)$.

In Fig. 1, the resonance curve (heavy line) is plotted for $\gamma = 0.04$, $p = 0.01$, $q = 0.03$.

Note that J_1J_2 intersects the right half parabola P_2 at two points: J_2 and $I_2\big(\Delta_2 = \frac{3\gamma}{4}a_*^2 + 2p; a^2 = a_*^2\big)$; the latter corresponds to three dephases $\big(\pi, \pm\frac{2\pi}{3}\big)$. Also note that the line $a^2 = a_*^2$ (i.e. the non equivalence line $T = 4p^2 - q^2a^2 = 0$) passes through I_2 and intersects the left-half parabola P_1 at $I_1\big(\Delta = \frac{3\gamma}{4}a_*^2 - 4p; a^2 = a_*^2\big)$.

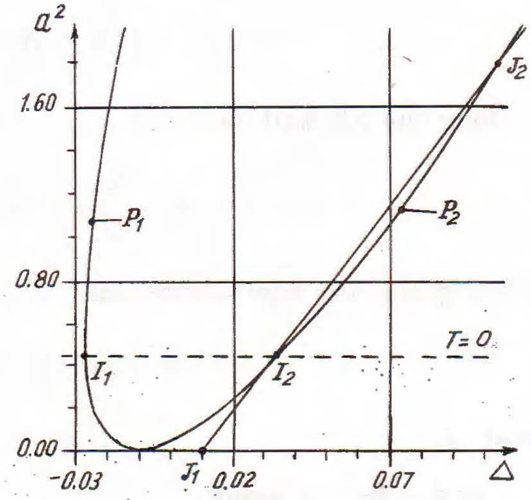

Fig. 1

§2. The indirect method - The associated resonance curve

As in [3] we use the transformation:

$$\begin{cases} f = (2p\cos\theta + qa)f_0 - 2p\sin\theta \cdot g_0, \\ g = 2p\sin\theta \cdot f_0 + (2p\cos\theta - qa)g_0 = 0, \end{cases} \tag{2.1}$$

whose matrix is

$$\{T\} = \begin{Bmatrix} 2p\cos\theta + qa & -2p\sin\theta \\ 2p\sin\theta & 2p\cos\theta - qa \end{Bmatrix}. \tag{2.2}$$

The determinant $T = 4p^2 - q^2a^2$ depends only on a. The associated equations are:

$$f = A\sin\theta = 0, \quad g = H\cos\theta - K = 0, \tag{2.3}$$

where

$$\begin{aligned} A &= \Big\{2p\Big[p - \Big(\Delta - \frac{3\gamma}{4}a^2\Big)\Big] + \frac{1}{2}q^2a^2\Big\} = -\frac{1}{2}(T + 4pX), \\ H &= \Big\{2p\Big[p + \Big(\Delta - \frac{3\gamma}{4}a^2\Big)\Big] - \frac{3}{2}q^2a^2\Big\} = \frac{1}{2}(3T + 4pX), \\ K &= qa\Big(\Delta - \frac{3\gamma}{4}a^2 - 2p\Big) = qaX. \end{aligned} \tag{2.4}$$

Recall that $T = 0$ is the non equivalence line.

The associated equations (2.3) can also be solved directly:

1/ For $A \neq 0$ (out off the line $A = 0$ i.e. the line (2.3)), we obtain:

- either $\theta = \theta_1 = 0$ along

$$H - K = 2\big(X + 3p + \frac{3}{2}qa\big)(2p - qa) = 0, \tag{2.5}$$

i.e. along the left-half parabola

$$P_1 : X + 3p + \frac{3}{2}qa = \Delta - \frac{3\gamma}{4}a^2 + p + \frac{3}{2}qa = 0, \tag{2.6}$$

and along the non equivalence line

$$2p - qa = 0 \quad (\text{i.e. } a^2 = a_*^2), \tag{2.7}$$

except I_2

- or $\theta = \theta_2 = \pi$ along

$$H + K = 2(2p + qa)\Big(X + 3p - \frac{3}{2}qa\Big) = 0, \tag{2.8}$$

i.e. along the right-half parabola

$$P_2 : X + 3p - \frac{3}{2}qa = \Delta - \frac{3\gamma}{4}a^2 + p - \frac{3}{2}qa = 0, \tag{2.9}$$

except I_2 and J_2

2/ For $A = 0$, $H \neq 0$ we obtain

$$\theta = \pm\theta_3 = \pm\arccos\frac{K}{H} = \pm\arccos\Big(\frac{-4qa}{4p}\Big), \tag{2.10}$$

with restriction $qa \leq 4p$ or $a^2 \leq 4a_*$ i.e. $\theta = \pm\theta_3$ along the segment J_1J_2 except I_2

3/ For $A = 0$, $H = 0$ i.e. at I_2 (the intersection point of two lines $A = 0$ and $H = 0$) the dephase θ is arbitrary.

Thus, the associated resonance curve C differs enough from the original one C_0:

- Except I_1 and I_2, all other points of the non equivalence line $T = 0$ are strange representative points.

- At I_2, except three values $\Big(\pi, \pm\frac{2\pi}{3}\Big)$, all other values of θ (which is arbitrary!) are strange dephases.

Using the procedure in [2], we obtain some more useful remarks.

Three characteristic determinants are:

$$\begin{aligned} D &= \begin{vmatrix} A & 0 \\ 0 & H \end{vmatrix} = AH = -\frac{1}{4}(T + 4pX)(3T + 4pX), \\ D_1 &= \begin{vmatrix} 0 & 0 \\ K & H \end{vmatrix} \equiv 0, \\ D_2 &= \begin{vmatrix} A & 0 \\ 0 & K \end{vmatrix} = AK = -\frac{1}{2}(T + 4pX)qaX. \end{aligned} \tag{2.11}$$

The critical region $D = 0$ consists of two lines

$$A = 0 \quad \text{(the line (2.3)) and} \quad H = 0 \tag{2.12}$$

and the compatible ensemble $D = D_1 = D_2 = 0$ is the line

$$A = 0 \quad \text{(the line (2.3)).} \tag{2.13}$$

The associated frequency - amplitude relationship is

$$\begin{aligned} W(\Delta, a^2) = D_2^2 - D^2 &= A^2(K^2 - H^2) = A^2\Big\{q^2a^2X^2 - \frac{1}{4}(3T + 4pX)^2\Big\} = \\ &= -A^2\Big\{(4p^2 - q^2a^2)X^2 + 6pXT + \frac{9}{4}T^2\Big\} = \\ &= -TA^2\Big(X^2 + 6pX + \frac{9}{4}T\Big) = 0. \end{aligned} \tag{2.14}$$

The factor T corresponds to the non equivalence line $T = 0$; it is an ordinary branch of the associated resonance curve (except I_2).

The last factor of $W(\Delta, a^2)$ can be factorized as:

$$X^2 + 6pX + \frac{9}{4}T = \left(X + 3p + \frac{3}{2}qa\right)\left(X + 9p - \frac{3}{2}qa\right) \tag{2.15}$$

and corresponds to the "parabola" $P = P_1 \cup P_2$ with two left and right half parabolas P_1, P_2 (except I_2 and J_2). The parabola P is also an ordinary branch of the associated resonance curve C.

The double factor A^2 corresponds to the compatible line (2.3). Along this compatible line, we must verify the trigonometrical condition $H^2 \leq K^2$ which leads to the segment J_1J_2. Thus J_1J_2 forms the critical part C_2 of the associated resonance curve C. Along J_1J_2 we have two critical dephases $\pm\theta_3$, except at I_3, the critical dephase is arbitrary.

§3. The indirect method - Elimination of strange elements

The results obtained in §1, §2 show that we have to demonstrate two following propositions:

1 - Except I_1, in the non equivalence line $T = 0$ all (other) ordinary representative points of the associated resonance curve C are strange representative ones.

2 - At I_2, except $\left(\pi, \pm\frac{2\pi}{3}\right)$, all (other) dephases are strange critical dephases.

The method of artificial dephase can be applied again with necessary modifications: since $D_1 \equiv 0$, another definition of the artificial dephase is chosen; the calculus at limit depends on trajectories along which we approach I_2.

First, let us examine ordinary representitative points in the non equivalence line. By $I(\Delta_*, a_*^2)$ and θ_* we denote the point of interest and its (associated) dephase. At I, we have

$$D_* = D(\Delta_*, a_*^2) = -4p^2X_*^2 < 0, \tag{3.1}$$

$$\sin\theta_* = \left(\frac{D_1}{D}\right)_* = 0, \ \cos\theta_* = \left(\frac{D_2}{D}\right)_* = \frac{qa_*}{2p} = 1 \text{ i.e. } \theta_* = 0. \tag{3.2}$$

By $N(\Delta, a^2)$ we denote an arbitrary point in the neighbourhood of I but out off $T = 0$. At N we have $T \neq 0$ and - by continuity - $D < 0$.

We introduce an angle $\bar{\theta}$ called artificial dephase, defined as

$$\cos\bar{\theta} = \frac{D_2}{D}, \quad \sin\bar{\theta} = \pm\sqrt{1 - \frac{D_2^2}{D^2}} \tag{3.3}$$

(the sign before radical is arbitrary chosen). Obviously, when N tends to I, the artificial dephase introduced tends to $\theta_* = 0$.

At N, since $T \neq 0$, from (2.1), we can express (f_0, g_0) as combinations of (f, g):

$$f_0(\Delta, a, \overline{\theta}) = \frac{1}{T}\Big\{(2p\cos\overline{\theta} - qa)f + 2p\sin\overline{\theta}\cdot g\Big\}, \qquad g_0(\Delta, a, \overline{\theta}) = \frac{1}{T}\Big\{-2p\sin\overline{\theta}\cdot f + (2p\cos\overline{\theta} + qa)g\Big\}. \tag{3.4}$$

Using (2.1), (3.3), (2.11), (2.4), regarding that $HD_2 - KD = 0$, we can transform (3.4) as:

$$f_0(\Delta, a, \overline{\theta}) = \frac{1}{TD}(2pD_2 - qaD)A\sin\overline{\theta} = \frac{(2pAK - qaAH)A\sin\overline{\theta}}{T\cdot AH} = = \frac{A(2pK - qaH)\sin\overline{\theta}}{TH} = -\frac{3qa}{2}\cdot\frac{A\sin\overline{\theta}}{H}, \tag{3.5}$$

$$g_0(\Delta, a, \overline{\theta}) = \frac{1}{T}\Big\{-2p\sin\overline{\theta}\cdot A\sin\overline{\theta}\Big\} = -\frac{2pA}{T}\Big(1 - \frac{D_2^2}{D^2}\Big) = = \frac{2pA\cdot W}{TD^2} = \frac{-2pA}{D^2}W_0(\Delta, a^2), \tag{3.6}$$

where

$$W_0(\Delta, a^2) = \frac{W(\Delta, a^2)}{T} = -A^2\Big(X^2 + 6pX + \frac{9}{4}T\Big). \tag{3.7}$$

At limit ($\Delta \to \Delta_*$, $a \to a_*$, $\overline{\theta} \to \Delta_*$), since $I \neq I_2$, we have

$$\begin{aligned} &qa \to qa_* = 2p, \quad A \to A_* \neq 0, \quad H \to H_* \neq 0,\\ &D \to D_* \neq 0, \quad W_0(\Delta, a^2) \to W_0(\Delta_*, a_*^2),\\ &f_0(\Delta, a, \overline{\theta}) \to f_0(\Delta_*, a_*, \theta_*), \quad g_0(\Delta, a, \overline{\theta}) \to g_0(\Delta_*, a_*, \theta_*). \end{aligned} \tag{3.8}$$

Therefore

$$\begin{cases} f_0(\Delta_*, a_*, \theta_*) = 0,\\ g_0(\Delta_*, a_*, \theta_*) = -2p\dfrac{A_*}{D_*^2}W_0(\Delta_*, a_*^2), \end{cases} \tag{3.9}$$

$I(\Delta_*, a_*^2)$ and θ_* form an element of the original resonance curve C_0 if and only if $f_0(\Delta_*, a_*, \theta_*)$ and $g_0(\Delta_*, a_*, \theta_*)$ simultaneously vanish. From (3.9), it follows that the required conditions lead to $W_0(\Delta_*, a_*^2) = 0$ i.e. $I(\Delta_*, a_*^2)$ must be an intersection point of the non-equivalence line $T = 0$ and the curve $W_0(\Delta, a^2) = 0$ (the point I_1).

Finally, we examine the critical point $I_2\Big(\Delta_2 = \frac{3\gamma}{4}a_*^2 + 2p; a_*^2\Big)$. In this case, the limit value of $\overline{\theta}$ and, consequently, those of $f_0(\Delta, a, \overline{\theta})$, $g(\Delta, a, \overline{\theta})$ depend on the trajectory along which N tends to I_2.

Let us denote: Γ - the trajectory of interest; $a^2 = a^2(\Delta)$ - the equation of Γ; $k = \left.\dfrac{da^2}{d\Delta}\right|_{I_2}$ - the slope of Γ at I_2.

Considering T, X, A, H, K as functions of Δ (a^2 replaced by $a^2(\Delta)$) we can easily obtain their developments in the neighbourhood of I_2:

$$\begin{aligned}
T &= 4p^2 - q^2a^2 = -kq^2(\Delta - \Delta_*),\\
X &= \Delta - \frac{3\gamma}{4}a^2 - 2p = \frac{3\gamma}{4}(k_X - k)(\Delta - \Delta_*),\\
A &= -\frac{1}{2}(T + 4pX) = -\frac{1}{2}(q^2 + 3\gamma p)(k_A - k)(\Delta - \Delta_*),\\
H &= \frac{1}{2}(3T + 4pX) = \frac{3}{2}(q^2 + \gamma p)(k_H - k)(\Delta - \Delta_*),\\
K &= qaX = qa_*\frac{3\gamma}{4}(k_K - k)(\Delta - \Delta_*) = \frac{3\gamma p}{2}(k_K - k)(\Delta - \Delta_*),
\end{aligned} \tag{3.10}$$

where $k_X = k_K$, k_A, k_H are slopes at I_2 of the lines $X = 0$, $K = 0$, $A = 0$, $H = 0$, respectively:

$$k_X = k_K = \frac{4}{3\gamma}, \quad k_A = \frac{4p}{q^2 + 3\gamma p}, \quad k_H = \frac{4p}{3(q^2 + \gamma p)}. \tag{3.11}$$

Following developments are also useful

$$\begin{aligned}
D &= AH = -\frac{3}{4}(q^2 + 3\gamma p)(q^2 + \gamma p)(k_A - k)(k_H - k)(\Delta - \Delta_*)^2,\\
D_2 &= AK = -\frac{3\gamma p}{4}(q^2 + 3\gamma p)(k_A - k)(k_H - k)(\Delta - \Delta_*)^2,\\
&X + 3p + \frac{3}{2}qa = 6p + \ldots,\\
&X + 3p - \frac{3}{2}qa = \frac{3(2\gamma p + q^2)}{8p}(k_{P_2} - k)(\Delta - \Delta_*),
\end{aligned} \tag{3.12}$$

where k_{P_2} is the slope of the right half parabola P_2 at I_2

$$k_{P_2} = \frac{8p}{3(2\gamma p + q^2)}. \tag{3.13}$$

By θ_k we denote the limit of $\overline{\theta}$: $\theta_k = \lim \overline{\theta}$ (N approaches I_2 along the trajectory Γ with slope k at I_2). We have

$$\cos\theta_k = \lim \cos\overline{\theta} = \frac{\gamma p}{q^2 + \gamma p} \cdot \frac{k_K - k}{k_H - k}. \tag{3.14}$$

Particularly:

if $k = 0$ $\cos\theta_0 = 1$ i.e. $\theta_0 = 0$ ($k = 0$ is the slope of $T = 0$),

if $k = k_{P_2}$ $\cos\theta_{k_{P_2}} = -1$ i.e. $\theta_{k_{P_2}} = \pi = \theta_2$,

If $k = k_A$ $\cos\theta_{k_A} = -\dfrac{1}{2}$ i.e. $\theta_{k_A} = \pm\dfrac{2\pi}{3}$.

Returning to (2.5), using the development of A and H, we obtain at limit

$$f_0(\Delta_2, a_*, \theta_k) = +9p\sin\theta_k \cdot \frac{q^2 + 3\gamma p}{q^2 + \gamma p} \cdot \frac{k_A - k}{k_H - k}. \tag{3.15}$$

It follows that $f_0(\Delta_2, a_*, \theta_k)$ vanishes if either $\theta_k = \theta_0 = 0$ or $\theta_k = \theta_{k_{P_2}} = \pi$ or $k = k_A$ i.e. $\theta_k = \theta_{k_A} = \pm\dfrac{2\pi}{3}$. At limit, the expression (3.6) can be written as:

$$\begin{aligned} g_0(\Delta_2, a_*, \theta) &= \frac{2p(X + 3p + \frac{3}{2}qa)}{H^2} A(X + 3p = \frac{3}{2}qa) \\ &= \frac{-2p(q^2 + 3\gamma p)(2\gamma p + q^2)}{(q^2 + \gamma p)^2 (k_H - k)^2}(k_H - k)(k_{P_2} - k). \end{aligned} \tag{3.16}$$

Evidently $g_0(\Delta_2, a_*, \theta_*)$ vanishes if either $k = k_A$ i.e. $\theta_k = \theta_{k_A} = \pm\dfrac{2\pi}{3}$ or $k = k_{P_2}$ i.e. $\theta_k = \theta_{k_{P_2}} = \pi$.

We obtain thus the known result: at I_2 only $\theta = \pi$ and $\theta = \pm\dfrac{2\pi}{3}$ are "original" dephases, all other associated dephases are strange.

Remark 1. From the demonstration we can conclude that three original dephases at $I_2\left(\theta = \pi, \pm\dfrac{2\pi}{3}\right)$ coincide with the limit values of the associated dephases if we approach I_2 by moving along associated branches passing through I_2.

Remark 2. The existence of the (real) artificial dephase $\bar\theta$ requires $\left|\dfrac{K}{H}\right| \le 1$ which is equivalent to

$$T\Big\{\Delta - \Big(\frac{3\gamma}{4}a^2 - p - \frac{3}{2}qa\Big)\Big\}\Big\{\Delta - \Big(\Delta - \Big(\frac{3\gamma}{4}a^2 - p + \frac{3}{2}qa\Big)\Big\} \ge 0. \tag{3.17}$$

Thus, the point N must be chosen in shaded domain I, II, III shown in Fig. 2.

Remark 3. Starting from (II, III), N tends to I_2, the slope $k \in (-\infty, 0) \cup [k_2, +\infty)$ and the graph of $\cos\theta_k$ is of the form shown in Fig. 3. Thus, for arbitrary given associated dephases θ_k, there always exists corresponding trajectories Γ with the slope k at I_2 so that $\lim\bar\theta = \theta_k = \theta_*$.

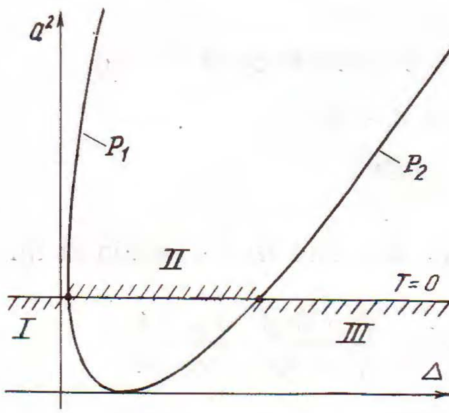

Fig. 2

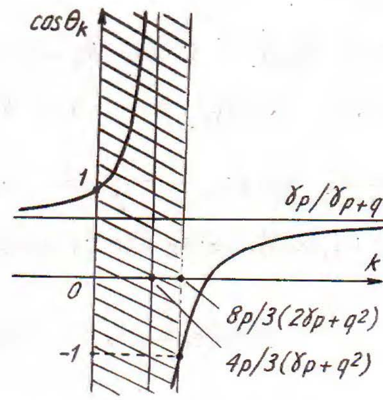

Fig. 3

Conclusion

We have identified the resonance curve of a quasi-linear system subjected to linear and quadratic pasrametric excitations in the case without damping. The direct and indirect methods have been used. Although the non-equivalence line contains critical representative point at which strange dephases exist, the original resonance can also be obtained from the associated one.

This publication is completed with financial support from The Council for Natural Sciences of Vietnam.

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Received September 15, 1998

LẬP ĐƯỜNG CỘNG HƯỞNG CỦA HỆ CHỊU HAI KÍCH ĐỘNG THÔNG SỐ BẬC NHẤT VÀ BẬC HAI (TRƯỜNG HỢP KHÔNG CẢN)

Xét hệ đã khảo sát ở [3] trong trường hợp không cản, $h = 0$. Đường không tương đương chứa điểm tới hạn do các pha liên hợp là bất kỳ. Phương pháp pha nhân tạo được áp dụng để loại các pha tới hạn lạ.