GLOBAL ANALYSIS IN THE NUMERICAL TREATMENT OF THE SKYRME EQUATION

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ABSTRACT

The Skyrme equation used widely in Physics is singular at the boundary r = 0. The singularity causes uncontrollable instabilities in the numerical solutions. This paper presents a new computa- tional schema to overcome this difficulty to give the solutions with an arbitrarily high accuracy by combining the numerical methods with a global analysis.

Keywords: computational physics, numerical computation, differential equation

1. INTRODUCTION

It was Skyrme who has shown that nucleons can be modeled as solitons of a non-linear differential equation [1]. In the static and spherically symmetric case, it simplifies to the Skyrme equation

$$F''(r) = \frac{G(r)}{K(r)} \qquad , \qquad G(r) = 2\sin^2\big(F(r)\big) + \frac{r^2}{4} ,$$

$$K(r) = \sin\big(2F(r)\big)\left(-\big(F'(r)\big)^2 + \frac{\sin^2\big(F(r)\big)}{r^2} + \frac{1}{4}\right) - \frac{rF'(r)}{2} + \frac{m^2 r^2}{4}\sin\big(F(r)\big) \qquad (1)$$

for the real-valued function F (r) of the radial coordinate r. The solutions of the Skyrme equation can be found numerically with the following boundary condition

$$F(0) = N\pi \quad , \quad F(\infty) = 0 \qquad (2)$$

where N is an integer. Adkins, Nappi and Witten [2, 3] have shown the numerical solution of Eq.(1) describes nucleons within 30 % errors.

The idea of Skyrme has become very popular, but the solutions of the Skyrme equation have not been studied systematically, except the approximated analytic solution of Atyiah-Manton [4]. It is not clear whether topologically non-stable solutions exist besides the topologically stable ones. In the recent numerical experiments [5], we have shown that the solutions presented in [2, 3] are problematic in the infinitesimal neighborhood of the origin r = 0, and the numerical results are not reliable.

In this paper, we will conduct a systematic schema by combining the numerical treatment with a global analysis, leading to more reliable numerical results

2. THE NUMERICAL EXPERIMENTS

Using the forward shooting method [6] starting from the boundary point F(0) = Nπ, we can obtain with an arbitrarily high precision the numerical solutions which oscillate with damping around the F(r) = 0 axis. These solutions have infinite energy.

We can obtain the finite energy solution by following procedure: At the large value of the variable r, Eq.(1) simplifies to

$$F''(r) = m^2 \sin(F(r)) - \frac{2}{r} F'(r) \tag{3}$$

which has the following asymptotic solution at large r

$$F(r) = C\,\frac{e^{-mr}}{r} \tag{4}$$

where C is an integration constant. We can use the backward shooting method starting from the point r = R = 20 with the asymptotic formula (4). Varying the value of C, one finds a solution within a certain chosen error as shown in Fig. 1 for both cases m = 0 and m ≠ 0.

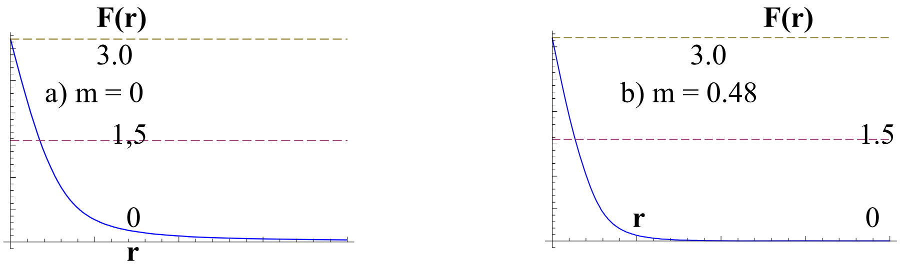

Figure 1. The profile function of a skyrmion with m = 0 (a) m = 0,48 (b).

These solutions become unstable, if a too high precision is required. It can be explained as follows:

Let us draw the solution curve with m = 0 in Fig. 1 with a magnification, in the neighborhood r = 0 to the distant of 10−11 in Fig. 2.

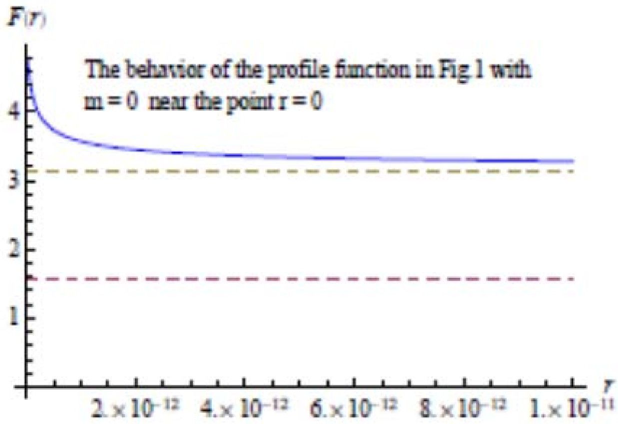

Figure 2. Magnification of the curve m = 0 in Fig.1a in the neighborhood of r = 0.

The above curve goes very close to F (0) = π, but turns sharply to the points F (0) = 3π/2. So, the topological charge of the solution derived from this curve is not 1. If a too high precision is required, it is very difficult to reach the point F (0) = π by the backward shooting method.

But such a solution still exists. We can demonstrate its existence by drawing all possible solution trajectories near the real solution as in Fig. 3.

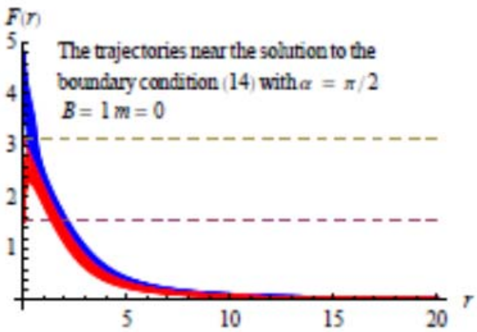

Figure 3. The trajectories near the solution to the boundary condition with α = π/2.

Since most solution trajectories of Eq. (1) swing between F (0) = π/2 or F (0) = 3π/2 just by a very small change in initial values of the parameter C. The trajectories can go as close to the point F (0) = π as possible, but in the last step they turn to up or down direction sharply. The trajectory which separates the up going and down going trajectories is the solution going to the point F(0) = π.

3. GLOBAL ANALYSIS

3.1. Boundary conditions and asymptotic solutions

Equation (2) is still not a complete initial or boundary condition for Eq. (1). In the numerical experiments, we have observed that the trajectories of Eq. (1) are attracted to the points F (0) = kπ/2 with a deviation less than 10 %. The attempts to increase the precision in the neighborhood of r = 0 often result in instabilities. Fortunately, we are able to prove the following proposition for the asymptotic behavior of all possible solutions of the Skyrme equation near the origin r = 0.

Proposition 1. *The solutions of the Skyrme equation must have one of the following asymptotic formulas*

$$F(r) = \frac{(4k+1)\pi}{2} + \beta r - \frac{3m^2}{4} r^2 - \frac{\beta}{24} r^3 + \frac{m^2}{24} r^4 + \frac{\beta}{640} r^5 - \frac{m^2}{1080} r^6 \quad (5)$$

$$F(r) = \frac{(4k+3)\pi}{2} + \beta r + \frac{3m^2}{4} r^2 - \frac{\beta}{24} r^3 - \frac{m^2}{24} r^4 + \frac{\beta}{640} r^5 + \frac{m^2}{1080} r^6 \quad (6)$$

$$F(r) = n\pi + \beta r + \frac{\beta m^2}{12(4\beta^2+1)} r^3 \quad (7)$$

Proof. Let us introduce the parameter a and the function Y (r) as follows

$$a = \sin(F(0)) \quad (8)$$

$$Y(r) = F(r) - F(0) \quad (9)$$

We have the limit

$$\lim_{r \to 0}\big(Y(r)\big) = 0 \quad (10)$$

In the case a ≠ 0, Eq. (1) have the following asymptotic form in the neighborhood of r = 0

$$Y''(r) = \frac{m^2}{8a} r^2 - \frac{r}{4a^2} Y'(r) + \frac{a\sqrt{1-a^2}}{r^2} \quad (11)$$

Eq. (11) has the following analytic solution

$$Y(r) = C[2] + \frac{1}{4} a m^2 r^2 + C[1] a \sqrt{2\pi}\, Erf\left(\frac{r}{2\sqrt{2a}}\right)$$

$$+ \frac{1}{2a}\left(\sqrt{1-a^2} - 2a^2 m^2\right) r^2 \, {}_2F_2\left(1, 1; 3/2, 2; -\frac{r^2}{8a^2}\right) - a\sqrt{1-a^2}\ln(r) \quad (12)$$

where C[1] and C[2] are two arbitrary integration parameters. The Gauss error function $Erf(x)$ is defined in Ref.[7] as follows

$$Erf(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(2n+1)} = \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216} + \cdots\right) \tag{13}$$

$$Erf(0) = 0 \tag{14}$$

The generalized hyper-geometric function ${}_2F_2(1,1;b,2;x)$ is defined in Ref.[8] as follows

$${}_2F_2(1,1;b,2;x) = \sum_{n=0}^{\infty} \frac{n!\,x^n}{b_n\,(n+1)!} \tag{15}$$

$${}_2F_2(1,1;b,2;0) = 1 \tag{16}$$

where the Pochhammer symbol b_n is defined as follows

$$b_0 = 1 \;\; ; \;\; b_n = b(b+1)\ldots.(b+n-1) \tag{17}$$

Since $\lim_{r\to 0} \ln(r) = -\infty$, the condition (10) requires C[2] = 0 and a = ±1, which means that F (0) = (2k + 1)π/2, if F (0) ≠ nπ. Eq.(12) leads to the following asymptotic formula of the profile function F (r)

$$F(r) = (2k+1)\frac{\pi}{2} + \frac{1}{4}\sin\left(\frac{(2k+1)\pi}{2}\right)m^2r^2 + \sqrt{2}\,C[1]\,Erf\left(\frac{1}{2\sqrt{2}}\right) - \sin\left(\frac{(2k+1)\pi}{2}\right)m^2r^2\;{}_2F_2\left(1,1;\frac{3}{2},2;-\frac{r^2}{8}\right) \tag{18}$$

In the infinitesimal neighborhood of r = 0, the Gauss error and generalized hyper-geometric functions can be approximated as follows

$$Erf\left(\frac{r}{2\sqrt{2}}\right) \approx \frac{r}{\sqrt{2\pi}}\left(1 - \frac{r^2}{24} + \frac{r^4}{640}\right) \tag{19}$$

$${}_2F_2\left(1,1;\frac{3}{2},2;-\frac{r^2}{8}\right) \approx 1 - \frac{r^2}{24} + \frac{r^4}{1080} \tag{20}$$

From Eqs. (19) - (20), we obtain the asymptotic formulas (5) and (6). In the case a = 0, the following approximations can be used when r → 0

$$\frac{\sin(F(r))}{r} \to F'(0) = \beta \quad ;$$

$$\frac{\sin(2F(r))}{r} \to 2\beta \quad ;$$

$$\frac{1}{4} + \frac{\sin^2(F(r))}{r^2} - (F'(r))^2 \to \frac{1}{4} \tag{21}$$

$$\frac{r^2}{4} + 2\sin^2(F(r)) - \frac{r^2}{4}\left(1 + \frac{8\sin^2(F(r))}{r^2}\right) \to \frac{r^2}{4}(1 + 8\beta^2) \tag{22}$$

where β is a finite value number. Eq.(1) is now reduced to the following asymptotic form

$$Y''(r) = (1 + 8\beta^2)^{-1}\left((m^2 + 2)r\beta \quad 2\frac{Y'(r)}{r}\right) \tag{23}$$

Equation (23) has the following analytic solution

$$Y(r) = r\beta + \frac{m^2 r^3 \beta}{12(1 + 4\beta^2)} \tag{24}$$

satisfying Y (0) = 0 and Y ′(0) = F ′(0) = β. Eq. (24) implies the asymptotic formula (7).

In summary, at the origin r = 0, the solutions of the Skyrme equation must have the values F (0) = nπ/2. Beside the solution with the usual boundary value F (0) = nπ and the asymptotic formula (7), there are solutions with the boundary values F (0) = (4k + 1)π/2 and F (0) = (4k + 3)π/2 and the asymptotic formulas (5) and (6) respectively. In the asymptotic formulas, there is only one free parameter β = F ′(0) to choose to satisfy the second boundary condition as r → ∞.

Let us consider the cases k = 1 and n = 1 in Eqs.(5-7), we have three families of solutions with the boundary condition at the origin r = 0 F (0) = π/2, π, 3/2π. As we have seen in the numerical experiments, the topologically stable solution with the boundary condition F (0) = π is the limit between the other two families of solution with the boundary conditions F (0) = π/2 and F (0) = 3π/2.

In the next section, we will examine the energy of these solutions and see that only the solution with the boundary condition F (0) = π is energetically stable.

3.2. Energy finiteness and skyrmions

The energetically stable solutions must have a finite energy, which is given with the following formula [1-3]

$$M = \frac{4\pi F_\pi}{e} M_0 = \frac{4\pi F_\pi}{e}\int_0^R \Bigg(\frac{\sin^2(F(r))}{4} + \frac{(rF'(r))^2}{8} + \frac{m^2 r^2}{4}\big(1 - \cos(F(r))\big) + \sin^2(F(r))\left(\frac{\sin^2(F(r))}{2r^2} + (F'(r))^2\right)\Bigg) \tag{25}$$

where F_π = 186M eV is the pion decay constant and e is the parameter of the Skyrme model satisfying the Balachandra's bound [9]

$$1.57 < e < 4.17 \tag{26}$$

The finiteness of the expression (25) has two implications. Firstly, the finiteness of the last term in the limit $r \to 0$ implies that $\sin(F(r)) \to 0$. Hence, for the skyrmions we have the following boundary value

$$F(0) = N\pi \tag{27}$$

In other words, Eq.(25) selects out the skyrmions from the infinite energy solutions, which have the boundary condition $F(0) = (N+1/2)\pi$. Secondly, in the limit $r \to \infty$, the first term in the integrand must tend to zero to keep the energy finite, which means that $F(\infty) = k\pi$. So, the condition $F(\infty) = 0$ is not necessary to keep for the skyrmion solutions. Instead we can look for the skyrmion solution with the following boundary condition

$$F(0) = N\pi \quad ; \quad F(\infty) = k\pi \tag{28}$$

where N and k are integers. The difference N−k can be interpreted as the conserved baryon number.

In the article [5], the skyrmions which are confined within a finite radius R are also studied with possible applications in the dense hadronic matter.

4. NUMERICAL CALCULATIONS COMBINED WITH THE GLOBAL ANALYSIS

In the light of the global analysis presented in Sect 3, we can understand the results of the numerical experiments of Sect 2.

First, we understand why the numerical backward shooting method can hit the solution easily with an asymptotic formula at the large value of $r \to \infty$.

Secondly, we understand that the obtained numerical solutions are not the real finite energy skyrmions. If we increase the accuracy, the obtained trajectories will turn to other boundary values in the small neighborhood of $r = 0$. This infinitesimal behavior will make the energy infinite. Thus the numerically obtained solutions are not stable both topologically and energetically.

Thirdly, the topologically and energetically stable solutions in fact exist as a limit between the two neighboring unstable solution families. This can be seen in Fig. 3.

Lastly, having in mind the asymptotic formulas (7) in principle, we can use the numerical forward shooting method in the small neighborhood of $r = 0$ and change the value of $\beta = F'(0)$ until the second boundary condition $F(\infty) = k\pi$ is satisfied.

In Fig. 4, we choose the case of $F(\infty) = \pi$, $F(0) = 2\pi$ and $m = 0.48$, the numerically obtained solution has a complicated damping oscillating behavior.

However, using the energy formula (25), we can show numerically that the energy of this solution is not finite. So, it is not energetically stable. The attempts to vary the value of β to avoid the oscillating damping solution in search of the stable solutions have not given the converged results.

To compromise between the above numerical methods, we have used the forward shooting method in a small neighborhood of $r = 0$ and the backward one outside of it. This solution has a good large distant behavior, while in the small neighborhood, the solution depends on one parameter $\beta = F'(0)$. In principle, we can vary the value of β until, the derivatives of two curves coincide at a "patching" distant $R = 0.01$ for example.

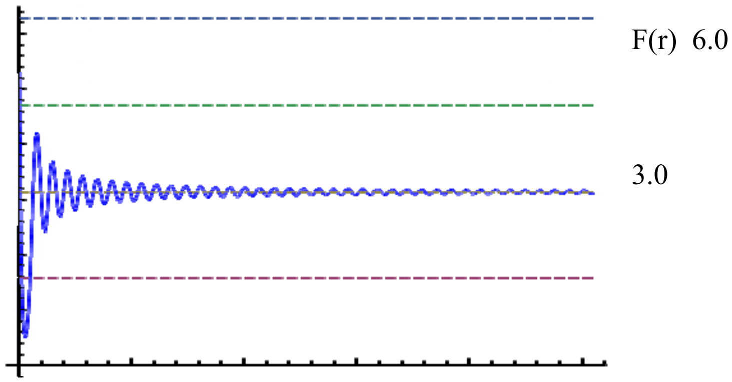

Figure 4. The damping behavior of a solution with a topological charge B = 1, m = 0.48, a = π.

In order to make the algorithm convergent, we have used the above ”patched” solution as the initial values for a combined backward and forward finite-difference numerical schema. Unfortunately, the used numerical schema does not converge.

5. DISCUSSION

In this paper, we have shown the issues in the numerical treatment of the Skyrme equation in a systematical way. A global analysis helps us to understand the numerically obtained results and the irregular behavior of the solutions. The topologically and energetically stable solutions are difficult to achieve. However, they can be approximated by the unstable with an arbitrarily high accuracy. In the light of the global analysis, we can distinguish the unstable solutions with the accumulated numerical errors of the stable one.

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TÓM TẮT

PHÂN TÍCH TOÀN CỤC TRONG XỬ LÍ SỐ ĐỐI VỚI PHƯƠNG TRÌNH SKYRME

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Phương trình Skyrme được sử dụng rộng rãi trong Vật lí có kì dị tại biên r = 0. Kì dị gây ra mất ổn định. Bài báo đưa ra một khung tính toán mới để vượt qua khó khăn này để đưa ra các lời giải với độ chính xác cao tùy ý bằng cách phồ hợp các phương pháp tính số với phân tích toàn cục

Từ khóa: vật lí tính toán, tính số, phương trình vi phân.