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NONLINEAR FREE VIBRATION OF MICROBEAMS PARTIALLY SUPPORTED BY FOUNDATION USING A THIRD-ORDER FINITE ELEMENT FORMULATION

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Abstract. Geometrically nonlinear free vibration of microbeams partially supported by a three-parameter nonlinear elastic foundation is studied in this paper. Equations of motion based on the modified couple stress theory (MCST) and a refined third-order shear deformation beam theory are derived using Hamilton's principle, and they are solved by a finite element formulation. The validity of the derived formulation is verified by comparing the present results with the published data for the case of the microbeams fully resting on the foundation. Numerical investigation is carried out to show the effects of the length scale parameter, the aspect ratio, the nondimensional amplitude and the boundary conditions on the nonlinear free vibration behavior of the microbeams. The obtained numerical results reveal that the foundation supporting length plays an important role on the vibration of the microbeams, and the influence of the foundation supporting length on the frequency ratio is dependent on the boundary conditions. It is also shown that the frequency ratio is decreased by the increase of the length scale, regardless of the boundary condition and the initial deflection. The influence of the nonlinear foundation stiffness on the ratio of nonlinear frequency to linear frequency of the microbeams is also studied and discussed.

Keywords: Microbeam, modified couple stress theory, refined third-order beam theory, nonlinear elastic foundation, nonlinear free vibration.

Classification numbers: 5.2.4, 5.4.2

1. INTRODUCTION

Thanks to the advanced technologies, the micro/nanoelectromechanical systems (MEMS/NEMS) can now be easily manufactured from various materials. The main structures used in the MEMS/NEMS are beams, plates and shells. Due to the small size effect, the classical continuum theories (CCTs) are not sufficient to model mechanical behavior of these microstructures. Other theories such as the higher-order continuum theories (HCTs) have been developed to accompany a material length scale parameter (MLSP) [1] in modeling mechanical behavior of these microstructures. The HCTs have been adopted by many researchers in analyzing the MEMS/NEMS equipped with beams/plates/shells [2 - 4]. A review of the HCTs for analysis of microstructures can be found in [5].

The modified couple stress theory (MCST) developed by Yang *et al*. [4] for nonlinear vibration analysis of microbeams can be considered as the most popular HCTs. The theory includes only one MLSP, and the couple stress tensor is symmetric. Wang *et al*. [6] presented a nonlinear free vibration analysis of Euler-Bernoulli microbeams on the basis of the MCST and von Kármán geometrically nonlinear theory. This problem was also studied by Ke *et al*. [7], but for microbeams made from functionally graded material. Static bending, postbuckling and free vibration of nonlinear microbeams were investigated by Xia *et al*. [8], in which the nonlinear model was considered within the context of non-classical continuum mechanics via the introduction of a material length scale parameter.

The effect of nonlinear elastic foundation support on free vibration of microstructures has been reported by several authors. Şimşek [9] studied nonlinear bending and free vibration of microbeams on a nonlinear elastic foundation using MCST and He's variational method. The nonlinear forced vibration analysis of a higher-order shear deformable functionally graded microbeam fully resting on a nonlinear elastic foundation based on modified couple stress theory was investigated by Debabrata [10].

To the authors' best knowledge, the nonlinear free vibration of microbeams partially supported by a nonlinear elastic foundation has not been reported in the literature, and it is studied in the present work. Based on the modified couple stress theory (MCST) and a refined third-order shear deformation beam theory, the governing equations and associated boundary conditions for the microbeams are derived from Hamilton's principle and they are solved by a finite element formulation. The verification of the derived formulation is performed, and then a parametric study is carried out to highlight the effects of the aspect ratio, amplitude, the material length scale and the boundary conditions on the nonlinear frequencies of the microbeams. It is worthy to note that in addition to the influence of the partial foundation support on the vibration of the microbeams, the third-order shear deformation theory employed for the first time in geometric nonlinear analysis herein is the novel point of the present paper.

2. MATHEMATICAL MODEL

An isotropic microbeam of length L, rectangular cross section ($b\times h$), partially supported by a foundation, as depicted in Figure 1, is considered. The foundation considered herein is a nonlinear foundation model stiffness of the Winkler elastic medium k_w, Pasternak elastic medium k_s and nonlinear elastic medium k_{NL} [9]. It is assumed that the beam is supported by the foundation from the left end, and the supporting length is L_F. The Cartesian system (x, y, z) in Figure 1 is chosen such that the x-axis is on the mid-plane and along the length, while the y-axis is along the width and the z-axis directs upwards.

The refined third-order shear deformation theory [11], in which the transverse displacement is split into bending and shear parts, is adopted herewith. According to the theory, the displacements of a point in x, y and z directions, $u_1(x,z,t)$, $u_2(x,z,t)$ and $u_3(x,z,t)$, respectively, are given by

$$
\begin{aligned}
&u_1(x,z,t)=u_0(x,t)-z\cdot w_{b,x}(x,t)-\left(\frac{5z^3}{3h^2}-\frac{z}{4}\right)w_{s,x}(x,t)\\
&u_2(x,z,t)=0,\ u_3(x,z,t)=w_b(x,t)+w_s(x,t)
\end{aligned}
\tag{1}
$$

where $u_0(x,t)$ is the axial displacement of a point on the x-axis; $w_b(x,t)$ and $w_s(x,t)$ are, respectively, the bending and shear components of the transverse displacement. A subscript comma in Eq. (1) and hereafter is used to denote the derivative with respect to the followed variable, e.g. $w_{b,x}=\partial w_b/\partial x$.

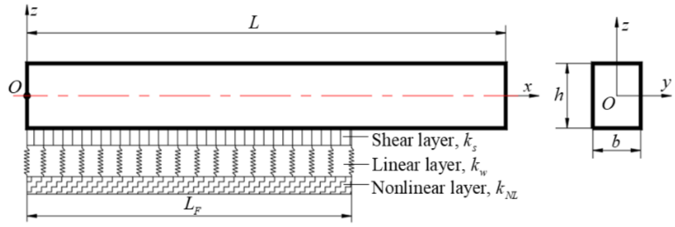

Figure 1. Geometry of an isotropic microbeam partially supported by a nonlinear elastic foundation.

The strain components based on the von-Kármán's nonlinear strain–displacement relationship resulted from Eq. (1) are of the forms

$$
\begin{aligned}
&\varepsilon_{xx}=u_{1,x}+\frac{1}{2}u_{3,x}^2=u_{0,x}-zw_{b,xx}-f\,w_{s,xx}+\frac{1}{2}\left(w_{b,x}+w_{s,x}\right)^2\\
&\varepsilon_{zz}=\frac{1}{2}u_{1,z}^2=\frac{1}{2}\left(w_{b,x}+f_{,z}w_{s,x}\right)^2,\ \gamma_{xz}=2\varepsilon_{xz}=u_{1,z}+u_{3,x}=gw_{s,x}
\end{aligned}
\tag{2}
$$

with

$$
f=\frac{5z^3}{3h^2}-\frac{z}{4};\ \ g=1-f_{,z} \tag{3}
$$

The constitutive equations based on linear behavior of the material are

$$
\begin{Bmatrix}\sigma_{xx}\\ \sigma_{zz}\\ \tau_{xz}\end{Bmatrix}=\begin{bmatrix}C_{11} & C_{12} & 0\\ C_{12} & C_{11} & 0\\ 0 & 0 & C_{22}\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{zz}\\ \gamma_{xz}\end{Bmatrix},\ C_{11}=\frac{E}{1-\upsilon^2},\ C_{12}=\upsilon C_{11},\ C_{22}=\frac{1-\upsilon}{2}C_{11} \tag{4}
$$

where E and υ are the Young's modulus and Poisson's ratio of the beam material.

Based on the modified couple stress theory proposed by Yang *et al*. [4], the strain energy U in a deformed linear elastic body occupying a volume V can be written in the form

$$
U=\frac{1}{2}\int_V\left(\boldsymbol{\sigma}:\boldsymbol{\varepsilon}+\mathbf{m}:\boldsymbol{\chi}\right)\mathrm{d}V \tag{5}
$$

where $\boldsymbol{\sigma}$ is the classical stress tensor; $\boldsymbol{\varepsilon}$ is the strain tensor; $\mathbf{m}$ is the deviatoric part of the couple stress tensor, and $\boldsymbol{\chi}$ is the symmetric curvature tensor. These tensors can be written in the form

$$
\boldsymbol{\sigma}=\begin{bmatrix}\sigma_{xx} & 0 & \tau_{xz}\\ 0 & 0 & 0\\ \tau_{xz} & 0 & \sigma_{zz}\end{bmatrix},\ \boldsymbol{\varepsilon}=\begin{bmatrix}\varepsilon_{xx} & 0 & \varepsilon_{xz}\\ 0 & 0 & 0\\ \varepsilon_{xz} & 0 & \varepsilon_{zz}\end{bmatrix} \tag{6}
$$

$$
\mathbf{m}=2l^2\mu\boldsymbol{\chi} \tag{7}
$$

$$\boldsymbol{\chi} = \frac{1}{2}[\nabla\boldsymbol{\theta} + (\nabla\boldsymbol{\theta})^T] \tag{8}$$

with l is the material length scale parameter which reflects the effect of the couple stress, μ is the Lame's constant, and $\boldsymbol{\theta}$ is the rotation vector, defined by

$$\boldsymbol{\theta} = \frac{1}{2}\mathrm{curl}(\mathbf{u}) \tag{9}$$

with $\mathbf{u} = [u_1, u_2, u_3]$ is the vector of displacements.

Substitution of Eq. (1) into (9) yields

$$\boldsymbol{\theta} = [\theta_x, \theta_y, \theta_z]^T;\ \theta_y = -w_{b,x} - \frac{1}{2}(1 + f_{,z})w_{s,x};\ \theta_x = \theta_z = 0 \tag{10}$$

From Eqs. (8) and (10), the expression for the non-zero components of the symmetric curvature tensor can be written in the form

$$\boldsymbol{\chi} = \begin{bmatrix} 0 & \chi_{xy} & 0 \\ \chi_{xy} & 0 & \chi_{yz} \\ 0 & \chi_{yz} & 0 \end{bmatrix};\ \chi_{xy} = -\frac{1}{2}w_{b,xx} - \frac{1}{4}(1 + f_{,z})w_{s,xx};\ \chi_{yz} = -\frac{1}{4}f_{,zz}\cdot w_{s,x} \tag{11}$$

The equations of motion for the free vibration of the microbeam are derived from Hamilton's principle as [12]

$$\delta\int_{t_1}^{t_2}(T - U - U_f)\mathrm{d}t = 0 \tag{12}$$

where T and U are, respectively, the kinetic and strain energies of the microbeam, and U_f is the strain energy stored in the foundation.

From Eq. (1), the first variation of the kinetic energy on the time interval $[t_1, t_2]$ is

$$\delta\int_{t_1}^{t_2} T\mathrm{d}t = \int_{t_1}^{t_2}\int_0^L \begin{pmatrix} \rho A(\dot{u}_0\delta\dot{u} + (\dot{w}_b + \dot{w}_s)(\delta\dot{w}_b + \delta\dot{w}_s) - \\ -\rho J(\dot{w}_{b,x}\delta\dot{w}_{b,x} + \dfrac{1}{84}\dot{w}_{s,x}\delta\dot{w}_{s,x}) \end{pmatrix}\mathrm{d}t\mathrm{d}x \tag{13}$$

where an over dot denotes the derivative with respect to the time variable t, and ρ is the mass density of the microbeam.

The first variation of the strain energy induced by the nonlinear foundation is as follows

$$\begin{aligned} \delta\int_{t_1}^{t_2} U_f\mathrm{d}t &= \int_{t_1}^{t_2}\int_0^{L_F}\left(k_w w\delta w + k_s w_{,x}\delta w_{,x} + k_{NL}w^3\delta w\right)\mathrm{d}x\mathrm{d}t \\ &= \int_{t_1}^{t_2}\int_0^{L_F}\begin{pmatrix} k_w(w_b + w_s)\delta(w_b + w_s) + k_s(w_{b,x} + w_{s,x})\delta(w_{b,x} + w_{s,x}) + \\ +k_{NL}(w_b + w_s)^3\delta(w_b + w_s) \end{pmatrix}\mathrm{d}x\mathrm{d}t \end{aligned} \tag{14}$$

The first variation of the strain energy of the microbeam on the time interval $[t_1, t_2]$ can be written as

$$\delta\int_{t_1}^{t_2} U dt = \int_{t_1}^{t_2}\int_V (\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{zz}\delta\varepsilon_{zz} + \tau_{xz}\delta\gamma_{xz} + 2\mathrm{m}_{xy}\delta\chi_{xy} + 2\mathrm{m}_{yz}\delta\chi_{yz})\mathrm{d}V\mathrm{d}t =$$

$$= \int_{t_1}^{t_2}\int_0^L \left\{ \begin{aligned}
&\frac{EA}{(1-\upsilon^2)}\Bigg\{\left[u_{0,x} + \frac{1}{2}(1+\upsilon)w_{b,x}^2 + \frac{1}{6}(6+\upsilon)w_{b,x}w_{s,x} + \frac{1}{12}(6+\upsilon)w_{s,x}^2\right]\delta u_{0,x}\\
&+\left[u_{0,x} + \frac{1}{2}(1+\upsilon)w_{b,x}^2 + \frac{1}{6}(6+\upsilon)w_{b,x}w_{s,x} + \frac{1}{12}(6+\upsilon)w_{s,x}^2\right]w_{b,x}\delta w_{b,x}\\
&+\left[u_{0,x} + \frac{1}{2}(1+\upsilon)w_{b,x}^2 + \frac{1}{6}(6+\upsilon)w_{b,x}w_{s,x} + \frac{1}{12}(6+\upsilon)w_{s,x}^2\right]w_{s,x}\delta w_{b,x}\\
&+\left[u_{0,x} + \frac{1}{2}(1+\upsilon)w_{b,x}^2 + \frac{1}{6}(6+\upsilon)w_{b,x}w_{s,x} + \frac{1}{12}(6+\upsilon)w_{s,x}^2\right]w_{b,x}\delta w_{s,x}\\
&+\left[u_{0,x} + \frac{1}{2}(1+\upsilon)w_{b,x}^2 + \frac{1}{6}(6+\upsilon)w_{b,x}w_{s,x} + \frac{1}{12}(6+\upsilon)w_{s,x}^2\right]w_{s,x}\delta w_{s,x}\\
&+\left[\upsilon u_{0,x} + \frac{1}{2}(1+\upsilon)w_{b,x}^2 + \frac{1}{6}(6+\upsilon)w_{b,x}w_{s,x} + \frac{1}{12}(6+\upsilon)w_{s,x}^2\right]w_{b,x}\delta w_{b,x}\\
&+\left[\frac{1}{6}\upsilon u_{0,x} + \frac{1}{12}(1+\upsilon)w_{b,x}^2 + \frac{1}{6}(1+\upsilon)w_{b,x}w_{s,x}\right]w_{s,x}\delta w_{b,x}\\
&+\left[\frac{1}{168}(9+14\upsilon)w_{s,x}^2\right]w_{s,x}\delta w_{b,x} + \frac{5}{12(1-\upsilon)}w_{s,x}\delta w_{s,x}\\
&+\left[\frac{1}{6}\upsilon u_{0,x} + \frac{1}{12}(1+\upsilon)w_{b,x}^2 + \frac{1}{6}(1+\upsilon)w_{b,x}w_{s,x}\right]w_{b,x}\delta w_{s,x}\\
&+\left[\frac{1}{6}\upsilon u_{0,x} + \frac{1}{12}(1+\upsilon)w_{b,x}^2 + \frac{1}{84}(9+14\upsilon)w_{b,x}w_{s,x}\right]w_{s,x}\delta w_{s,x}\\
&+\left[\frac{1}{168}(9+14\upsilon)w_{s,x}^2\right]w_{b,x}\delta w_{s,x} + \frac{1}{252}(11+21\upsilon)w_{s,x}^2 w_{s,x}\delta w_{s,x}\Bigg\}\\
&+\frac{EJ}{1-\upsilon^2}\left\{w_{b,xx}\delta w_{b,xx} + \frac{1}{84}w_{s,xx}\delta w_{s,xx}\right\}\\
&+\frac{EAl^2}{(1+\upsilon)}\Bigg\{\left(\frac{1}{2}w_{b,xx} + \frac{7}{24}w_{s,xx}\right)\delta w_{b,xx} + \left(\frac{1}{4}w_{b,xx} + \frac{7}{48}w_{s,xx}\right)\delta w_{s,xx}\\
&+\frac{1}{24}\left(w_{b,xx} + w_{s,xx}\right)\delta w_{s,xx} + \frac{25}{24h^2}w_{s,x}\delta w_{s,x}\Bigg\}
\end{aligned}\right\} \mathrm{d}x\mathrm{d}t \tag{15}$$

with $A = b\times h$ and $J = bh^3/12$ are, respectively, the area and the inertia moment of the cross-section.

Substituting Eqs. (13), (14) and (15) into Eq. (12) and integrating by parts, the governing equations of motion in terms of the displacements for the microbeam can be obtained by setting the coefficients of the vitual displacements δu, δw_b and δw_s to zero, and they have the following forms

$$\frac{EA}{12(1-\upsilon^2)}\begin{pmatrix} 12u_{0,xx}+12(1+\upsilon)w_{b,x}w_{b,xx}+2(6+v)w_{s,x}w_{s,xx} \\ +2(6+\upsilon)(w_{b,xx}w_{s,x}+w_{b,x}w_{s,xx}) \end{pmatrix}=A\rho\ddot{u}_0 \tag{16}$$

$$\begin{aligned}
&\frac{EA}{1-\upsilon}\left(u_{0,x}w_{b,xx}+u_{0,xx}w_{b,x}\right)+\frac{EA(6+\upsilon)}{6(1-\upsilon^2)}\left(u_{0,x}w_{s,xx}+u_{0,xx}w_{s,x}\right)+\frac{3EA}{1-\upsilon}w_{b,x}^2w_{b,xx}+\\
&+\frac{7EA}{4(1-\upsilon)}w_{b,x}^2w_{s,xx}+\frac{7EA}{2(1-\upsilon)}w_{b,x}w_{b,xx}w_{s,x}+\frac{EA(11\upsilon+21)}{6(1-\upsilon^2)}w_{b,x}w_{s,x}w_{s,xx}+\\
&+\frac{EA(11\upsilon+21)}{12(1-\upsilon^2)}w_{b,xx}w_{s,x}^2-\left(\frac{EJ}{1-\upsilon^2}+\frac{EAl^2}{2(1+\upsilon)}\right)w_{b,xxxx}-k_w(w_b+w_s)+\\
&+k_s(w_{b,xx}+w_{s,xx})-k_{NL}(w_b+w_s)^2+\frac{EA(28\upsilon+93)}{56(1-\upsilon^2)}w_{s,xx}w_{s,x}^2-\frac{7EAl^2}{24(1+\upsilon)}w_{s,xxxx}=\\
&=A\rho\ddot{w}_b+J\rho\ddot{w}_{b,xx}+A\rho\ddot{w}_s
\end{aligned} \tag{17}$$

$$\begin{aligned}
&\frac{EA(\upsilon+6)}{6(1-\upsilon^2)}\left(u_{0,x}(w_{b,xx}+w_{s,xx})+u_{0,xx}(w_{b,x}+w_{s,x})\right)+\frac{7EA}{4(1-\upsilon)}w_{b,x}^2w_{b,xx}+\\
&+\frac{EA(11\upsilon+21)}{12(1-\upsilon^2)}(w_{b,x}^2w_{s,xx}+2w_{b,x}w_{b,xx}w_{s,x})-\frac{7EAl^2}{24(1+\upsilon)}w_{b,xxxx}+\\
&+\frac{EA(28\upsilon+93)}{56(1-\upsilon^2)}(w_{s,x}^2w_{b,xx}+2w_{s,x}w_{s,xx}w_{b,x})-\left(\frac{EJ}{84(1-\upsilon^2)}+\frac{3EAl^2}{16(1+\upsilon)}\right)w_{s,xxxx}-\\
&-k_w(w_b+w_s)+k_s(w_{b,xx}+w_{s,xx})-k_{NL}(w_b+w_s)^3+\frac{42EA(42\upsilon+137)}{84(1-\upsilon^2)}w_{s,x}^2w_{s,xx}=\\
&=A\rho\ddot{w}_b+A\rho\ddot{w}_s+\frac{1}{84}J\rho\ddot{w}_{s,xx}
\end{aligned} \tag{18}$$

where double over dots denotes the second-order differential with respect to time.

Four types of boundary conditions, namely simply-supported (SS), clamped-clamped (CC), clamped-free (CF) and clamped-simply supported (CS) are considered herein. The constraints for these boundaries are as follows.

- For SS: $u_0 = w_b = w_s = 0$ at $x = 0$ and $w_b = w_s = 0$ at $x = L$
- For CC: $u_0 = w_b = w_s = w_{b,x} = w_{s,x} = 0$ at $x = 0$ and $w_b = w_s = w_{b,x} = w_{s,x} = 0$ L (19)
- For CF: $u_0 = w_b = w_s = w_{b,x} = w_{s,x} = 0$ at $x = 0$
- For CS: $u_0 = w_b = w_s = w_{b,x} = w_{s,x} = 0$ at $x = 0$ and $w_b = w_s = 0$ at $x = L$

3. SOLUTION METHOD

Finite element method is used herein to solve the equations of motion (16)-(18). To this end, the microbeam is assumed to be divided into a number of elements with length of l_e. Noting that the beam should be divided to get $L_f = NE_f \cdot l_e$ with NE_f is an integer and is the number of elements for the supporting foundation. A two-node beam element with five degree of freedom per node is considered herewith. The vector of the nodal displacements for the element is defined as follows

$$\underset{10\times1}{\{\mathbf{q}_e\}} = \{\mathbf{u}_0 \;\; \mathbf{w}_b \;\; \mathbf{w}_s\}^T \tag{20}$$

where $\mathbf{u}_0$, $\mathbf{w}_b$ and $\mathbf{w}_s$ are, respectively, the element vectors of nodal axial, bending transverse and shear transverse displacements with

$$\mathbf{u}_0 = \{u_{01} \; u_{02}\}^T, \quad \mathbf{w}_b = \{w_{b1} \, w_{b1,x} \, w_{b2} \;\; w_{b2,x}\}^T, \quad \mathbf{w}_s = \{w_{s1} \, w_{s1,x} \, w_{s2} \;\; w_{s2,x}\}^T \tag{21}$$

Linear polynomials are used to interpolate the axial displacement u from its nodal values, while Hermite cubic polynomials are employed for the transverse displacements w_b and w_s as

$$u_0 = \mathbf{N}\mathbf{u}_0, \quad w_b = \mathbf{H}\mathbf{w}_b, \quad w_s = \mathbf{H}\mathbf{w}_s \tag{22}$$

where $\mathbf{N} = \{N_1 \; N_2\}$ and $\mathbf{H} = \{H_1 \; H_2 \; H_3 \; H_4\}$ are, respectively, the matrices of linear and Hermite shape functions as [13].

With the interpolation and using the Galerkin finite element method [14] to Eqs. (16)-(18), one can obtain the following discrete equations

$$\begin{aligned}
&\sum^{NE}\int_0^{l_e}\Big\{\mathbf{N}^T A\rho\mathbf{N}\ddot{\mathbf{u}}_0 - \frac{EA}{12(1-\upsilon^2)}\Big(-12\mathbf{N}_{,x}^T\mathbf{N}_{,x}\mathbf{u}_0 + 12(1+\upsilon)(\mathbf{H}_{,x}\mathbf{w}_b)(\mathbf{N}^T\mathbf{H}_{,xx}\mathbf{w}_b) \\
&+2(6+\upsilon)(\mathbf{H}_{,x}\mathbf{w}_s)(\mathbf{N}^T\mathbf{H}_{,xx}\mathbf{w}_s) + 2(6+\upsilon)(\mathbf{H}_{,xx}\mathbf{w}_b)(\mathbf{N}^T\mathbf{H}_{,x}\mathbf{w}_s) \\
&+(\mathbf{H}_{,x}\mathbf{w}_b)(\mathbf{N}^T\mathbf{H}_{,xx}\mathbf{w}_s)\Big)\Big\}\mathrm{d}x = 0
\end{aligned} \tag{23}$$

$$\begin{aligned}
&\sum^{NE}\int_0^{l_e}\Big\{A\rho(\mathbf{H}^T\mathbf{H}\ddot{\mathbf{w}}_b) + J\rho(\mathbf{H}^T\mathbf{H}_{,xx}\ddot{\mathbf{w}}_b) + A\rho(\mathbf{H}^T\mathbf{H}\ddot{\mathbf{w}}_s) - \Big(-\frac{7EAl^2}{24(1+\upsilon)}\mathbf{H}_{,xx}^T\mathbf{H}_{,xx}\mathbf{w}_s \\
&+\frac{3EA}{1-\upsilon}(\mathbf{H}_{,x}\mathbf{w}_b)^2(\mathbf{H}^T\mathbf{H}_{,xx}\mathbf{w}_b) + \frac{EA}{1-\upsilon}\Big((\mathbf{H}^T\mathbf{N}_{,x}\mathbf{u}_0)(\mathbf{H}_{,xx}\mathbf{w}_b) + (\mathbf{H}_{,x}^T\mathbf{N}_{,x}\mathbf{u}_0)(\mathbf{H}_{,x}\mathbf{w}_b)\Big) \\
&+\frac{7EA}{4(1-\upsilon)}(\mathbf{H}_{,xx}\mathbf{w}_s)(\mathbf{H}_{,x}\mathbf{w}_b)(\mathbf{H}^T\mathbf{H}_{,x}\mathbf{w}_b) - \Big(\frac{EJ}{1-\upsilon^2} + \frac{EAl^2}{2(1+\upsilon)}\Big)(\mathbf{H}_{,xx}^T\mathbf{H}_{,xx}\mathbf{w}_b) + \\
&+\frac{EA(6+\upsilon)}{6(1-\upsilon^2)}\Big((\mathbf{H}^T\mathbf{N}_{,x}\mathbf{u}_0)(\mathbf{H}_{,xx}\mathbf{w}_s) - (\mathbf{H}_{,x}^T\mathbf{N}_{,x}\mathbf{u}_0)(\mathbf{H}_{,x}\mathbf{w}_s)\Big) + k_s(\mathbf{H}^T\mathbf{H}_{,xx}\mathbf{w}_b + \mathbf{H}^T\mathbf{H}_{,xx}\mathbf{w}_s) \\
&+\frac{7EA}{2(1-\upsilon)}(\mathbf{H}_{,x}\mathbf{w}_s)(\mathbf{H}_{,x}\mathbf{w}_b)(\mathbf{H}^T\mathbf{H}_{,xx}\mathbf{w}_b) + \frac{EA(11\upsilon+21)}{6(1-\upsilon^2)}(\mathbf{H}_{,xx}\mathbf{w}_s)(\mathbf{H}_{,x}\mathbf{w}_s)(\mathbf{H}^T\mathbf{H}_{,x}\mathbf{w}_b) \\
&\frac{EA(11\upsilon+21)}{12(1-\upsilon^2)}(\mathbf{H}_{,x}\mathbf{w}_s)^2(\mathbf{H}^T\mathbf{H}_{,xx}\mathbf{w}_b) - k_{NL}(\mathbf{H}\mathbf{w}_b + \mathbf{H}\mathbf{w}_s)^2(\mathbf{H}^T\mathbf{H}\mathbf{w}_b + \mathbf{H}^T\mathbf{H}\mathbf{w}_s) \\
&+\frac{EA(28\upsilon+93)}{56(1-\upsilon^2)}(\mathbf{H}^T\mathbf{H}_{,xx}\mathbf{w}_s)(\mathbf{H}_{,x}\mathbf{w}_s)^2 - k_w(\mathbf{H}^T\mathbf{H}\mathbf{w}_b + \mathbf{H}^T\mathbf{H}\mathbf{w}_s)\Big)\Big\}\mathrm{d}x = 0
\end{aligned} \tag{24}$$

$$\begin{aligned}
&\sum^{NE}\int_0^{l_e}\Bigg\{A\rho\mathbf{H}^T\mathbf{H}\ddot{\mathbf{w}}_b + A\rho\mathbf{H}^T\mathbf{H}\ddot{\mathbf{w}}_s + \frac{1}{84}J\rho\mathbf{H}^T\mathbf{H}_{,xx}\ddot{\mathbf{w}}_s \\
&-\left(\frac{7EA}{4(1-\upsilon)}(\mathbf{H}_{,x}\mathbf{w}_b)^2(\mathbf{H}^T\mathbf{H}_{,xx}\mathbf{w}_b)-\left(\frac{EJ}{84(1-\upsilon^2)}+\frac{3EAl^2}{16(1+\upsilon)}\right)\mathbf{H}^T_{,xx}\mathbf{H}_{,xx}\mathbf{w}_s\right. \\
&+\frac{EA(\upsilon+6)}{6(1-\upsilon^2)}\left(\mathbf{H}^T\mathbf{N}_{,x}\mathbf{u}_0(\mathbf{H}_{,xx}\mathbf{w}_b+\mathbf{H}_{,xx}\mathbf{w}_s)-\mathbf{H}^T_{,x}\mathbf{N}_{,x}\mathbf{u}_0(\mathbf{H}_{,x}\mathbf{w}_b+\mathbf{H}_{,x}\mathbf{w}_s)\right) \\
&+\frac{42EA(42\upsilon+137)}{84(1-\upsilon^2)}(\mathbf{H}_{,x}\mathbf{w}_s)^2(\mathbf{H}^T\mathbf{H}_{,xx}\mathbf{w}_s) \\
&+\frac{EA(11\upsilon+21)}{12(1-\upsilon^2)}((\mathbf{H}_{,x}\mathbf{w}_b)^2(\mathbf{H}^T\mathbf{H}_{,xx}\mathbf{w}_s)+2(\mathbf{H}_{,x}\mathbf{w}_b)(\mathbf{H}_{,xx}\mathbf{w}_b)(\mathbf{H}^T\mathbf{H}_{,x}\mathbf{w}_s)) \\
&+\frac{EA(28\upsilon+93)}{56(1-\upsilon^2)}((\mathbf{H}_{,x}\mathbf{w}_s)^2(\mathbf{H}^T\mathbf{H}_{,xx}\mathbf{w}_b)+2(\mathbf{H}_{,x}\mathbf{w}_s)(\mathbf{H}_{,xx}\mathbf{w}_s)(\mathbf{H}^T\mathbf{H}_{,x}\mathbf{w}_b)) \\
&-\frac{7EAl^2}{24(1+\upsilon)}(\mathbf{H}^T_{,xx}\mathbf{H}_{,xx}\mathbf{w}_b)+k_s(\mathbf{H}^T\mathbf{H}_{,xx}\mathbf{w}_b+\mathbf{H}^T\mathbf{H}_{,xx}\mathbf{w}_s)-k_w\mathbf{H}^T\mathbf{H}\mathbf{w}_b-k_w\mathbf{H}^T\mathbf{H}\mathbf{w}_s \\
&\left.-k_{NL}(\mathbf{H}\mathbf{w}_b+\mathbf{H}\mathbf{w}_s)^2(\mathbf{H}^T\mathbf{H}\mathbf{w}_b+\mathbf{H}^T\mathbf{H}\mathbf{w}_s)\right)\Bigg\}\mathrm{d}x=0
\end{aligned} \tag{25}$$

One can write Eqs. (23) - (25) in a matrix form as

$$\sum^{NE}\left(\left[\mathbf{M}_e\right]\left\{\ddot{\mathbf{q}}_e\right\}+\left[\mathbf{K}_e\right]\left\{\mathbf{q}_e\right\}\right)=0 \tag{26}$$

where $\left[\mathbf{M}_e\right]$ and $\left[\mathbf{K}_e\right]$ are the element mass and stiffness matrices, respectively, and they have the forms as

$$\left[\mathbf{M}_e\right]=\begin{bmatrix}\mathbf{m}^e_{11} & 0 & 0\\ 0 & \mathbf{m}^e_{22} & \mathbf{m}^e_{23}\\ 0 & \mathbf{m}^e_{32} & \mathbf{m}^e_{33}\end{bmatrix},\left[\mathbf{K}_e\right]=\begin{bmatrix}\mathbf{k}^e_{11} & \mathbf{k}^e_{12} & \mathbf{k}^e_{13}\\ \mathbf{k}^e_{21} & \mathbf{k}^e_{22} & \mathbf{k}^e_{23}\\ \mathbf{k}^e_{31} & \mathbf{k}^e_{32} & \mathbf{k}^e_{33}\end{bmatrix} \tag{27}$$

with

$$\begin{aligned}
&\mathbf{m}^e_{11}=\int_0^{L_e}\mathbf{N}^TA\rho\mathbf{N}\mathrm{d}x,\quad \mathbf{m}^e_{22}=\int_0^{L_e}(\mathbf{H}^TA\rho\mathbf{H}+\mathbf{H}^TJ\rho\mathbf{H}_{,xx})\mathrm{d}x,\quad \mathbf{m}^e_{23}=\int_0^{L_e}\mathbf{H}^TA\rho\mathbf{H}\mathrm{d}x,\\
&\mathbf{m}^e_{32}=\left(\mathbf{m}^e_{23}\right)^T,\ \ \mathbf{m}^e_{3,3}=\int_0^{L_e}\left(\mathbf{H}^TA\rho\mathbf{H}-\mathbf{H}^TJ\rho\mathbf{H}_{,xx}\right)\mathrm{d}x,
\end{aligned} \tag{28}$$

and

$$\mathbf{k}_{11}^{e}=\int_{0}^{L_e}\frac{EA\mathbf{N}_{,x}^{T}\mathbf{N}_{,x}}{(1-\upsilon^{2})}\mathrm{d}x,\ \mathbf{k}_{12}^{e}=\int_{0}^{L_e}\frac{EA(\mathbf{H}_{,x}\mathbf{w}_{b})}{(1-\upsilon)}\mathbf{N}^{T}\mathbf{H}_{,xx}\mathrm{d}x,$$

$$\mathbf{k}_{13}^{e}=-\int_{0}^{L_e}\frac{EA}{12(1-\upsilon^{2})}\begin{pmatrix}2(6+v)(\mathbf{H}_{,x}\mathbf{w}_{s})(\mathbf{N}^{T}\mathbf{H}_{,xx}\mathbf{w}_{s})+\\ 2(6+\upsilon)(\mathbf{H}_{,xx}\mathbf{w}_{b})(\mathbf{N}^{T}\mathbf{H}_{,x}\mathbf{w}_{s})+(\mathbf{H}_{,x}\mathbf{w}_{b})(\mathbf{N}^{T}\mathbf{H}_{,xx}\mathbf{w}_{s})\end{pmatrix}\mathrm{d}x \tag{29}$$

$$\mathbf{k}_{21}^{e}=-\int_{0}^{l_e}\begin{pmatrix}\frac{EA}{1-\upsilon}\left((\mathbf{H}^{T}\mathbf{N}_{,x})(\mathbf{H}_{,xx}\mathbf{w}_{b})+(\mathbf{H}_{,x}^{T}\mathbf{N}_{,x})(\mathbf{H}_{,x}\mathbf{w}_{b})\right)+\\ +\frac{EA(6+\upsilon)}{6(1-\upsilon^{2})}\left((\mathbf{H}^{T}\mathbf{N}_{,x})(\mathbf{H}_{,xx}\mathbf{w}_{s})-(\mathbf{H}_{,x}^{T}\mathbf{N}_{,x})(\mathbf{H}_{,x}\mathbf{w}_{s})\right)\end{pmatrix}\mathrm{d}x, \tag{30}$$

$$\mathbf{k}_{22}^{e}=-\int_{0}^{l_e}\begin{pmatrix}\frac{3EA}{1-\upsilon}(\mathbf{H}_{,x}\mathbf{w}_{b})^{2}(\mathbf{H}^{T}\mathbf{H}_{,xx})+\frac{7EA}{2(1-\upsilon)}(\mathbf{H}_{,x}\mathbf{w}_{s})(\mathbf{H}_{,x}\mathbf{w}_{b})(\mathbf{H}^{T}\mathbf{H}_{,xx})\\ +\frac{7EA}{4(1-\upsilon)}(\mathbf{H}_{,xx}\mathbf{w}_{s})(\mathbf{H}_{,x}\mathbf{w}_{b})(\mathbf{H}^{T}\mathbf{H}_{,x})-k_{NL}\mathbf{H}^{T}(\mathbf{H}\mathbf{w}_{b}+\mathbf{H}\mathbf{w}_{s})\mathbf{H}\\ -\left(\frac{EJ}{1-\upsilon^{2}}+\frac{EAl^{2}}{2(1+\upsilon)}\right)(\mathbf{H}_{,xx}^{T}\mathbf{H}_{,xx})-k_{w}(\mathbf{H}^{T}\mathbf{H})\\ +\frac{EA(11\upsilon+21)}{12(1-\upsilon^{2})}(\mathbf{H}_{,x}\mathbf{w}_{s})^{2}(\mathbf{H}^{T}\mathbf{H}_{,xx})+k_{s}(\mathbf{H}^{T}\mathbf{H}_{,xx})\\ +\frac{EA(11\upsilon+21)}{6(1-\upsilon^{2})}(\mathbf{H}_{,xx}\mathbf{w}_{s})(\mathbf{H}_{,x}\mathbf{w}_{s})(\mathbf{H}^{T}\mathbf{H}_{,x})\end{pmatrix}\mathrm{d}x, \tag{31}$$

$$\mathbf{k}_{23}^{e}=-\int_{0}^{l_e}\begin{pmatrix}-\frac{7EAl^{2}}{24(1+\upsilon)}\mathbf{H}_{,xx}^{T}\mathbf{H}_{,xx}-k_{NL}\mathbf{H}^{T}(\mathbf{H}\mathbf{w}_{b}+\mathbf{H}\mathbf{w}_{s})\mathbf{H}+\\ -k_{w}(\mathbf{H}^{T}\mathbf{H})+k_{s}(\mathbf{H}^{T}\mathbf{H}_{,xx})+\frac{EA(28\upsilon+93)}{56(1-\upsilon^{2})}(\mathbf{H}^{T}\mathbf{H}_{,xx})(\mathbf{H}_{,x}\mathbf{w}_{s})^{2}\end{pmatrix}\mathrm{d}x \tag{32}$$

$$\mathbf{k}_{31}^{e}=-\int_{0}^{l_e}\left(\frac{EA(\upsilon+6)}{6(1-\upsilon^{2})}\left(\mathbf{H}^{T}\mathbf{N}_{,x}(\mathbf{H}_{,xx}\mathbf{w}_{b}+\mathbf{H}_{,xx}\mathbf{w}_{s})-\mathbf{H}_{,x}^{T}\mathbf{N}_{,x}(\mathbf{H}_{,x}\mathbf{w}_{b}+\mathbf{H}_{,x}\mathbf{w}_{s})\right)\right)\mathrm{d}x \tag{33}$$

$$\mathbf{k}_{32}^{e}=-\int_{0}^{l_e}\begin{pmatrix}\frac{7EA}{4(1-\upsilon)}(\mathbf{H}_{,x}\mathbf{w}_{b})^{2}(\mathbf{H}^{T}\mathbf{H}_{,xx})-k_{NL}(\mathbf{H}\mathbf{w}_{b}+\mathbf{H}\mathbf{w}_{s})^{2}(\mathbf{H}^{T}\mathbf{H})\\ +\frac{EA(28\upsilon+93)}{56(1-\upsilon^{2})}((\mathbf{H}_{,x}\mathbf{w}_{s})^{2}(\mathbf{H}^{T}\mathbf{H}_{,xx})+2(\mathbf{H}_{,x}\mathbf{w}_{s})(\mathbf{H}_{,xx}\mathbf{w}_{s})(\mathbf{H}^{T}\mathbf{H}_{,x}))\\ -\frac{7EAl^{2}}{24(1+\upsilon)}(\mathbf{H}_{,xx}^{T}\mathbf{H}_{,xx})+k_{s}\mathbf{H}^{T}\mathbf{H}_{,xx}-k_{w}\mathbf{H}^{T}\mathbf{H}\end{pmatrix}\mathrm{d}x \tag{34}$$

$$\mathbf{k}_{33}^{e}=-\int_{0}^{l_e}\begin{pmatrix}\frac{42EA(42\upsilon+137)}{84(1-\upsilon^{2})}(\mathbf{H}_{,x}\mathbf{w}_{s})^{2}(\mathbf{H}^{T}\mathbf{H}_{,xx})-(k_{NL}(\mathbf{H}\mathbf{w}_{b}+\mathbf{H}\mathbf{w}_{s})^{2}-k_{w})\mathbf{H}^{T}\mathbf{H}\\ -\left(\frac{EJ}{84(1-\upsilon^{2})}+\frac{3EAl^{2}}{16(1+\upsilon)}\right)\mathbf{H}_{,xx}^{T}\mathbf{H}_{,xx}-\frac{7EAl^{2}}{24(1+\upsilon)}(\mathbf{H}_{,xx}^{T}\mathbf{H}_{,xx})+k_{s}\mathbf{H}^{T}\mathbf{H}_{,xx}\\ +\frac{EA(11\upsilon+21)}{12(1-\upsilon^{2})}((\mathbf{H}_{,x}\mathbf{w}_{b})^{2}(\mathbf{H}^{T}\mathbf{H}_{,xx})+2(\mathbf{H}_{,x}\mathbf{w}_{b})(\mathbf{H}_{,xx}\mathbf{w}_{b})(\mathbf{H}^{T}\mathbf{H}_{,x}))\end{pmatrix}\mathrm{d}x \tag{35}$$

The element stiffness and mass matrices are better to be derived in terms of the natural coordinate $\xi = -1 + 2x/l_e$ with $1 \le \xi \le 1$, $0 \le x \le l_e$, and $\mathrm{d}x = l_e \mathrm{d}\xi / 2$. The highest order of the integrals in Eqs. (28)-(35) is six, and thus Gauss quadrature with 4 points along the element length can be used to exactly**.**

Assuming a harmonic form for the vector of nodal displacements, the discrete equation of motion (26) can be written in the form

$$\left([\mathbf{K}] - \omega^2 [\mathbf{M}]\right)\bar{\mathbf{D}} = 0 \tag{36}$$

where [**M]** and [**K]** are, respectively, the global mass matrix and stiffness matrix; ω and $\bar{\mathbf{D}}$ are, respectively, the frequency and the eigenvector of the nodal displacements corresponding to an eigenvalue.

A direct iterative algorithm is used herein to obtain nonlinear frequencies from Eq. (36). In the algorithm, the linearization is used to calculate the nonlinear terms from the previous iteration solution. For example, the terms $(\mathbf{H}_{,x}\mathbf{w}_b)^2(\mathbf{H}^T\mathbf{H}_{,xx})$ and $(\mathbf{H}_{,x}\mathbf{w}_b)(\mathbf{H}^T_{,x}\mathbf{N}_{,x})$ can be linearized as

$$(\mathbf{H}_{,x}\mathbf{w}_b)^2(\mathbf{H}^T\mathbf{H}_{,xx}) \approx \left[(\mathbf{H}_{,x}\mathbf{w}_b)^2\right]_k (\mathbf{H}^T\mathbf{H}_{,xx}); \quad (\mathbf{H}_{,x}\mathbf{w}_b)(\mathbf{H}^T_{,x}\mathbf{N}_{,x}) \approx \left[\mathbf{H}_{,x}\mathbf{w}_b\right]_k (\mathbf{H}^T_{,x}\mathbf{N}_{,x}) \tag{37}$$

where the term in the square bracket is evaluated using the solution known from the *k*-th iteration. The procedure for the nonlinear algorithm contains three steps as follows [6]

Step 1. Neglecting nonlinear terms in the stiffness matrix [**K**] of Eq. (36), the linear stiffness matrix $[\mathbf{K}]_L$ is obtained and the corresponding linear eigenvalue problem is solved.

Step 2. The linear eigenvectors obtained in Step 1 are appropriately scaled up such that the maximum transverse displacement is equal to a given vibration amplitude. Then, the scaled normalized linear eigenvectors are used to evaluate the nonlinear stiffness matrix $[\mathbf{K}]_{NL}$. The nonlinear eigenvalues and eigenvectors are obtained from the updated eigensystem (36).

Step 3. The eigenvector is scaled up again and Step 2 is repeated until the relative error between the eigenvalues obtained from two consecutive iterations *i* and *i*+1 satisfies the prescribed convergence criteria as

$$\frac{\left|\omega_{NL}^{i+1} - \omega_{NL}^{i}\right|}{\omega_{NL}^{i}} \le \varepsilon_0 \tag{38}$$

where ω_{NL}^k is the frequency at iteration *k* (*k* = *i*, *i*+1) and ε_0 is a small value number, which is set to be 10^{-5} in this work.

4. NUMERICAL INVESTIGATION

Numerical investigation is carried out in this section to study the effect of various parameters on the nonlinear free vibration behavior of microbeams. To this end, an aluminum microbeam with L/h = 100, b=2h, l = 17.6 µm and the material properties are [3]: E=70 GPa, ρ=2702 kg/m^3, υ=0.3.

The following dimensionless parameters are, respectively, used for the fundamental frequency, foundation stiffness, deflection, length scale and foundation supporting length

$$\omega_{\text{ratio}} = \frac{\omega_{NL}}{\omega_L},\ K_w = \frac{k_w L^4}{EJ},\ K_s = \frac{k_s L^2}{EJ},\ K_{NL} = \frac{k_{NL} r^2 L^4}{EJ},\ \bar{w} = \frac{w}{r},\ \eta = \frac{l}{h},\ \alpha_F = \frac{L_f}{L} \tag{39}$$

where $r = \sqrt{A/J}$.

Before computing the nonlinear frequency ratios of the microbeam, the accuracy and convergence of the derived formulation are firstly verified. To this end, Table 1 shows the convergence for the SS and CC microbeams with $w_{\max}/h = 0.2, 0.4, 0.6 \text{ and } 0.8$. The results in this work are also compared with that of Refs. [6] and [8] for several cases. Noting that both Refs. [6] and [8] used $w_{\max}/h$ for evaluating the nonlinear frequency ratios, which is diferent from (39). As seen from the Table 1, the convergence of the derived formulation is achieved by using just four elements for the majority of cases and six elements for the case of $\eta = 2 \text{ and } w_{\max}/h = 0.6 \text{ or } 0.8$, regardless of the boundary conditions and the length scale parameter η. Also, a good agreement between the result of the present work with that of Refs. [6, 8] can be seen from Table 1. Because of this convergence, six elements are used for the microbeams in all the computations reported below. Noting that the number in brackets in Table 1 is the number of iterations required for the convergence condition in Eq. (38), and the maximum number of iterations is seven for the cases in the table.

Table 1. Convergence of the derived formulation in evaluating the nonlinear frequency ratio values ω_{ratio} of microbeam without foundation.

B.C.	η	$w_{\max}/h$	*NE*				Refs.	
			2	4	6	8	[6]	[8]
SS	1	0.2	1.0075 (3)	**1.0078** (3)	**1.0078** (3)	**1.0078** (3)	1.0084	1.0084
		0.4	1.0297 (3)	**1.0308** (3)	**1.0308** (4)	**1.0308** (4)	1.0330	1.0330
		0.6	1.0655 (4)	**1.0678** (4)	**1.0678** (5)	**1.0678** (5)	1.0729	1.0727
		0.8	1.1135 (5)	**1.1170** (5)	**1.1170** (6)	**1.1170** (6)	1.1264	1.1259
	2	0.2	1.0190 (4)	**1.0197** (4)	**1.0197** (5)	**1.0197** (5)	1.0213	1.0213
		0.4	1.0736 (4)	**1.0762** (5)	**1.0762** (6)	**1.0762** (6)	1.0828	1.0824
		0.6	1.1584 (5)	1.1625 (5)	**1.1627** (6)	**1.1627** (7)	1.1780	1.1776
		0.8	1.2658 (5)	1.2714 (6)	**1.2716** (7)	**1.2716** (7)	1.2997	1.3064
CC	1	0.2	1.0017 (3)	**1.0018** (3)	**1.0018** (4)	**1.0018** (5)	-	-
		0.4	1.0070 (4)	**1.0073** (4)	**1.0073** (4)	**1.0073** (5)	-	-
		0.6	1.0157 (4)	**1.0163** (4)	**1.0163** (5)	**1.0163** (6)		
		0.8	1.0276 (5)	**1.0287** (5)	**1.0287** (6)	**1.0287** (7)	-	-
	2	0.2	1.0045 (4)	**1.0046** (5)	**1.0046** (5)	**1.0046** (5)	-	-
		0.4	1.0177 (4)	**1.0184** (5)	**1.0184** (5)	**1.0184** (5)	-	-
		0.6	1.0395 (5)	1.0409 (5)	**1.0408** (6)	**1.0408** (6)		
		0.8	1.0692 (5)	1.0714 (5)	**1.0710** (6)	**1.0710** (7)	-	-

Since the data of nonlinear frequency ratios for the microbeam partially supported by the nonlinear foundation is not available in the literature, the accuracy of the derived formulation is verified herewith by comparing the nonlinear frequency ratio of a microbeam fully resting on the nonlinear foundation obtained in the present work with that of Ref. [9] as shown in Table 2 for the SS and CC microbeams. A good agreement between the result of the present work with that of Ref. [9] is noted from Table 2, regardless of the boundary conditions, the value of η and the foundation stiffness.

Table 2. Verification study for nonlinear frequency ratio values ω_{ratio} for SS and CC microbeams with nonlinear foundation.

B.C.	η	$\bar{w}_{max}$	$K_w = 100$				$K_w = 200$			
			K_s=50, K_{NL} =100		K_s=100, K_{NL}=200		K_s=50, K_{NL} =100		K_s=100, K_{NL}=200	
			Present	[9]	Present	[9]	Present	[9]	Present	[9]
SS	0.25	1	1.0521	1.0505	1.0535	1.0525	1.0459	1.0445	1.0496	1.0486
		2	1.1940	1.1893	1.2003	1.1964	1.1723	1.1678	1.1863	1.1825
		3	1.3973	1.3902	1.4125	1.4037	1.3555	1.3486	1.3856	1.3772
		4	1.6383	1.6303	1.6664	1.6509	1.5750	1.5670	1.6255	1.6106
	0.5	1	1.0468	1.0453	1.0502	1.0492	1.0418	1.0404	1.0467	1.0457
		2	1.1757	1.1709	1.1882	1.1846	1.1577	1.1532	1.1759	1.1723
		3	1.3624	1.3545	1.3890	1.3811	1.3275	1.3200	1.3650	1.3574
		4	1.5858	1.5761	1.6302	1.6166	1.5325	1.5231	1.5936	1.5804
	0.75	1	1.0401	1.0387	1.0454	1.0445	1.0363	1.0350	1.0425	1.0416
		2	1.1517	1.1470	1.1712	1.1678	1.1381	1.1338	1.1609	1.1576
		3	1.3160	1.3078	1.3556	1.3486	1.2893	1.2816	1.3355	1.3287
		4	1.5152	1.5044	1.5788	1.5670	1.4740	1.4636	1.5480	1.5365
	1	1	1.0333	1.0321	1.0400	1.0392	1.0307	1.0295	1.0378	1.0370
		2	1.1273	1.1230	1.1520	1.1489	1.1176	1.1136	1.1438	1.1409
		3	1.2679	1.2601	1.3178	1.3115	1.2486	1.2412	1.3017	1.2956
		4	1.4411	1.4301	1.5204	1.5101	1.4109	1.4003	1.4955	1.4854
CC	0.25	1	1.0269	1.0273	1.0326	1.0315	1.0252	1.0255	1.0311	1.0301
		2	1.1037	1.1052	1.1251	1.1209	1.0971	1.0986	1.1194	1.1156
		3	1.2209	1.2240	1.2648	1.2558	1.2074	1.2106	1.2532	1.2452
		4	1.3678	1.3733	1.4389	1.4233	1.3464	1.3520	1.4209	1.4068
	0.5	1	1.0206	1.0208	1.0267	1.0260	1.0196	1.0197	1.0257	1.0250
		2	1.0801	1.0809	1.1031	1.1004	1.0761	1.0769	1.0992	1.0968
		3	1.1724	1.1743	1.2202	1.2144	1.1641	1.1661	1.2122	1.2069
		4	1.2903	1.2937	1.3679	1.3579	1.2769	1.2805	1.3553	1.3461
	0.75	1	1.0148	1.0149	1.0206	1.0201	1.0143	1.0143	1.0200	1.0195
		2	1.0581	1.0584	1.0801	1.0783	1.0560	1.0563	1.0777	1.0761
		3	1.1264	1.1273	1.1727	1.1689	1.1219	1.1229	1.1678	1.1643
		4	1.2155	1.2171	1.2916	1.2851	1.2081	1.2099	1.2836	1.2775
	1	1	1.0107	1.0106	1.0156	1.0153	1.0104	1.0104	1.0153	1.0149
		2	1.0420	1.0421	1.0611	1.0599	1.0408	1.0410	1.0598	1.0586
		3	1.0921	1.0924	1.1330	1.1303	1.0897	1.0901	1.1301	1.1276
		4	1.1586	1.1593	1.2267	1.2222	1.1546	1.1554	1.2219	1.2176

Table 3. Nonlinear frequency ratio values ω_{ratio} of microbeam partially supported by the nonlinear foundation with K_w = *200*, K_s*=100*, K_{NL}*=200.*

B.C.	α_F	$\eta=0.5$				$\eta=1$			
		$\bar{w}_{\max}$				$\bar{w}_{\max}$			
		1	2	3	4	1	2	3	4
CF	0.25	1.0289(4)	1.1059(6)	1.2119(7)	1.3321(9)	1.0119(4)	1.0458(5)	1.0973(6)	1.1616(6)
	0.50	1.0198(4)	1.0720(6)	1.1427(8)	1.2220(9)	1.0086(4)	1.0333(5)	1.0713(6)	1.1196(7)
	0.75	1.0162(4)	1.0630(5)	1.1357(8)	1.2289(9)	1.0153(4)	1.0593(5)	1.1270(7)	1.2129(9)
SS	0.25	1.0199(4)	1.0756(6)	1.1580(8)	1.2586(8)	1.0110(4)	1.0430 (5)	1.0934 (5)	1.1589 (6)
	0.50	1.0390(4)	1.1452(6)	1.2962(9)	1.4718(9)	1.0293(4)	1.1109(5)	1.2309(7)	1.3755(9)
	0.75	1.0693(4)	1.2553(6)	1.5178(9)	1.8268(9)	1.0485(4)	1.1826(4)	1.3782(5)	1.6142(7)
CS	0.25	1.0199(4)	1.0746(6)	1.1541(8)	1.2490(10)	1.0084(4)	1.0327(5)	1.0708(6)	1.1199(7)
	0.50	1.0253(4)	1.0964(6)	1.2029(6)	1.3334(10)	1.0142(4)	1.0555(4)	1.1202(5)	1.2035(6)
	0.75	1.0532(5)	1.1990(6)	1.4089(7)	1.6600(9)	1.0309(4)	1.1184(4)	1.2511(4)	1.4168(6)
CC	0.25	1.0080(3)	1.0314(4)	1.0688(5)	1.1179(5)	1.0035(3)	1.0141(3)	1.0313(4)	1.0548(6)
	0.50	1.0190(4)	1.0728(5)	1.1540(7)	1.2533(9)	1.0103(3)	1.0402(4)	1.0873(5)	1.1482(6)
	0.75	1.0307(4)	1.1180(5)	1.2503(6)	1.4154(7)	1.0167(3)	1.0652(4)	1.1416(5)	1.2407(5)

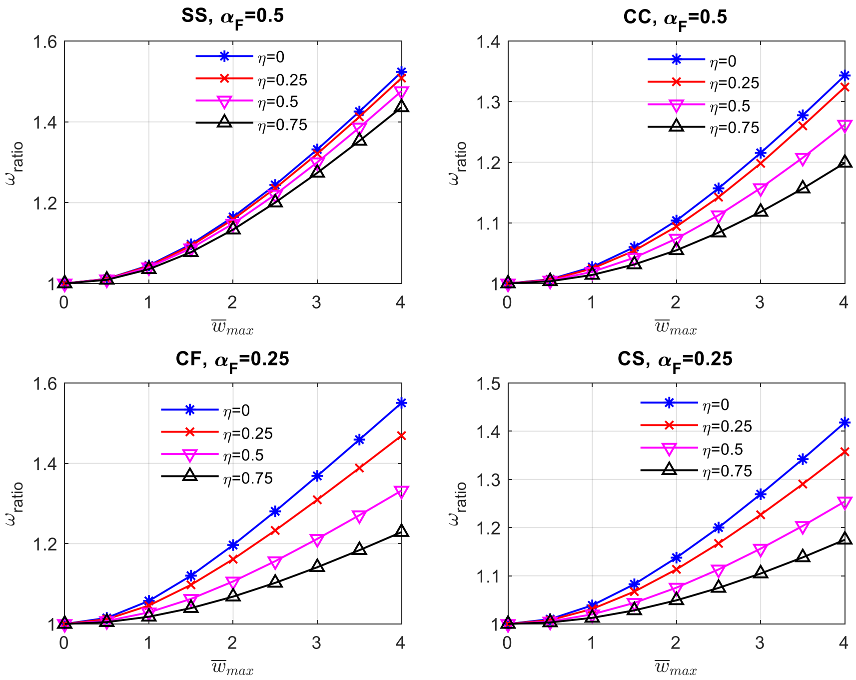

Figure 2. Dimensionless amplitude $\bar{w}_{1\max}$ versus nonlinear frequency ratio ω_{ratio} of microbeams for K_w = *200*, K_s*=100*, K_{NL}*=200.*

Table 3 lists the nonlinear frequency ratios of the SS, SC, CC, and CF microbeams partially supported by the nonlinear foundation with K_w = *200*, K_s*=100*, K_{NL}*=200*, $\eta = 0.5,\ 1.0$, and various values of α_F, $\bar{w}_{\max}$. As seen from the table, the frequency ration in the table decreases by the increase of $\bar{w}_{\max}$, and by decreasing η, regardless of the boundary conditions. The influence of the foundation supporting parameter α_F on the nonlinear frequency ratio is, however, dependent on the boundary conditions. The frequency ratio of the SS, CS and CC beams increases with increasing α_F, but that of CF beam decreases with increasing α_F. The dependence of the nonlinear frequency ratio upon η can be explained by the change of the effect of the material length scale *l*, as seen from Eqs. (5) and (7), and this leads to the change of the microbeam energy *U*.

Figure 2 shows the influence of the dimensionless material length scale parameter η on the nonlinear frequency ratio ω_{ratio}. In this figure, the nonlinear frequency ratio is plotted as a function of the dimensionless vibration amplitude ($\bar{w}_{\max}$) for the various values of the length scale parameter, η = 0, 0.25, 0.50, 0.75, 1), and for K_w = 200, K_s = 100, K_{NL} = 200, α_F = 0.5, 0.25. It is evident that an increase in the scale parameter leads to a decrease in the nonlinear frequency ratio, although both the linear and the nonlinear vibration frequencies increase with scale parameter. This situation can be interpreted by the fact that the increase in the linear frequency due to the scale parameter is larger than the increase in the nonlinear frequency. This fact holds true for all considered end conditions.

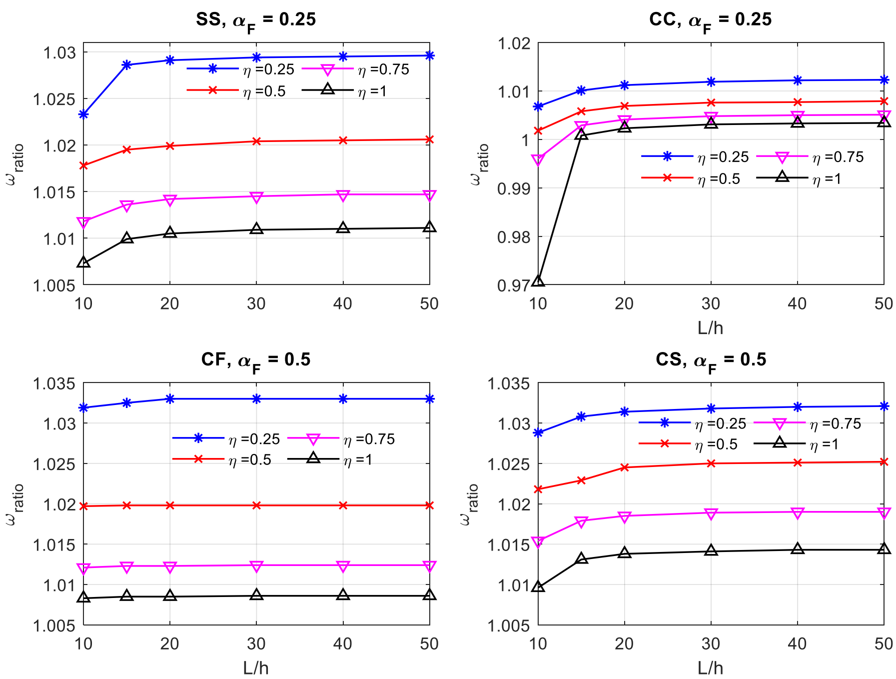

Figure 3. Aspect ratio L/h versus nonlinear frequency ratio ω_{ratio} of microbeams for K_w = *200*, K_s=*100*, K_{NL}=*200* and $\bar{w}_{\max} = 1$.

The effect of the aspect ratio *L/h* on the nonlinear frequency ratio of the microbeams is illustrated in Figure 3 for the SS, CC, CF, CS partially supported by the nonlinear elastic foundation with α_F= 0.25, 0.5 and $\bar{w}_{\max} = 1$. The nonlinear frequency ratio, as seen from the figure, steadily increases with increasing the aspect ratio, and the increase is the most significant for *L/h* less than 20. The result in Figure 3 also shows the ability of the finite element formulation derived in the present work in modeling the shear deformation effect of the microbeam.

4. CONCLUSIONS

The nonlinear free vibration analysis of microbeams partially supported on the nonlinear elastic foundation has been presented on the basis of the MCST and the refined third-order shear deformation beam theory. The equations of motion are derived from Hamilton's principle and they are solved by a finite element formulation. Using an iterative procedure, the nonlinear frequency ratios have been computed for microbeams with various boundary conditions, and the effects of the length scale parameter, the aspect ratio and the nondimensional amplitude on the nonlinear frequency ratio have been studied in detail and highlighted. It has been shown that the foundation supporting length plays an important role on the vibration of the microbeams, and the influence of the foundation supporting length on the frequency ratio is dependent on the boundary conditions. The effects of the stiffness of the nonlinear foundation and the aspect ratio on the ratio of the nonlinear frequency to linear frequency of the microbeams have also been studied and discussed. The method proposed in this paper can be extended to nonlinear free vibration analysis of microbeams made of new composite materials such as functionally graded materials and carbon nanotubes reinforced composite materials, which are widely used in MEMS/ NEMS nowadays.

CRediT authorship contribution statement. Le Cong Ich: Methodology, Software, Writing- Original draft preparation. Tran Quang Dung: Investigation. Nguyen Van Chinh: Data curation. Lam Van Dung: Validation. Nguyen Dinh Kien: Supervision, Writing- Reviewing and Editing.

Declaration of competing interest. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

REFERENCES

1. Fleck N., Muller G., Ashby M. and Hutchinson J. Strain gradient plasticity: Theory and experiment. Acta Metall Mater **42** (2) (1994) 475–487.
2. Neff P. and Forest S. - A geometrically exact micro-morphic model for elastic metallic foams accounting for affine microstructure. Modelling, existence of minimizers, identification of moduli and computational results. J Elast **87** (2–3) (2007) 239–276.
3. Simsek M. and Reddy J. N. - Bending and vibration of functionally graded microbeams using a new higher order beam theory and the modified couple stress theory, International Journal of Engineering Science **64** (2013) 37-53.
4. Yang F. A. C. M., Chong A. C. M., Lam D. C. C. and Tong P. - Couple stress-based strain gradient theory for elasticity. International journal of solids and structures, **39** (10) (2002) 2731-2743.
5. Thai H-T, Vo TP, Nguyen TK, Kim S. A review of continuum mechanics models for size-dependent analysis of beams and plates. Compos Struct **177** (2017) 196–219.
6. Wang Y. G., Lin W. H. and Liu N. - Nonlinear free vibration of a microscale beam based on modified couple stress theory. Physica E: Low-dimensional Systems and Nanostructures, **47** (2013), 80-85.

7. Ke L. L., Wang Y. S., Yang, J. and Kitipornchai, S. - Nonlinear free vibration of size-dependent functionally graded microbeams. International Journal of Engineering Science, **50** (1) (2012), 256-267.

8. Xia W., Wang L. and Yin L. - Nonlinear non-classical microscale beams: static bending, postbuckling and free vibration. International Journal of Engineering Science, **48** (12) (2010), 2044-2053.

9. Şimşek M. - Nonlinear static and free vibration analysis of microbeams based on the nonlinear elastic foundation using modified couple stress theory and He’s variational method. Composite Structures, **112** (2014), 264-272.

10. Das D. - Nonlinear forced vibration analysis of higher order shear-deformable functionally graded microbeam resting on nonlinear elastic foundation based on modified couple stress theory. Proceedings of the Institution of Mechanical Engineers, Part L: Journal of Materials: Design and Applications, **233** (9) (2019), 1773-1790.

11. Shimpi R. P. and Patel H. G. - Free vibrations of plate using two variable refined plate theories. Journal of Sound and Vibration **296** (4-5) (2006) 979-999.

12. Reddy J. N. - Energy principles and variational methods in applied mechanics (2nd ed.), John Wiley & Sons, New Jersey, 2002, 542 p.

13. Vo T. P., Thai H. T., Nguyen T. K., Maheri A. and Lee J. - Finite element model for vibration and buckling of functionally graded sandwich beams based on a refined shear deformation theory. Eng Struct **64** (2014) 12–22.

14. Cook R. R., Malkus D. S., Plesha M. E., Witt R. J. - Concepts and applications of finite element analysis, 4th ed. John Willey & Sons, New Jersey (2002).