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FORCED TRANSVERSE VIBRATIONS OF FRACTIONAL VISCOELASTIC EULER-BERNOULLI BEAM USING THE MODAL ANALYSIS METHOD

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Abstract. In the paper, the forced transverse vibration of fractional viscoelastic Euler-Bernoulli is studied. Based on the fractional relationship of stress and strain, the partial differential equation describing transverse vibration of Euler-Bernoulli viscoelastic beam is considered. The Riemann-Liouville fractional derivative of the order $0<\alpha<1$ and $0<\beta<1$ is used. Using the Ritz-Galerkin method, the fractional partial derivative equation describing the vibration of the beam is transformed into a system of differential equations containing fractional derivatives. The dynamic response of a simply supported fractional viscoelastic beam to a harmonic concentrated force is calculated in detail. The forced vibration solution of the beam is determined using the harmonic balancing method. The solution to the vibration equations is determined analytically, while dynamic responses are calculated numerically. The effects of fractional–order parameters on the vibration amplitude-time curves are investigated. From the calculation results, we can see that the lower the α parameter is, the larger the vibration amplitude. This is consistent with our logic thinking. A comparison between the approximately analytical solution and the numerical one shows a good agreement, and the correctness of the obtained results is therefore verified.

Keywords: Vibration, fractional beam, fractional differential equation, dynamic response.

Classification numbers: 5.4.2, 5.5.2.

1. INTRODUCTION

Fractional-order calculus has almost the same long history as the traditional integer-order calculus, and it was presented more than 300 years ago [1 - 4]. Fractional-order system has a great influence on the things in nature which could be seen, touched, and controlled. Fractional-order calculus is slowly developed because it had no obviously practical application for a long time due to the relatively low calculation level in early time. In recent years, fractional-order calculus was paid more and more attention from researchers in different fields and became an international hot research topic. Fractional-order calculus has been studied extensively in the fields of material science, robot dynamics, electrochemistry, viscoelastic theory, automatic control theory, fluid mechanics, and so forth.

A differential equation is called the fractional differential equation if it includes at least one no integer order derivative in the expression. Ordinary differential equations involving fractional differential operators of Riemann-Liouville's type or of Caputor's type are known to have many potential applications in mathematical modeling, in areas like mechanics and in the life sciences [5 - 9].

Works on system dynamics with fractional-order derivative may be divided into several groups, one of which is the qualitative analysis of the number and stability of solutions. For example, Machado and Galhano [10] analyzed statistical dynamics of many micromechanical masses and found the existence of both integer and fractional properties in the global dynamics. Li *et al*. [11] studied a range of stable parameters of the simplified Mathieu-type equation with fractional-order derivative. Wang and Hu [12] as well as Wang and Du [13] investigated a linear single degree-of-freedom oscillator with fractional-order derivative and found some important phenomena.

Due to the complexity of fractional-order derivatives, numerical investigation on the complicated nonlinear dynamics phenomena such as bifurcation, chaos and synchronization becomes another interesting research topic in the dynamical system with fractional-order derivative. Atanackovic and Stankovic [14] proposed a modified numerical procedure to solve fractional-order differential equations, and the test on several examples verified the efficiency of the method. Cao *et al*. [15] simulated the fractional-order Duffing equation and investigated the effect of the fractional order on system dynamics using phase diagram, bifurcation diagram and Poincare map. Sheu *et al*. [16] solved the fractional damped Duffing equation by turning it into a set of fractional integral equations. Wu *et al*. [17], Chen and Chen [18] and Lu [19] studied the synchronization in fractional-order nonlinear systems.

The investigation by analytical way was also important in dynamical system, and there were some important works on analytical investigation on dynamical system with the fractional-order derivative. Wahi and Chatterjee [20] studied an oscillator with fractional-order derivative and time-delay. Padovan and Sawicki [21], Borowiec *et al*. [22], Huang and Jin [23] also investigated differential fractional-order system and presented important results using an analytical approach. Using the averaging method, Shen *et al*. studied resonance of Duffing oscillator [24] and of Van der Pol oscillator [25] with fractional-order derivatives. Khang and Chien [26] studied the subharmonic resonance of Duffing oscillator with fractional-order derivatives. Using the asymptotic methods in the theory of nonlinear oscillations, Khang *et al*. [27, 28] have studied the resonance oscillations in third order systems involving fractional-order derivative and have received some new effects.

The use of fractional relations of stress and strain to calculate vibration of continuous systems is rather limited. Based on the stress-strain constitutive relation of the beam material proposed by Bagley and Torvik [29], Freundlich has studied the dynamic response of a simply supported viscoelastic beam of a fractional derivative type subjected to a moving force load [30, 31]. Paola *et al*. studied the establishment of the equations of motion of Euler-Bernoulli beams [32]. Pirotta *et al*. [33] investigated the establishment of the vibration equation of the Timoshenko beam by generalizing the vibration equation of the Euler-Bernoulli beam.

Using the fractional relation of stress and strain, the governing vibration equation of the fractional Euler-Bernoulli beam is established in this paper. Then the dynamic response of a simply supported fractional beam is calculated. This study is organized in four sections. In Section 2, an analytical formulation for the transverse vibration of a fractional viscoelastic Euler-Bernoulli beam is presented. In Section 3, the dynamic response of a simply supported fractional viscoelastic beam subjected to a harmonic concentrated force is calculated in detail. Section 4 includes some concluding remarks of the study.

2. ANALYTICAL FORMULATION

Let us consider the model of an Euler-Bernoulli beam of the length l and the flexural rigidity EI as shown in Figure 1, where w(x, t) is the dynamic deflection, and p(x, t) is the distributed force.

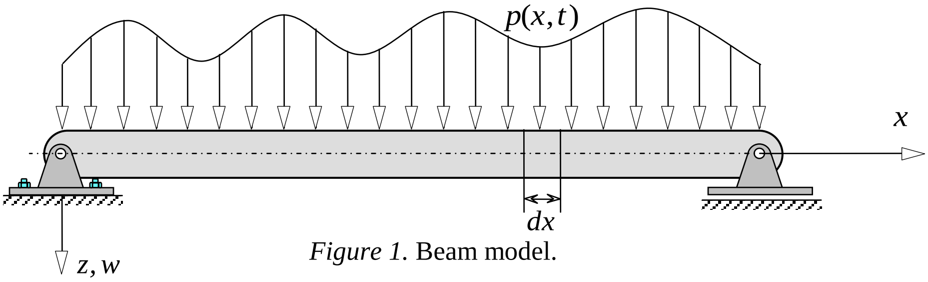

Figure 1. Beam model.

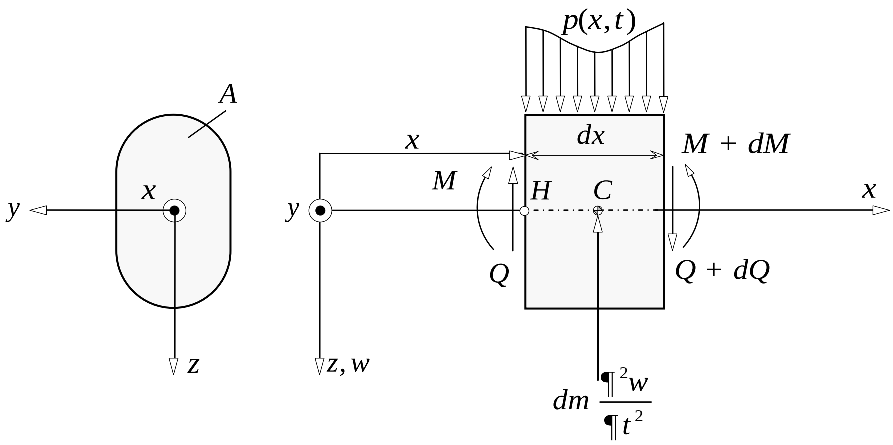

Figure 2. Forces in an element of the beam.

The governing equation of motion and boundary conditions of a thin beam has been derived by considering an element of the beam as shown in Fig. 2. Assuming the deformation to be small enough such that the shear deformation is much smaller than w(x,t), a summation of forces in the x direction yields

$$-\rho A\frac{\partial^2 w}{\partial t^2} - Q + Q + \frac{\partial Q}{\partial x}dx + p(x,t)dx = 0 \tag{1}$$

Next, the moments acting on the element dx about the y-axis are summed. This yields

$$-M_y + M_y + \frac{\partial M_y}{\partial x}dx - \left(Q + \frac{\partial Q}{\partial x}dx\right)\frac{dx}{2} - Q\frac{dx}{2} = 0 \tag{2}$$

In Eqs. (1) and (2) M_y is the bending moment, Q is the shear force, r is the mass of a volume unit, and A is the cross-sectional area assumed to be constant.

Simplifying Eq. (1) yields

$$\rho A\frac{\partial^2 w}{\partial t^2}-\frac{\partial Q}{\partial x} = p(x,t) \tag{3}$$

Since *dx* is assumed to be very small, $(dx)^2$ is assumed to be almost zero, so that Eq. (2) yields

$$\frac{\partial M_y}{\partial x}-Q=0 \quad \Rightarrow \quad Q=\frac{\partial M_y}{\partial x} \tag{4}$$

For the Euler-Bernoulli beam**,** the fractional constitutive law of the viscoelastic material has the following form [5]

$$\sigma_{xx}(x,t)=E_\alpha D_t^\alpha \varepsilon_{xx}(x,t)=\mu_\alpha E D_t^\alpha \varepsilon_{xx}(x,t), \qquad 0<\alpha<1 \tag{5}$$

where a the fractional-order, m_a the endogenous damping parameter, E the elastic modulus, s_{xx} the stress, and e_{xx} the strain.

In Eq. (5) D_t^a is the Riemann-Liouville derivative that is defined by the following form [1-4]

$$D_t^\alpha x(t)=\frac{d^n}{dt^n}\left(\frac{d^{-(n-\alpha)}x(t)}{dt^{-(n-\alpha)}}\right)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-\tau)^{n-\alpha-1}x(\tau)d\tau \tag{6}$$

Another choice is the Caputo definition

$${}_a^C D_t^\alpha x(t)=\frac{1}{\Gamma(n-q)}\int_a^x (t-\tau)^{n-\alpha-1}\left[\frac{d^n}{d\tau^n}x(\tau)\right]d\tau \tag{7}$$

In both cases $(n - 1) < a < n$.

In the Eq. (6) and Eq. (7), $\Gamma(n-q)$ is called Gamma function, and is defined by the following form

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt\,.$$

From [10] the axial strain $\varepsilon(x,z,t)$ which is related to bending deformation $w(x,t)$, is rewritten as

$$\varepsilon_{xx}(x,z,t)=-\frac{\partial^2 w}{\partial x^2}z \tag{8}$$

Substituting Eq. (8) into Eq. (5) yields

$$\sigma_{xx}(x,t)=\mu_\alpha E D_t^\alpha\left(-\frac{\partial^2 w}{\partial x^2}z\right)=-\mu_\alpha Ez\, D_t^\alpha\left[\frac{\partial^2 w(x,t)}{\partial x^2}\right]=-\mu_\alpha Ez\frac{\partial^2}{\partial x^2}\left[D_t^\alpha w(x,t)\right] \tag{9}$$

From mechanics of materials, the beam sustains a bending moment $M(x,t)$ which is related to the beam deflection, or bending deformation, $w(x,t)$, by

$$M_y = \int_A z\sigma_{xx} dA \tag{10}$$

Substituting Eq. (9) into Eq. (10) yields

$$M_y = -\mu_\alpha E \frac{\partial^2}{\partial x^2}\left[D_t^\alpha w(x,t)\right]\int z^2 dA \tag{11}$$

The cross-sectional area moment of inertia about the y-axis is defined by the formula

$$I = \int_A z^2 dA \tag{12}$$

Substituting Eq. (12) into Eq. (11) yields

$$M_y = -\mu_\alpha EI \frac{\partial^2}{\partial x^2}\left[D_t^\alpha w(x,t)\right] \tag{13}$$

Substituting Eq. (13) into Eq. (4) yields

$$Q = -\mu_\alpha EI \frac{\partial^3}{\partial x^3}\left[D_t^\alpha w(x,t)\right] \tag{14}$$

Substituting Eq. (14) into Eq. (3) yields

$$\mu_\alpha EI \frac{\partial^4}{\partial x^4}[D_t^\alpha w(x,t)] + \rho A \frac{\partial^2 w}{\partial t^2} = p(x,t) \tag{15}$$

It is assumed that the external damping is described by a fractional derivative and has the following form

$$F_C = \rho A \mu_\beta \frac{d^\beta w}{dt^\beta} \tag{16}$$

In which A is the cross-sectional area of the beam, and m_b is the external fractional damping parameter.

Equation (15) is then shown in the following form

$$\rho A \frac{\partial^2 w}{\partial x^2} + \mu_\alpha EI \frac{\partial^4}{\partial x^4}\left[D_t^\alpha w(x,t)\right] + \rho A \mu_\beta \frac{d^\beta w}{dt^\beta} = p(x,t) \tag{17}$$

The beam vibration equation (17) can be rewritten as

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI\mu_\alpha \frac{d^\alpha}{dt^\alpha}\left(\frac{\partial^4 w}{\partial x^4}\right) + \rho A \mu_\beta \frac{d^\beta w}{dt^\beta} = p(x,t) \tag{18}$$

Using the Ritz-Galerkin method, the transverse deflection of the beam can be described in the following form

$$w(x,t) = \sum_{i=1}^{\infty} W_i(x) q_i(t) \tag{19}$$

where $W_i(x)$ are the eigenfunctions and $q_i(t)$ are the modal coordinates [34].

From Eq. (19) the corresponding derivatives are evaluated bellow

$$\frac{d^\alpha}{dt^\alpha}\left(\frac{\partial^4 w}{\partial x^4}\right)=\sum_{i=0}^{\infty}W_i^{(IV)}(x)D_t^\alpha[q_i(t)] \tag{20}$$

$$D_t^\alpha[q_i(t)]=\frac{d^\alpha q_i(t)}{dt^\alpha}=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{q_i(t)}{(t-\tau)^\alpha}d\tau \tag{21}$$

$$\frac{d^\beta w}{dt^\beta}=\sum_{i=0}^{\infty}W_i(x)\frac{d^\beta q_i(t)}{dt^\beta}=\sum_{i=0}^{\infty}W_i(x)\frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t\frac{q_i(t)}{(t-\tau)^\beta}d\tau \tag{22}$$

$$\frac{\partial^2 w}{\partial t^2}=\sum_{i=0}^{\infty}W_i(x)\ddot{q}_i(t) \tag{23}$$

Substituting Eqs. (20), (21), (22) and (23) into Eq. (18) yields

$$\sum_{i=0}^{\infty}\left\{EI\mu_\alpha W_i^{(IV)}(x)D_t^\alpha[q_i(t)]+\rho A W_i(x)\ddot{q}_i(t)+\rho A\mu_\beta W_i(x)D_t^\beta[q_i(t)]\right\}=p(x,t) \tag{24}$$

Multiplying Eq. (24) by $W_k(x)$ yields

$$\sum_{i=0}^{\infty}\left\{\ddot{q}_i(t)+\mu_\alpha\frac{EI}{\rho A}\frac{W_i^{(IV)}(x)}{W_i(x)}D_t^\alpha[q_i(t)]+\mu_\beta D_t^\beta[q_i(t)]\right\}W_i(x)W_k(x)=\frac{p(x,t)W_k(x)}{\rho A} \tag{25}$$

Next, integrating over x from 0 to *l*, and using the orthogonality property of eigenfunctions, after some mathematical transformations, the differential equations of the k-th mode of the modal coordinates are obtained

$$\ddot{q}_k(t)+\mu_\alpha\omega_k^2D_t^\alpha[q_k(t)]+\mu_\beta D_t^\beta[q_k(t)]=\frac{\int_0^l p(x,t)W_k(x)dx}{\rho A\int_0^l W_k^2(x)dx} \tag{26}$$

Using the notations

$$h_k(t)=\frac{\int_0^l p(x,t)W_k(x)dx}{\rho A\int_0^l W_k^2(x)dx} \tag{27}$$

It follows from Eq. (26) that

$$\ddot{q}_k(t)+\mu_\alpha\omega_k^2D_t^\alpha[q_k(t)]+\mu_\beta D_t^\beta[q_k(t)]=h_k(t),\quad k=1,2,... \tag{28}$$

3. DYNAMIC RESPONSE OF A SIMPLY SUPPORTED FRACTIONAL VISCOELASTIC BEAM TO A HARMONIC CONCENTRATED FORCE

Transverse vibration of Bernoulli-Euler homogeneous isotropic fractional simply supported beam to a harmonic concentrated is investigated in this section (Fig. 3). The eigenfunctions of the simply supported beam have the following form [34]

$$W_k(x) = \sin(k\pi x / l) \tag{29}$$

Using the Delta-Dirac function the distributed force $p(x,t)$ has following form

$$p(x,t) = F_0 \cos\Omega t \; \delta(x-a) \tag{30}$$

Substituting Eq. (29) and Eq. (30) into Eq. (26) and doing some mathematical transformations, we get a system of equations

$$\ddot{q}_k(t) + \mu_\alpha \omega_k^2 D_t^\alpha[q_k(t)] + \mu_\beta D_t^\beta[q_k(t)] = b_k \cos\Omega t, \; k = 1,2,3,... \tag{31}$$

where: $b_k = \dfrac{2F_0 \sin(k\pi a / l)}{\rho A l}$,

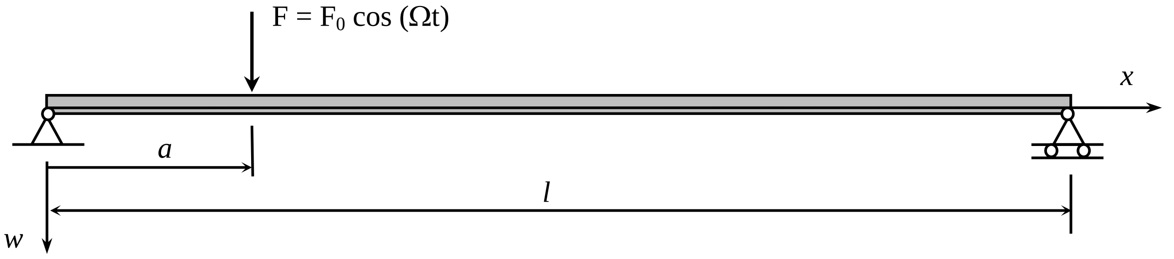

Fig. 3. The beam under harmonic concentrated force.

The analytical solution of each k-th equation in Eqs. (31) is assumed as

$$q_k(t) = A_k \sin(\Omega t + \delta_k) \tag{32}$$

Using the relations [28]

$$\begin{aligned} D_t^\alpha[\cos\Omega t] &= \Omega^\alpha \cos(\Omega t + \frac{\alpha\pi}{2}) \\ D_t^\alpha[\sin\Omega t] &= \Omega^\alpha \sin(\Omega t + \frac{\alpha\pi}{2}) \end{aligned} \tag{33}$$

we have the corresponding derivatives

$$\begin{aligned} D_t^\alpha[q_k(t)] &= A_k \Omega^\alpha \sin(\Omega t + \delta_k + \frac{\alpha\pi}{2}) \\ D_t^\beta[q_k(t)] &= A_k \Omega^\beta \sin(\Omega t + \delta_k + \frac{\beta\pi}{2}) \end{aligned} \tag{34}$$

From Eq. (32) we have

$$\ddot{q}_k(t) = -A_k \Omega^2 \sin(\Omega t + \delta_k) \tag{35}$$

Substitution of Eqs. (32), (34) and (35) into Eq. (31) we get the equation

$$\begin{aligned} -A_k\Omega^2 \sin(\Omega t + \delta_k) + \mu_\alpha \omega_k^2 A_k \Omega^\alpha \sin(\Omega t + \delta_k + \frac{\alpha\pi}{2}) \\ + \mu_\beta A_k \Omega^\beta \sin(\Omega t + \delta_k + \frac{\beta\pi}{2}) = b_k \cos\Omega t \end{aligned} \tag{36}$$

Using trigonometric formulas

$$\begin{aligned}\sin(x_1 \pm x_2) &= \sin x_1.\cos x_2 \pm \cos x_1.\sin x_2 \\ \cos(x_1 \pm x_2) &= \cos x_1.\cos x_2 \mp \sin x_1.\sin x_2\end{aligned} \tag{37}$$

we have

$$\begin{aligned}&\sin(\Omega t + \delta_k + \frac{\alpha\pi}{2}) = \sin(\Omega t + \delta_k).\cos\frac{\alpha\pi}{2} + \cos(\Omega t + \delta_k).\sin\frac{\alpha\pi}{2} \\ &\sin(\Omega t + \delta_k + \frac{\beta\pi}{2}) = \sin(\Omega t + \delta_k).\cos\frac{\beta\pi}{2} + \cos(\Omega t + \delta_k).\sin\frac{\beta\pi}{2} \\ &\cos\Omega t = \cos(\Omega t + \delta_k - \delta_k) = \cos(\Omega t + \delta_k).\cos\delta_k + \sin(\Omega t + \delta_k).\sin\delta_k\end{aligned} \tag{38}$$

Substituting Eqs. (38) into Eq. (36) yields

$$\begin{aligned}&-A_k\Omega^2\sin(\Omega t + \delta_k) + \mu_\alpha\omega_k^2 A_k\Omega^\alpha\cos\frac{\alpha\pi}{2}.\sin(\Omega t + \delta_k) + \mu_\alpha\omega_k^2 A_k\Omega^\alpha\sin\frac{\alpha\pi}{2}.\cos(\Omega t + \delta_k) \\ &\quad + \mu_\beta A_k\Omega^\beta\cos\frac{\beta\pi}{2}.\sin(\Omega t + \delta_k) + \mu_\beta A_k\Omega^\beta\sin\frac{\beta\pi}{2}.\cos(\Omega t + \delta_k) \\ &\quad = b_k\sin\delta_k\sin(\Omega t + \delta_k) + b_k\cos\delta_k\cos(\Omega t + \delta_k)\end{aligned} \tag{39}$$

By equating the coefficients of the sin and cos of the right and left sides of Eq. (39) we get the equations

$$\begin{aligned}&A_k(-\Omega^2 + \mu_\alpha\omega_k^2\Omega^\alpha\cos\frac{\alpha\pi}{2} + \mu_\beta\Omega^\beta\cos\frac{\beta\pi}{2}) = b_k\sin\delta_k \\ &A_k(\mu_\alpha\omega_k^2\Omega^\alpha\sin\frac{\alpha\pi}{2} + \mu_\beta\Omega^\beta\sin\frac{\beta\pi}{2}) \qquad = b_k\cos\delta_k\end{aligned} \tag{40}$$

Squaring the two sides of Eq. (40) and adding them together, we obtain

$$A_k = \frac{b_k}{\sqrt{(-\Omega^2 + \mu_\alpha\omega_k^2\Omega^\alpha\cos\frac{\alpha\pi}{2} + \mu_\beta\Omega^\beta\cos\frac{\beta\pi}{2})^2 + (\mu_\alpha\omega_k^2\Omega^\alpha\sin\frac{\alpha\pi}{2} + \mu_\beta\Omega^\beta\sin\frac{\beta\pi}{2})^2}} \tag{41}$$

From Eq. (41) and Eq. (42), we get the formulas to calculate the phase angles

$$\sin\delta_k = \frac{-\Omega^2 + \mu_\alpha\omega_k^2\Omega^\alpha\cos\frac{\alpha\pi}{2} + \mu_\beta\Omega^\beta\cos\frac{\beta\pi}{2}}{\sqrt{(-\Omega^2 + \mu_\alpha\omega_k^2\Omega^\alpha\cos\frac{\alpha\pi}{2} + \mu_\beta\Omega^\beta\cos\frac{\beta\pi}{2})^2 + (\mu_\alpha\omega_k^2\Omega^\alpha\sin\frac{\alpha\pi}{2} + \mu_\beta\Omega^\beta\sin\frac{\beta\pi}{2})^2}} \tag{42}$$

$$\cos\delta_k = \frac{\mu_\alpha\omega_k^2\Omega^\alpha\sin\frac{\alpha\pi}{2} + \mu_\beta\Omega^\beta\sin\frac{\beta\pi}{2}}{\sqrt{(-\Omega^2 + \mu_\alpha\omega_k^2\Omega^\alpha\cos\frac{\alpha\pi}{2} + \mu_\beta\Omega^\beta\cos\frac{\beta\pi}{2})^2 + (\mu_\alpha\omega_k^2\Omega^\alpha\sin\frac{\alpha\pi}{2} + \mu_\beta\Omega^\beta\sin\frac{\beta\pi}{2})^2}} \tag{43}$$

The solution of Eq. (31) has the following form

$$q_k(t) = A_k\sin(\Omega t + \delta_k) = A_k\cos\delta_k.\sin\Omega t + A_k\sin\delta_k.\cos\Omega t \tag{44}$$

Using Eq. (19), the steady-state response of the beam has the form

$$w(x,t) = \sum_{k=1}^{N} A_k \sin(\Omega t + \delta_k) \sin \frac{k\pi x}{l} \tag{45}$$

It should be noted that in the sum of (45) the $\sin(\pi x / l)$ term has a dominant preponderance. So if we make N large, the sum (45) doesn't change much. Usually, one chooses N = 3 or N = 5. Parameters of the examined beam are chosen as follows: l = 20 m; ρ = 7600 kg/m^3; A = 0.002 m^2; E = 2.1×10^{11} N/m^2 ; I = 3.953×10^{-6} m^4; z_{max} = 0.077 m; $M_z = I / z_{max}$; a = 5 m; F = F_0 cos(Ωt) with F_0 = 100 N; $\Omega = 10\pi$ 1/s, N=5. In which I (m^4) is the inertial moment of sectional area of the beam, M_z is the bending moment.

Some calculation results by analytical method are shown in Figures 4-6. In Figures 4, 5 and 6 are plotted the transverse vibrations of beam at cross section x = 10 m with β = 0.85; μ_α= 0.9; μ_β= 0.3 and different coefficients α.

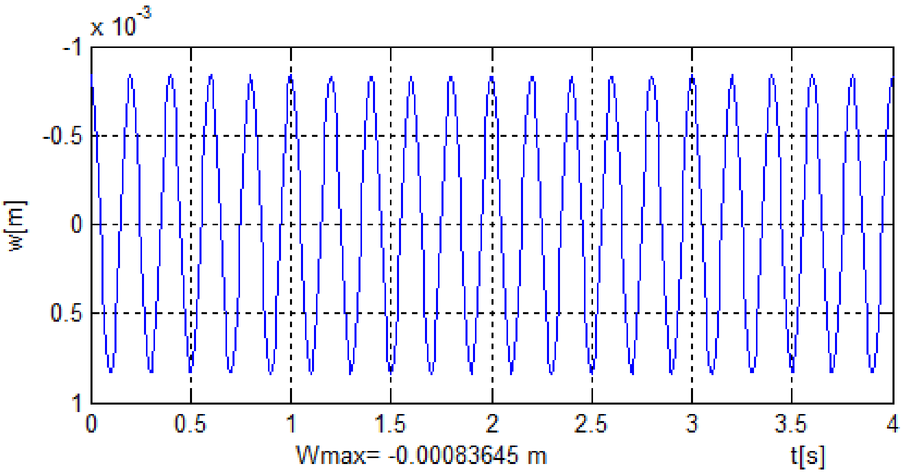

Figure 4. Transvere vibrations of beam at cross section x = 10 m, α = 0.

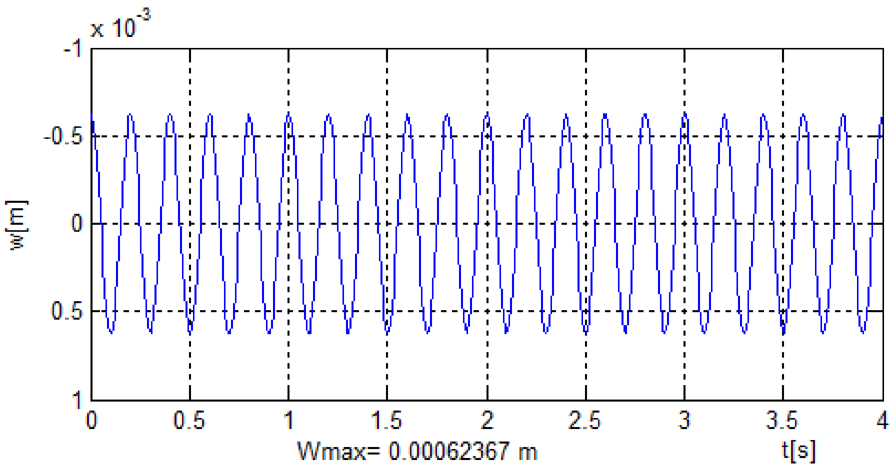

Figure 5. Transvere vibration of beam at cross section x = 10 m, α = 0.2

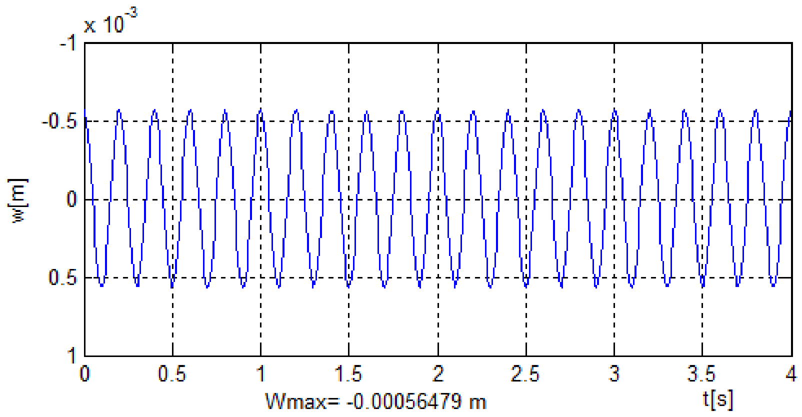

Figure 6. Transvere vibration of beam at cross section x = 10 m, α = 0.4.

In Figure 7 is the deflection of the beam at time t = 0.8 s, while in Figure 8 is the graph of the beam's stress at time t = 0.8 s, in which the solid line is the result calculated by the analytical method, the dashed line is the result calculated by the numerical method [35, 36]. We see that the results calculated using numerical method are consistent with those calculated using analytical method.

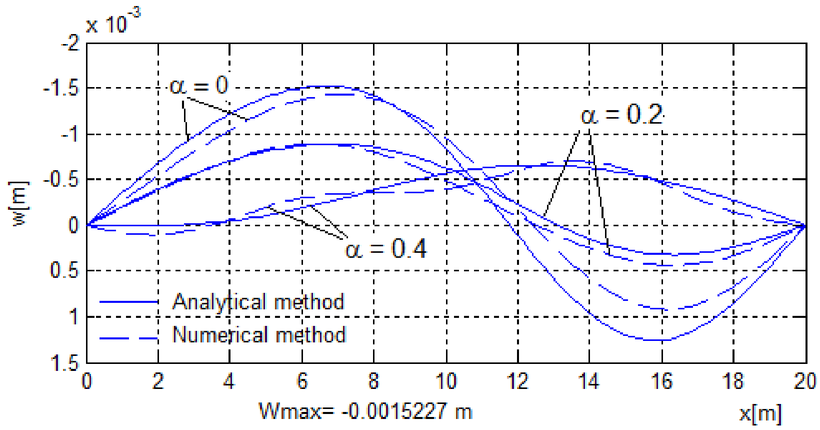

Figure 7. Dynamic deflection of the beam at time t = 0.8 s.

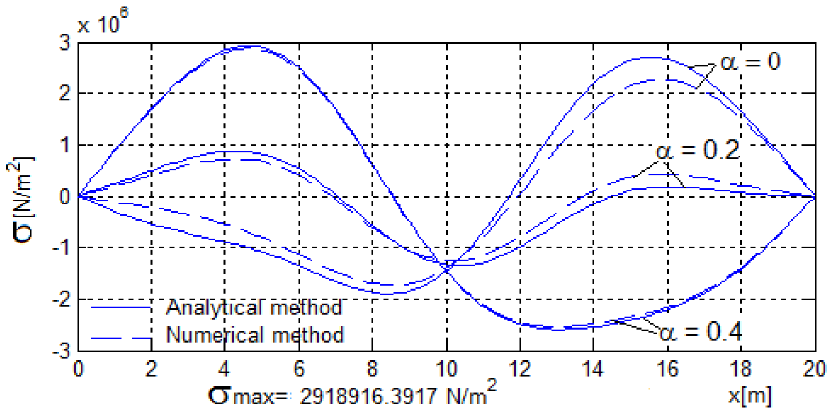

Figure 8. The stress of the beam at time t = 0.8 s.

4. CONCLUSIONS

Fractional derivatives have been used to describe the behavior of viscoelastic materials. In this paper, the vibration equations of the fractional viscoelastic Euler-Bernoulli beam are presented. Using the Ritz-Galerkin method, the fractional partial derivative equation describing the beam motion is transformed into a system of differential equations containing fractional derivatives.

As an applicable example, bending vibrations of a simply supported fractional viscoelastic beam were studied and calculated. Using the analytical method, the bending vibrations of the beam were calculated in detail. Some results calculated by the analytical method were compared with those calculated by the numerical method. The results calculated by the two methods agree quite well.

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CRediT authorship contribution statement. Nguyen Van Khang: Theory, Writing & Editing manuscript. Nguyen Minh Phuong, Pham Thanh Chung: Numerical simulation, Review

Declaration of competing interest. The authors declare that we have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper**.**

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