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FINITE TIME STABILIZATION OF NON-AUTONOMOUS, NONLINEAR SECOND-ORDER SYSTEMS BASED ON MINIMUM TIME PRINCIPLE

Dang Van My*, Trinh Hoang Minh, Vu Thi Thuy Nga, Nguyen Doan Phuoc

Hanoi University of Science and Technology, 1 Dai Co Viet Road, Ha Noi, Viet Nam

*Email: *my.dangvan@hust.edu.vn*

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Abstract. This paper proposes a controller design method to stabilize a class of nonlinear, non-autonomous second-order systems in finite time. This method is developed based on exact-linearization and Pontryagin's minimum time principle. It is shown that the system can be stabilized in a finite time of which the upper bound can be chosen according to the initial states of the system. Simulation results are given to validate the theoretical analysis.

Keywords: finite time stabilization, non-autonomous systems, minimum time principle.

Classification numbers: 4.10, 5.8.

1. INTRODUCTION

In recent years, there has been increasing research interest in designing finite/fixed-time stabilization (FTS) laws for second-order control-affine systems with a single control input, i.e., the autonomous system modeled by

$$\dot{\underline{x}} = \underline{f}(\underline{x}) + \underline{h}(\underline{x})u \tag{1}$$

where $\underline{x} = (x_1, x_2)^T$ is the state vector, $\underline{f}(\underline{0}) = \underline{0}$ and $\underline{0}$ is the origin. Several notable works on this avenue include [1 - 6].

The problem of designing FTS laws has theoretical significance because of two reasons. First, a FTS control law can drive the states to the origin in a finite-time. Second, FTS is crucial for designing sliding mode controllers, since the states of the system must be driven to the sliding surface in a finite time [7 - 9]. In the literature, almost all existing solutions to the FTS problem hinge on the Lyapunov stability theory [6]. The common approach is to find a continuously differentiable, positive definite function $V(\underline{x})$ so that there is at least a state-feedback controller $u(\underline{x})$ making

$$\frac{dV}{dt} = \frac{\partial V}{\partial \underline{x}}\frac{d\underline{x}}{dt} = \frac{\partial V}{\partial \underline{x}}\left(\underline{f}(\underline{x}) + \underline{h}(\underline{x})u(\underline{x})\right) \tag{2}$$

satisfies

$$\frac{dV}{dt} = \begin{cases} < 0 & \text{if } 0 \le t < T \\ = 0 & \text{if } t \ge T \end{cases} \tag{3}$$

where T is a finite strictly positive number. The value T is also referred to as the stabilizing time of system (1) under the FTS controller $u(\underline{x})$ due to Krasovski-LaSalle invariance principle, it must held that $\underline{x}(t) = \underline{0}$ for $t \ge T$. The aforementioned approaches, however, share the same issue of finding a suitable Lyapunov candidate function. Until now, there is still no complete solution for constructing such a function

In this paper, we propose a novel approach to the FTS problem that does not invoke the Lyapunov stability theory. The theoretical foundation of the proposed approach is based on the minimum time optimal control theory in Pontryagin's maximum principle [10] and a suitable exact-linearization approach from differential geometry [11 - 13]. The proposed approach gives a simple approach to stabilize a class of autonomous, nonlinear second-order system in a finite time T. An explicit formula for the switching time and the stabilizing time T, which can be used as guidelines for performance design is also given. It is noted that minimum time control has been studied for second-order linear systems in [10, 14, 15]. The authors of [16] studied minimum-time control of second-order systems with a partly unknown nonlinear dynamic.

In the preliminary work of this paper, the [17], a minimum time controller was designed for autonomous, nonlinear second-order systems in strict feedback form, which appears to be structurally more simple than the terminal sliding mode controller presented in [18] also for second-order autonomous systems in feedback form. In this paper, the results of [17] will be enlarged to non-autonomous systems and additionally give a formula to determine the precise switching- and stabilizing time. In fact, we are not aware of any other work considering minimum time principle for stabilizing non-autonomous, nonlinear systems in finite time.

2. MAIN RESULTS

In this paper, we consider a non-autonomous, nonlinear second-order system with the following feedback structure

$$\begin{cases} \dot{x}_1 = f_1(x_1, t) + x_2 \\ \dot{x}_2 = f_2(x_1, x_2, t) + u \end{cases} \tag{4}$$

or in a compact form

$$\dot{\underline{x}} = \underline{f}(\underline{x}, t) + \underline{h}u \tag{5}$$

where

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ \underline{f}(\underline{x}, t) = \begin{pmatrix} f_1(x_1, t) + x_2 \\ f_2(\underline{x}, t) \end{pmatrix} \text{ and } \underline{h} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Furthermore, we assume here that $f_1(x_1, t)$ are smooth and $f_1(0, t) = 0, \ \forall t$. The finite time stabilization for this system (4) will be carried out here in two steps:

– First, the system (4) will be linearized exactly by using a state feedback controller (also called the feedback linearization or feedback linearization).
– And then, the obtained LTI will be time optimal controlled with a state feedback controller in the sense of minimizing the following free end time cost-function:

$$J = \int_0^T dt = T \to \min! \tag{6}$$

It is obviously that with this time optimal control problem, the transient time T, which is also the stabilizing time, has to be finite.

2.1 Exact linearization by state feedback control

Consider the change of variables $z_1 = x_1,\ z_2 = f_1(x_1,t) + x_2$ or i.e.

$$\underline{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \underline{m}(\underline{x},t) = \begin{pmatrix} x_1 \\ f_1(x_1,t) + x_2 \end{pmatrix} \tag{7}$$

which is clearly invertible, since the fact that

$$\det \frac{\partial \underline{m}}{\partial \underline{x}} = \det \begin{pmatrix} 1 & 0 \\ \times & 1 \end{pmatrix} = 1 \neq 0,\ \forall \underline{x} \quad \Rightarrow \quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underline{m}^{-1}(\underline{z},t) = \begin{pmatrix} z_1 \\ z_2 - f_1(z_1,t) \end{pmatrix}. \tag{8}$$

Then the system (4) can be rewritten as

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = r(\underline{x},t) + u \end{cases} \tag{9}$$

where

$$r(\underline{x},t) = \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x_1}\left[f_1(x_1,t) + x_2\right] + f_2(\underline{x},t)\,. \tag{10}$$

Hence, by using the time varying state-feedback controller

$$v = r(\underline{x},t) + u \quad \text{or} \quad u = v - r(\underline{x},t)$$

the original system (5) becomes

$$\dot{\underline{z}} = \begin{pmatrix} z_2 \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\underline{z} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v = A\underline{z} + \underline{b}v \ \text{ where } A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \underline{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{11}$$

which is linear and time-invariant in the whole state space $\underline{z}$.

2.2 Time optimal control

We come to the next task, i.e. to the finite time stabilization of the LTI system (11). From literatures, there are many methods available for solving this task associated with Lyapunov's theory [1 - 7]. However, belonging to the purpose that the stabilization time T could be adjusted flexibility, the usage of principle of minimum time optimal control appears to be preferred [10, 14, 15, 17]. Therefore, we will use this principle for carrying out the second task.

Based on the time optimal control principle for any starting point $\underline{z}_0$ to the fixed endpoint $\underline{z}_T = \underline{0}$ we obtain the state feedback time optimal controller for LTI system (11) as follows [17]:

$$v(\underline{z}) = \begin{cases} -k\,\mathrm{sgn}\,\varphi_z(\underline{z}) \ \text{ if } \ \varphi_z \neq 0 \\ k\,\mathrm{sgn}(z_1) \ \text{ if } \ \varphi_z(\underline{z}) = 0 \text{ and } z_1 \neq 0 \\ 0 \ \text{ otherwise} \end{cases} \tag{12}$$

where

$$\varphi_z(\underline{z}) = z_1 + \frac{1}{2k} z_2 |z_2| \tag{13}$$

and $k > 0$ is arbitrarily chosen. The time optimal controller (12) is quite the same state feedback controller given previously in [17] for autonomous systems. It is also shown furthermore in [10,17] that the bigger k is chosen, the faster stabilizing will be. Moreover, within (12) it is realizable that each optimal trajectory ends at the origin and could have maximum one switching point on the curve (13). It means that this curve contains all switching points of the control input $\underline{v}$ and/or end part of all optimal trajectories starting outside it. Therefore, from now onwards, we refer to the curve (13) as the "switching-curve".

Coming back to the original state space $\underline{x}$ we have the finite time optimal controller:

$$u(\underline{x}) = \begin{cases} -\left[k \operatorname{sgn} \varphi_x(\underline{x}) + r(\underline{x},t)\right] & \text{if } \varphi_x(\underline{x},t) \neq 0 \\ k \operatorname{sgn}(x_1) - r(\underline{x},t) & \text{if } \varphi_x(\underline{x},t) = 0 \text{ and } x_1 \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

with the switching-curve (13) as follows:

$$\varphi_x(\underline{x},t) = \varphi_z\left(\underline{m}(\underline{z},t)\right) = x_1 + \frac{1}{2k}\left(f_1(x_1,t) + x_2\right)\left|f_1(x_1,t) + x_2\right|. \tag{15}$$

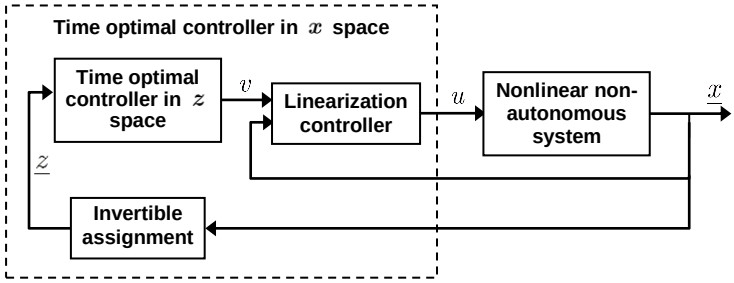

Figure 1. Control scheme for finite time stabilizing the non-autonomous nonlinear systems.

Figure 1 exhibits completely the structure of the proposed finite time stabilization control for second order non-autonomous nonlinear systems, where the transient time T is flexibility adjustable by choosing the control parameter k.

2.3 Determination of stabilizing time

In this subsection, we will give an explicit formula for determining the stabilizing time.

Lemma: *Let* $\underline{z}_0 = (\alpha, \beta)$ *be the initial states of the LTI system* (11). *The state feedback controller* (12) *stabilizes this in finite time* T *determined by*

$$T = \frac{\delta\beta + 2\sqrt{k\alpha\delta + \beta^2/2}}{k} \tag{16}$$

where

$$\delta = \left(1-|\gamma|\right)\operatorname{sgn}\beta + \gamma \;\; \textit{and} \;\; \gamma = \operatorname{sgn}\left(\alpha + \frac{1}{2k}\beta|\beta|\right).$$

Proof: Consider the following three cases with the illustration in Fig. 2.

• Case 1: $\underline{z}_0$ is on the switching-curve (13). In this case, $\gamma = 0$, $\delta = \operatorname{sgn}\beta$ and thus

$$k\alpha\delta + \frac{\beta^2}{2} = \delta\left(\alpha + \frac{1}{2k}\beta|\beta|\right) = 0\,.$$

The optimal control signal

$$v = -k\operatorname{sgn}\beta = -\delta k,\;\; \forall t \in [0,T]\,.$$

does not change its sign. The optimal trajectory can be calculated as follows:

$$\underline{z}(t) = e^{At}\underline{z}_0 + \int_0^t e^{A(t-\tau)}\underline{b}\,v\,d\tau = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} - \delta k\int_0^t \begin{pmatrix} 1 & t-\tau \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} d\tau$$

$$= \begin{pmatrix} \alpha + t\beta - \frac{\delta k}{2}t^2 \\ \beta - \delta k t \end{pmatrix}$$

Since the endpoint $\underline{z}_T = \underline{0}$ we have

$$\beta - \delta k T = 0 \quad \Leftrightarrow \quad T = \frac{\beta}{\delta k} = \frac{|\beta|}{k}$$

and this is consistent with (16).

• Case 2: $\underline{z}_0$ lies over the switching-curve (see Fig. 2):

$$\alpha + \frac{1}{2k}\beta|\beta| > 0\,.$$

In this case we have

$$\delta = \gamma = 1 \;\text{ and }\; k\alpha + \frac{1}{2}\beta^2 \geq k\alpha + \frac{1}{2}\beta|\beta| > 0\,.$$

The optimal input given by

$$v(t) = \begin{cases} -k & \text{for } \; 0 \leq t \leq t_1 \\ k & \text{for } \; t_1 < t \leq T \end{cases}$$

changes its sign one time as illustrated in Figure 2. Hence the first part of the optimal trajectory is given by:

$$\underline{z}(t) = \begin{pmatrix} -\frac{1}{2k}z_2^2(t) + \alpha + \frac{1}{2k}\beta^2 \\ -kt + \beta \end{pmatrix} \;\text{ for } 0 \leq t \leq t_1\,,$$

On the switching-curve AOB (Fig. 2), i.e. when $t = t_1$, we have

$$z_1 = \frac{1}{2k}z_2^2,\; z_2 < 0 \quad \Rightarrow \quad -\frac{1}{2k}z_2(t_1)^2 + \alpha + \frac{\beta^2}{2k} = \frac{1}{2k}z_2(t_1)^2$$

which deduces

$$t_1 = \frac{\beta + \sqrt{k\alpha + \frac{1}{2}\beta^2}}{k}.$$

Then, together with the result from Case 1 for $t_1 < t \le T$, the total stabilizing time is

$$T = t_1 + \frac{|z_2(t_1)|}{k} = \frac{\beta + 2\sqrt{k\alpha + \frac{1}{2}\beta^2}}{k}$$

and this is consistent with (16).

• Case 3: $\underline{z}_0$ lies under the switching-curve AOB. In this case we have

$$\delta = \gamma = -1 \text{ and } -k\alpha + \frac{1}{2}\beta^2 \ge -k\alpha - \frac{1}{2}\beta|\beta| > 0.$$

The optimal input

$$v(t) = \begin{cases} k & \text{for } 0 \le t \le t_1 \\ -k & \text{for } t_1 < t \le T \end{cases}$$

changes its sign one time as illustrated in Figure 2. Hence

$$\underline{z}(t) = \begin{pmatrix} \frac{1}{2k} z_2^2(t) + \alpha - \frac{1}{2k}\beta^2 \\ kt + \beta \end{pmatrix} \text{ for } 0 \le t \le t_1,$$

and on the switching-curve AOB, i.e. when $t = t_1$, we have similarly to Case 2:

$$t_1 = \frac{-\beta + \sqrt{-k\alpha + \frac{1}{2}\beta^2}}{k} \quad \Rightarrow \quad T = t_1 + \frac{|z_2(t_1)|}{k} = \frac{-\beta + 2\sqrt{-k\alpha + \frac{1}{2}\beta^2}}{k}$$

which is consistent with (16). ■

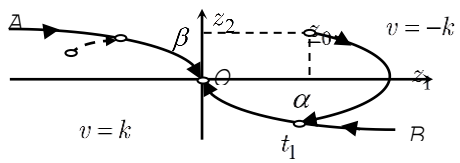

Figure 2. For the proof of Lemma 1.

Based on Lemma 1, we can prove the main result of this paper.

Theorem: *The state-feedback controller* (14) *stabilizes the nonlinear non-autonomous system* (5) *from any initial state* $\underline{x}_0$ *to the origin in finite time* T *determined in* (16).

Proof: Since the state-feedback controller (12) stabilizes the LTI system (11) in finite time T, we have $z_1(T) = z_1(T) = 0$. Substituting back to (8) we obtain:

$$\underline{x}(T)=\begin{pmatrix} z_1(T) \\ z_2(T)-f_1\big(z_1(T),T\big) \end{pmatrix}=\begin{pmatrix} 0 \\ f_1(0,T) \end{pmatrix}.$$

Hence, with $f_1(0,t)=0,\ \forall t$ we come finally to $\underline{x}(T)=\underline{0}$. ■

Remark 1: The formula (16) shows that the stabilizing time T for non-autonomous nonlinear systems can be made smaller by increasing k. ■

Remark 2: Consider a more general class of non-autonomous, nonlinear systems given as

$$\begin{cases} \dot{x}_1 = f_1(x_1,t)+g(x_2,t) \\ \dot{x}_2 = f_2(\underline{x},t)+h(u) \end{cases} \tag{17}$$

where $f_1(x_1,t)$ is smooth with $f_1(0,t)=0,\ \forall t$, $g(x_2,t)$ is smooth and invertible in an open region containing the origin,

$$D_g(x_2,t)=\frac{\partial g(x_2,t)}{\partial x_2}$$

satisfies $D_g \neq 0,\ \forall x_2,t$ and $h(u)$ is invertible. Under these assumptions, the corresponding transformation $\underline{z}=\underline{m}(\underline{x},t)$ for system (17) is defined by

$$\underline{z}=\underline{m}(\underline{x},t)=\begin{pmatrix} x_1 \\ f_1(x_1,t)+g(x_2,t) \end{pmatrix}$$

is invertible with its inverse

$$\underline{x}=\underline{m}^{-1}(\underline{z},t)=\begin{pmatrix} z_1 \\ g^{-1}\big(z_2-f_1(z_1(t),t)\big) \end{pmatrix}.$$

Then the following controller

$$h(u)=\frac{v-r(\underline{x},t)}{D_g(x_2,t)} \quad \text{with} \quad r(\underline{x},t)=\frac{\partial f_1}{\partial t}+\frac{\partial f_1}{\partial x_1}\big[f_1(x_1,t)+g(x_2,t)\big]+\frac{\partial g}{\partial t}+\frac{\partial g}{\partial x_2}f_2(\underline{x},t) \tag{18}$$

will linearize exactly the nonlinear non-autonomous system (17). The time optimal controller is designed as:

$$h(u)=\begin{cases} -\dfrac{k\,\mathrm{sgn}\,\varphi_x(\underline{x})+r(\underline{x},t)}{D_g(x_2,t)} & \text{if } \varphi_x(\underline{x})\neq 0 \\ 0 \quad \text{if } x_1=x_2=0 & \\ \dfrac{k\,\mathrm{sgn}\,\varphi_x(\underline{x})-r(\underline{x},t)}{D_g(x_2,t)} & \text{if } \varphi_x(\underline{x})=0,\ x_1\neq 0 \end{cases} \tag{19}$$

which stabilizes (17) in finite time, where

$$\varphi_x(\underline{x},t)=x_1+\frac{1}{2k}\big(f_1(x_1,t)+g(x_2,t)\big)\big|f_1(x_1,t)+g(x_2,t)\big|.$$ ■

3. SIMULATION RESULTS

3.1 Simulation 1

To illustrate the proposed FTS controller design method, consider the following system:

$$\begin{cases} \dot{x}_1 = t^2 x_1^3 + x_2 \\ \dot{x}_2 = e^t x_1^2 x_2 + u \end{cases} \tag{20}$$

Comparing with system (5), we have

$$\underline{f}(\underline{x},t) = \begin{pmatrix} t^2 x_1^3 + x_2 \\ e^t x_1^2 x_2 \end{pmatrix} \text{ and } \underline{h}(\underline{x}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

It is clearly, that this non-autonomous nonlinear can not be stabilized by applying any conventional methods related to Lyapunov's theory, because $\underline{f}(\underline{x},t)$ is not bounded in t.

The corresponding designed FTS controller for system (20) can be written as in (14) and (15) as follows:

$$\varphi_x(\underline{x}) = x_1 + \frac{1}{2k}\left(t^2 x_1^3 + x_2\right)\left|t^2 x_1^3 + x_2\right|$$
$$r(\underline{x},t) = 2t x_1^3 + 3t^2 x_1^2\left(t^2 x_1^3 + x_2\right) + e^t x_1^2 x_2$$

Figure 3 depicts the simulated states of the system with $k = 3.5$ and $\underline{x}_0 = (3\,,\,-5)^T$, obtained with simulation program **`FTS1.m`** (see Appendix). By using the formula (16), we obtain the switching time and the finite time stabilizing time of the closed-loop system as $t_1 \approx 1.8326s$ and $T \approx 2.2367s$. Thus, the simulation result exhibited in Fig. 3 is consistent with our analysis.

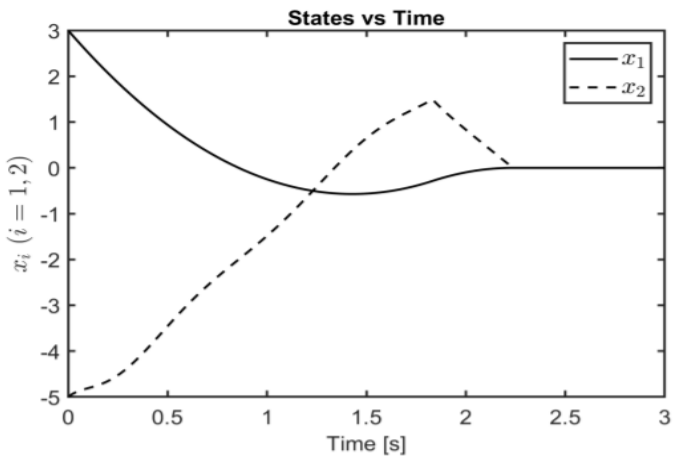

Figure 3. Simulation 1 - States x_1 (solid) and x_2 (dashed) vs time.

3.2 Simulation 2

Next, we consider the following system to illustrate the Remark 2:

$$\begin{cases} \dot{x}_1 = \exp(t^2) x_1^3 + \tan(x_2) \\ \dot{x}_2 = t\cos(t^2) x_1 x_2^2 + u \end{cases}. \tag{21}$$

For the system (21), it is observed that $g(x_2,t)=\tan(x_2)$ is invertible for $|x_2|<\pi/2$. Due the component $\exp(t^2)$ in $f_1(x_1,t)$, which increase very fast with time, it is completely unable to stabilize the system (21) asymptotically. Indeed, FTS controller should be designed to stabilize the state to the origin in finite time, so that the effect of $\exp(t^2)x_1^3$ is eliminated after the stabilizing time. The FTS controller is designed as in (19), with

$$\varphi_x(x)=x_1+\frac{1}{2k}\left(\exp(t^2)x_1^3+\tan(x_2)\right)\left|\exp(t^2)x_1^3+\tan(x_2)\right|$$

$$r(\underline{x},t)=2te^{t^2}x_1^3+3e^{t^2}x_1^2\left(e^{t^2}x_1^3+\tan(x_2)\right)+\frac{t\cos(t^2)x_1x_2^2}{\cos^2(x_2)}$$

Under the controllers designed as in (19) where k is chosen as 0.2, 1, 5, respectively, and with the initial state:

$$\underline{x}_0=(-1\,,\,1)^T\text{ , i.e. }\underline{z}_0=\underline{m}(\underline{x}_0,0)=(\alpha\,,\,\beta)^T=(-1\,,\,0.5574)^T\,,$$

we obtain the state trajectories $x_1(t)$, $x_2(t)$ of the corresponding closed-loop systems by using simulation program **`FTS2.m`** given in Appendix, which are depicted in Fig.4. The corresponding stabilizing time T is determined from (16) as $\approx 3.1741\ s$, $\approx 1.5923\ s$ and $\approx 0.7967\ s$. Thus, it can be observed from Fig. 4 that the obtained simulation results coincide completely with Theorem, Remarks 1 and 2.

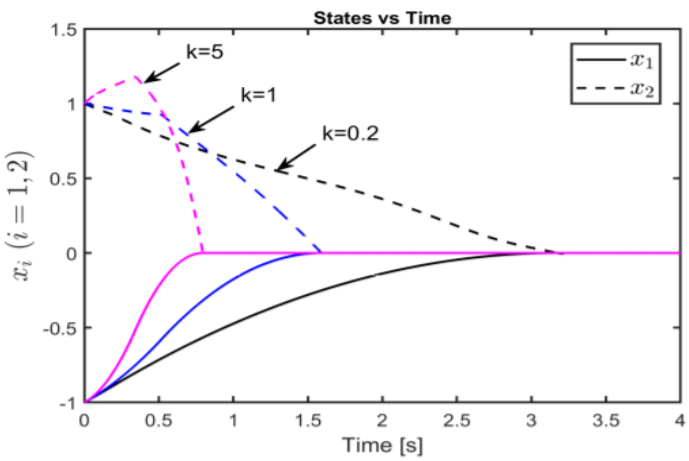

Figure 4. Trajectories of x_1 (solid) and x_2 (dashed) with different value k.

4. CONCLUSIONS

In this paper, a simple FTS controller design method based on exact linearization and Pontryagin's minimum time principle has been proposed for a class of non-autonomous, nonlinear second-order systems. The stabilizing time of the controller was also explicitly given. Our future work will be extending this FTS controller design method for nonlinear, non-autonomous, higher order systems, where the main challenge is determining the switching time of the optimal control input.

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CRediT authorship contribution statement. Author 1: Methodology, Implementation, Draft writting. Author 2: Implementation, Draft writting. Author 3: Methodology. Author 4: Methodology.

Declaration of competing interest. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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APPENDIX: SIMULATION PROGRAMS

FTS1.m

```
% FTS for non-autonomous, non-linear system
% dx(1)=f1+x(2), dx(2)=f2+u
clc
global k
k=3.5;
options = odeset('RelTol',1e-6,'AbsTol',1e-6);
[t,x]=ode45(@Sys1,[0 3],[3 -5],options);
plot(t,x(:,1),t,x(:,2)); legend('x1','x2');
```

This simulation program uses the following function named **Sys1.m** to declare simulated non-autonomous, non-linear system dynamic.

Sys1.m

```
function dx=Sys1(t,x)
    global k
    f1=t^2*x(1)^3;
    df1t=2*t*x(1)^3; df1x=3*t^2*x(1)^2;
    f2=exp(t)*x(1)^2*x(2);
    r=df1t+df1x*(f1+x(2))+f2;
    phi=x(1)+(f1+x(2))*abs(f1+x(2))/(2*k);
    if phi~=0
        v=-k*sign(phi);
    elseif x(1)~=0
        v=sign(x(1));
    else
        v=0;
    end
    u=v-r;
    dx=[f1+x(2);f2+u];
end
```

FTS2.m

```
% FTS for non-autonomous, non-linear system
clc
global k
% non-autonomous system with unbounded exponentially increasing dynamic
% dx(1)=f1+g, dx(2)=f2+u
k=1;
options = odeset('RelTol',1e-6,'AbsTol',1e-6);
[t,x]=ode45(@Sys2,[0 4],[-1 1],options);
plot(t,x(:,1),t,x(:,2)); legend('x1','x2');
```

The simulation program **FTS2.m** above uses the function named **Sys2.m** to declare simulated non-autonomous, non-linear system dynamic as follows.

Sys2.m

```
function dx=Sys2(t,x)
    global k
    f1=exp(t^2)*x(1)^3;
    df1t=2*t*exp(t^2)*x(1)^3; df1x=3*exp(t^2)*x(1)^2;
```

```
    g=tan(x(2)); dgx=1/cos(x(2))^2;
    f2=t*cos(t^2)*x(1)*x(2)^2;
    r=df1t+df1x*(f1+g)+dgx*f2;
    phi=x(1)+(f1+g)*abs(f1+g)/(2*k);
    if phi~=0
        v=-k*sign(phi);
    elseif x(1)~=0
        v=sign(x(1));
    else
        v=0;
    end
    u=(v-r)/dgx;
    dx=[f1+g;f2+u];
end
```