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# FREE VIBRATION OF BIDIRECTIONAL FUNCTIONALLY GRADED SANDWICH BEAMS PARTIALLY RESTING ON PASTERNAK FOUNDATION BASED ON A SINUSOIDAL THEORY

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**Abstract.** In this paper, free vibration of a bidirectional functionally graded sandwich (BFGSW) beams partially resting on a Pasternak foundation is studied. The beams with three layers, an axially functionally graded core and two bidirectional functionally graded face sheets, are made from a mixture of metal and ceramic. The material properties of the face sheets are considered to vary continuously in both the thickness and length directions by the power-law distributions, and they are estimated by Mori-Tanaka scheme. A sinusoidal shear deformation theory, in which the transverse displacement is split into bending and shear parts, is employed to derive energy expressions of the beam. A finite element formulation is formulated and employed to compute vibration characteristics. Numerical result reveals that the ratio of foundation support to the beam length plays an important role on the vibration behaviour, and the dependence of the frequencies upon the material grading indexes is governed by this ratio. Numerical investigation is carried out to highlight the effects of the material distribution, the layer thickness ratio, the foundation stiffness on the vibration characteristics of the beams. The influence of the aspect ratio on the frequencies of the beams and is also examined and discussed.

*Keywords:* BFSW beam, Pasternak foundation, sinusoidal theory, free vibration, finite element formulation.

*Classification numbers:* 5.4.2, 5.4.3, 5.4.5.

## 1. INTRODUCTION

Thanks to the advanced manufacturing methods [1], functionally graded materials (FGMs) initiated by Japanese scientists in mid-1980 [2] can now be incorporated into sandwich construction to improve performance of the structures. Nowadays, there is a speedy increase in the use of functionally graded sandwich (FGSW) structures in aerospace, energy, automotive, reactor industries and civil due to their high rigidity, low specific weight, excellent vibration characteristics and good fatigue properties. Many investigations on mechanical behaviour of

FGSW structures have been carried out recently, contributions that are most relevant to the present work are briefly discussed below.

Vo *et al*. [3] employed the finite element method to study free vibration and buckling of FGSW beams with power-law distribution of the material properties. The finite element formulation has been derived by the authors using the quasi-3D shear deformation theory, which taking the thickness stretching effect into account. Thai and Vo [4] considered static bending, buckling and free vibration of FGM plates using a new shear deformation theory, which ensures free shear stress at the upper and lower surfaces of the plate. In [5], Nguyen *et al*. employed Timoshenko beam theory to derive a finite element formulation for studying dynamic responses of two-directional FGM (2D-FGM) beams under a moving force. The beams [5] were assumed to be formed from four different constituent materials, two ceramics and two metals. Refined shear deformation theories were employed by Vo *et al*. in [6] to study vibration and buckling of FGM beams and FGSW beams. The influence of the layer thickness ratio, rules of material distribution on responses of the sandwich beams has been examined by the authors. A Ritz-based solution for buckling and free vibration analyses of FGSW beams with various boundary conditions was presented by Nguyen *et al*. [7] using a quasi-3D beam theory. Apetre and Sankar [8] investigated several available sandwich beam theories for their suitability in analysing sandwich plates with a functionally graded core. Amirian *et al*. [9] employed the element free Galerkin method and Galerkin formulation to investigate free vibration of sandwich beams with functionally graded core. Sakiyama *et al.* [10] studied free vibration of a sandwich beam with an elastic or viscoelastic core and arbitrary boundary conditions using the discrete Green function. Rahmani and Khalili [11] employed the high-order sandwich panel theory to study free vibration of sandwich beams with syntactic foam as a functionally graded flexible core. The state space approach was used by Trinh *et al*. [12] to investigate free vibration of FGSW beams. Karamanli [13] employed the symmetric smoothed particle hydrodynamics method to investigate bending of FGM sandwich beams with material properties varying in both the thickness and length directions. Bending behaviour of sandwich beams with a homogeneous core and two-directional FGM faces was also considered by Nguyen *et al*. [14] using a finite element formulation.

The effect of elastic foundation support on mechanical behaviour of structures has been reported by several authors. Regarding to the FGSW beams on elastic foundation, Su *et al*. [15] employed the modified Fourier series to study free vibration of FGSW beams supported by a Pasternak foundation. Chebyshev collocation method was used by Tossapanon and Wattanasakulpong [16] to solve buckling and vibration problems of FGSW beams on an elastic foundation. Zenkour *et al*. [17] studied bending behaviour of a functionally graded viscoelastic sandwich beam with elastic core resting on Pasternak elastic foundations using an analytical method. The Ritz method was employed in combination with Newmark method by Songsuwan *et al.* [18] to compute dynamic response of FGSW beams on an elastic foundation under the action of a moving harmonic load. However, it has been shown that the mechanical behaviour of structures partially supported by an elastic foundation is very different from that of the ones fully supported by the foundation. For instance, Eisenberger *et al*. [19] employed an analytical approach to show that the frequencies and mode shapes of beams partially supported by the elastic foundation significantly differ from that of the beams fully supported by the foundation. Based on a refined shear deformation theory, Le *et al*. [20] derived a finite element formulation for computing frequencies and mode shapes of FGSW plates partially resting on Pasternak foundation.

To the authors' best knowledge, the free vibration of bidirectional functionally graded sandwich (BFGSW) beams partially resting on Pasternak foundation has not been reported in the literature, and it is studied in the present work. The beams consist of three layers, a unidirectional FGM core and two skin layers of bidirectional FGM. The material properties of the skin layers are considered to vary continuously in both the length and thickness directions by power-law distributions, and they are estimated by Mori-Tanaka scheme. Based on a sinusoidal shear deformation theory, a finite element formulation is derived and employed to compute the frequencies and mode shapes of the beams. It is worthy to note that addition to the partial foundation support to the BFGSW beams, the sinusoidal shear deformation theory and the Mori-Tanaka scheme used in the study are new features of the present work. Using the derived formulation, vibration characteristics are evaluated, and the effects of the material distribution, the layer thickness ratio and the foundation support on the vibration characteristics of the beams are examined and highlighted.

## 2. MATERIALS AND METHODS

A BFGSW beam with length $L$, rectangular cross section ($b \times h$), partially supported by a foundation as depicted in Figure 1 is considered. The beam consists of three layers, an FGM core and two bidirectional FGM skin layers. The foundation is modelled herein as the Pasternak foundation, which is represented by Winkler springs of stiffness $k_W$ and a shear layer with stiffness $k_G$. The beam is partially supported by the foundation from the left end as shown in Figure 1, where $\alpha_F$ is the ratio of the supported part $L_F$ to the total beam length $L$. The Cartesian system ($x$, $z$) in Figure 1 is chosen such that the $x$-axis is on the mid-plane, while the $z$-axis directs upward. Denoting $z_0$, $z_1$, $z_2$ and $z_3$ are, respectively, the vertical coordinates of the bottom surface, two interfaces between the layers, and the top surface.

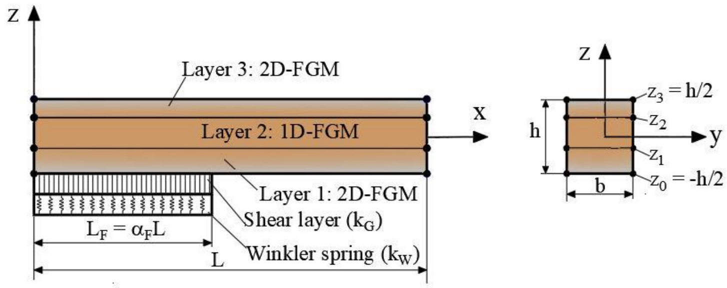


*Figure 1.* A BFGSW beam partially supported by a Pasternak foundation.

The beam is assumed to be formed from a mixture of ceramic and metal whose volume fraction varies according to [13]

$$\begin{cases} V_c = \left(\dfrac{z - z_0}{z_1 - z_0}\right)^{n_z} \left(1 - \dfrac{x}{2L}\right)^{n_x}, & \text{for } z \in [z_0, z_1] \\ V_c = \left(1 - \dfrac{x}{2L}\right)^{n_x}, & \text{for } z \in [z_1, z_2] \\ V_c = \left(\dfrac{z - z_3}{z_2 - z_3}\right)^{n_z} \left(1 - \dfrac{x}{2L}\right)^{n_x}, & \text{for } z \in [z_2, z_3] \end{cases} \tag{1}$$

and $V_m = 1 - V_c$. In Eq. (1), $n_x$ and $n_z$ are the power-law indexes which determine the material distribution through the thickness and length directions, respectively. The beam becomes homogeneous if $n_x = n_z = 0$. Figure 1 shows the distribution of $V_c$ and $V_m$ in the thickness and length direction for two pairs of the power-law indexes, $n_x = n_z = 0.5$ and $n_x = n_z = 5$, and $z_1 = -h/4$, $z_2 = h/4$.

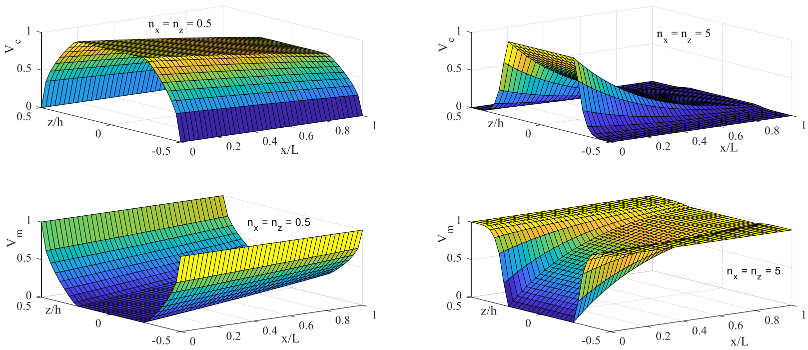


*Figure 2*. The distribution of $V_c$ and $V_m$ of the BFGSW beam for $z_1$ = -h/4 and $z_2$ = h/4.

Mori-Tanaka scheme [21] is employed herewith to evaluate the effective properties of the FGM layers. According to Mori-Tanaka scheme, the effective bulk modulus $K_f$ and shear modulus $G_f$ of the three layers of the beam are given by

$$\frac{K_f - K_m}{K_c - K_m} = \frac{V_c}{1+(1-V_c)(K_c-K_m)/(K_m+4G_m/3)}$$
$$\frac{G_f - G_m}{G_c - G_m} = \frac{V_c}{1+\{(1-V_c)(G_c-G_m)\}/\{G_m+G_m(9K_m+8G_m)/(6K_m+12G_m)\}} \tag{2}$$

where

$$K_c = \frac{E_c}{3(1-2\upsilon_c)},\ G_c = \frac{E_c}{2(1+\upsilon_c)},\ K_m = \frac{E_m}{3(1-2\upsilon_m)},\ G_m = \frac{E_m}{2(1+\upsilon_m)} \tag{3}$$

are the bulk and the shear moduli of the ceramic and metal at each point of the beam, respectively.

Noting that the effective mass density $\rho_f$ is defined by Voigt model as

$$\rho_f = (\rho_c - \rho_m)V_c + \rho_m \tag{4}$$

The effective Young's modulus $E_f$ and Poisson's ratio $\nu_f$ are computed via effective bulk modulus and shear modulus as

$$E_f = \frac{9K_f G_f}{3K_f + G_f},\ \upsilon_f = \frac{3K_f - 2G_f}{6K_f + 2G_f} \tag{5}$$

## 3. MATHEMATICAL MODEL

Based on the sinusoidal shear deformation theory [4], the transverse displacement is split into bending and shear parts, $w_b$ and $w_s$, and the displacements of a point in $x$ and $z$ directions, $u(x;z;t)$ and $w(x;t)$, respectively, are given by

$$u(x,z,t) = u_0(x,t) - zw_{b,x}(x,t) - \left[z - \frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right)\right]w_{s,x}(x,t)$$
$$w(x,t) = w_b(x,t) + w_s(x,t) \tag{6}$$

where $u_0(x,t)$, $w_b(x,t)$ and $w_s(x,t)$ are, respectively, the in-plane displacement in $x$-directions, bending and shear components of the transverse displacement of points on the neutral axis of the beam. In the above equation and hereafter, a subscript comma is used to denote the derivative with respect to the followed variable, e.g. $w_{b,x} = \partial w_b / \partial x$.

The strains resulted from Eq. are of the forms

$$\varepsilon_x = u_{0,x} - z w_{b,xx} - f(z) w_{s,xx},\ \gamma_{xz} = g(z) w_{s,x} \tag{7}$$

where $f(z) = z - \frac{h}{\pi}\sin\frac{\pi z}{h}$, $g(z) = 1 - f_{,z}(z)$.

The constitutive equations based on linear behaviour of the beam material are of the forms

$$\begin{Bmatrix} \sigma_x \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} E_f(x,z) & 0 \\ 0 & G_f(x,z) \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \gamma_{xz} \end{Bmatrix} \tag{8}$$

The finite element method is used herein to handle the variation of the beam rigidities along the beam length. To this end, the beam is assumed to be divided into a number of elements with length of $l$. Eqs. and give the strain energy for the element due to the beam deformation, $U_e^{B}$, in the form

$$\begin{aligned} U_e^{\mathrm{B}} &= \frac{1}{2}\int_{V_e} \begin{Bmatrix} \sigma_x \\ \tau_{xz} \end{Bmatrix}^T \begin{Bmatrix} \varepsilon_x \\ \gamma_{xz} \end{Bmatrix} dV = \frac{1}{2}\int_{V_e} \begin{Bmatrix} \varepsilon_x \\ \gamma_{xz} \end{Bmatrix}^T \begin{bmatrix} E_f(x,z) & 0 \\ 0 & G_f(x,z) \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \gamma_{xz} \end{Bmatrix} dV \\ &= \frac{1}{2} b \int_0^l \left\{ \begin{bmatrix} u_{0,x} \\ w_{b,xx} \\ w_{s,xx} \end{bmatrix}^T \begin{bmatrix} I_1 & I_2 & I_3 \\ I_2 & I_5 & I_4 \\ I_3 & I_4 & I_6 \end{bmatrix} \begin{bmatrix} u_{0,x} \\ w_{b,xx} \\ w_{s,xx} \end{bmatrix} + w_{s,x} I_7 w_{s,x} \right\} dx \end{aligned} \tag{9}$$

where $V_e$ is the volume of the element; $I_1, I_2, \dots I_6, I_7$ are the beam rigidities, defined as

$$\begin{aligned} \left(I_1, I_2, I_3, I_4, I_5, I_6\right) &= b\int_{-h/2}^{h/2} E_f(x,z) \left\{1, z, f(z), zf(z), z^2, f^2(z)\right\} dz \\ I_7 &= b\int_{-h/2}^{h/2} G_f(x,z) g^2(z) dz \end{aligned} \tag{10}$$

In Eq. and hereafter a superscript "$T$" denotes the transpose of a vector or a matrix.

The element strain energy resulted from the foundation deformation is of the form

$$\begin{aligned} U_e^{\mathrm{F}} &= \frac{1}{2} b \int_0^l \left( k_W w^2 + k_G w_{,x}^2 \right) dx = \frac{1}{2} b \int_0^l \left[ k_W \left(w_b + w_s\right)^2 + k_G \left(w_{b,x} + w_{s,x}\right)^2 \right] dx \\ &= \frac{1}{2} b \int_0^l \left\{ \begin{bmatrix} w_b \\ w_s \end{bmatrix}^T \begin{bmatrix} k_W & k_W \\ k_W & k_W \end{bmatrix} \begin{bmatrix} w_b \\ w_s \end{bmatrix} + \begin{bmatrix} w_{b,x} \\ w_{s,x} \end{bmatrix}^T \begin{bmatrix} k_G & k_G \\ k_G & k_G \end{bmatrix} \begin{bmatrix} w_{b,x} \\ w_{s,x} \end{bmatrix} \right\} dx \end{aligned} \tag{3}$$

The element kinetic energy resulted from Eq. is of the form

$$\begin{aligned} T_e &= \frac{1}{2}\int_{V_e} \rho_f \left( \dot{u}^2 + \dot{w}^2 \right) dV \\ &= \frac{1}{2}\int_{V_e} \rho_f \left[ \left\{ \dot{u}_0 - z\dot{w}_{b,x} - f(z)\dot{w}_{s,x} \right\}^2 + \left( \dot{w}_b^2 + \dot{w}_s^2 \right) \right] dV \\ &= \frac{1}{2} b \int_0^l \left\{ \begin{bmatrix} \dot{u}_0 \\ \dot{w}_b \\ \dot{w}_s \end{bmatrix}^T \begin{bmatrix} J_1 & J_2 & J_3 \\ J_2 & J_5 & J_4 \\ J_3 & J_4 & J_6 \end{bmatrix} \begin{bmatrix} \dot{u}_0 \\ \dot{w}_b \\ \dot{w}_s \end{bmatrix} + \begin{bmatrix} \dot{w}_b \\ \dot{w}_s \end{bmatrix}^T \begin{bmatrix} J_1 & J_1 \\ J_1 & J_1 \end{bmatrix} \begin{bmatrix} \dot{w}_b \\ \dot{w}_s \end{bmatrix} \right\} dx \end{aligned} \tag{4}$$

where an over dot denotes the derivative with respect the time variable $t$, and the mass moments

$J_1$, $J_2$, ... $J_6$ are defined as

$$\left(J_1, J_2, J_3, J_4, J_5, J_6\right) = \int_{-h/2}^{h/2} \rho_f(x,z)\left\{1,\ z,\ f(z),\ zf(z),\ z^2,\ f^2(z)\right\} dz \tag{5}$$

Noting that the rigidities $I_1$, $I_2$, ... $I_6$, $I_7$ and the mass moments $J_1$, $J_2$, ... $J_6$ as defined by Eqs.  and (5) are functions of $x$.

## 4. FINITE ELEMENT FORMULATION

A two-node beam element with five degree of freedom per node is considered herewith. Linear polynomials are used to interpolate the axial displacement $u$ from its nodal values, while Hermite cubic polynomials are employed for the transverse displacements $w_b$ and $w_s$ as

$$u_0 = \mathbf{N}\mathbf{u}^e, \quad w_b = \mathbf{H}\mathbf{w}_b^e, \quad w_s = \mathbf{H}\mathbf{w}_s^e \tag{6}$$

where

$$\mathbf{u}^e = \left\{u_{01}\ u_{02}\right\}^T, \quad \mathbf{w}_b^e = \left\{w_{b1}\ w_{b,x1}\ w_{b2}\ \ w_{b,x2}\right\}^T, \quad \mathbf{w}_s^e = \left\{w_{s1}\ w_{s,x1}\ w_{s2}\ \ w_{s,x2}\right\}^T \tag{7}$$

are, respectively, the element vectors of nodal axial, bending transverse and shear transverse displacements; $\mathbf{N} = \left\{N_1\ \ N_2\right\}$ and $\mathbf{H} = \left\{H_1\ \ H_2\ \ H_3\ \ H_4\right\}$ are, respectively, the matrices of linear and Hermite shape functions with

$$\begin{aligned} &N_1 = 1 - \frac{x}{l},\ N_2 = \frac{x}{l}, H_1 = 1 - 3\frac{x^2}{l^2} + 2\frac{x^3}{l^3},\ \ H_2 = x - 2\frac{x^2}{l} + \frac{x^3}{l^2} \\ &H_3 = 3\frac{x^2}{l^2} - 2\frac{x^3}{l^3}, \qquad H_4 = -\frac{x^2}{l} + \frac{x^3}{l^2} \end{aligned} \tag{8}$$

Using the above interpolations, one can write the element strain energy $U_e^{\mathrm{B}}$ in the forms

$$U_e^{\mathrm{B}} = \frac{1}{2}\begin{Bmatrix}\mathbf{u}^e \\ \mathbf{w}_b^e \\ \mathbf{w}_s^e\end{Bmatrix}^T \begin{bmatrix}\mathbf{k}_{uu} & \mathbf{k}_{ub} & \mathbf{k}_{us} \\ \mathbf{k}_{ub}^T & \mathbf{k}_{bb} & \mathbf{k}_{bs} \\ \mathbf{k}_{us}^T & \mathbf{k}_{bs}^T & \mathbf{k}_{ss}\end{bmatrix} \begin{Bmatrix}\mathbf{u}^e \\ \mathbf{w}_b^e \\ \mathbf{w}_s^e\end{Bmatrix} = \frac{1}{2}\mathbf{d}^T\mathbf{k}_e^{\mathrm{B}}\mathbf{d} \tag{9}$$

with

$$\underset{10\times1}{\mathbf{d}} = \left\{\mathbf{u}^e\ \ \mathbf{w}_b^e\ \ \mathbf{w}_s^e\right\}^T \tag{10}$$

is the vector of the nodal displacements for the element, and

$$\underset{10\times10}{\mathbf{k}_e^{\mathrm{B}}} = \begin{bmatrix}\mathbf{k}_{uu} & \mathbf{k}_{ub} & \mathbf{k}_{us} \\ \mathbf{k}_{ub}^T & \mathbf{k}_{bb} & \mathbf{k}_{bs} \\ \mathbf{k}_{us}^T & \mathbf{k}_{bs}^T & \mathbf{k}_{ss}\end{bmatrix} \tag{11}$$

is the element stiffness matrix; $\mathbf{k}_{uu}$, $\mathbf{k}_{bb}$ and $\mathbf{k}_{ss}$ are, respectively, the membrane, bending and shear stiffness matrices with the following forms

$$\begin{aligned} &\underset{2\times2}{\mathbf{k}_{uu}} = b\int_0^l\left(\mathbf{N}_{,x}^T I_1 \mathbf{N}_{,x}\right)dx, \underset{4\times4}{\mathbf{k}_{bb}} = b\int_0^l\left(\mathbf{H}_{,xx}^T I_5 \mathbf{H}_{,xx}\right)dx \\ &\underset{4\times4}{\mathbf{k}_{ss}} = b\int_0^l\left(\mathbf{H}_{,xx}^T I_6 \mathbf{H}_{,xx} + \mathbf{H}_{,x}^T I_7 \mathbf{H}_{,x}\right)dx \end{aligned} \tag{12}$$

and the coupling stiffness matrices $\mathbf{k}_{ub}$, $\mathbf{k}_{us}$ and $\mathbf{k}_{bs}$ have the forms

$$\underset{2\times4}{\mathbf{k}_{ub}} = b\int_0^l \left(\mathbf{N}_{,x}^T I_2 \mathbf{H}_{,xx}\right) dx,\ \underset{2\times4}{\mathbf{k}_{us}} = b\int_0^l \left(\mathbf{N}_{,x}^T I_3 \mathbf{H}_{,xx}\right) dx,\ \underset{4\times4}{\mathbf{k}_{bs}} = b\int_0^l \left(\mathbf{H}_{,xx}^T I_4 \mathbf{H}_{,xx}\right) dx \tag{13}$$

The strain energy $U_e^{\mathrm{F}}$ given by equation (3) can now be written in the form

$$U_e^{\mathrm{F}} = \frac{1}{2}\mathbf{d}^T \mathbf{k}_e^{\mathrm{F}} \mathbf{d} \tag{14}$$

where the element foundation stiffness $\mathbf{k}_e^{\mathrm{F}}$ has the form

$$\underset{10\times10}{\mathbf{k}_e^{\mathrm{F}}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathbf{k}_{bb}^{\mathrm{F}} & \mathbf{k}_{bs}^{\mathrm{F}} \\ 0 & \left(\mathbf{k}_{bs}^{\mathrm{F}}\right)^T & \mathbf{k}_{ss}^{\mathrm{F}} \end{bmatrix} \tag{15}$$

with

$$\underset{4\times4}{\mathbf{k}_{bb}^{\mathrm{F}}} = \mathbf{k}_{bs}^{\mathrm{F}} = \mathbf{k}_{ss}^{\mathrm{F}} = b\int_0^l \left(\mathbf{H}^T k_W \mathbf{H} + \mathbf{H}_{,x}^T k_G \mathbf{H}_{,x}\right) dx \tag{16}$$

The kinetic energy $T_e$ in Eq. (4) can also be written in the form

$$T_e = \frac{1}{2}\dot{\mathbf{d}}^T \mathbf{m}_e \dot{\mathbf{d}} = \frac{1}{2}\dot{\mathbf{d}}^T \begin{bmatrix} \mathbf{m}_{uu} & \mathbf{m}_{ub} & \mathbf{m}_{us} \\ \mathbf{m}_{ub}^T & \mathbf{m}_{bb} & \mathbf{m}_{bs} \\ \mathbf{m}_{us}^T & \mathbf{m}_{bs}^T & \mathbf{m}_{ss} \end{bmatrix} \dot{\mathbf{d}} \tag{17}$$

where $\mathbf{m}_e$ is the element mass matrix formed from the following sub-matrices

$$\begin{aligned} &\underset{2\times4}{\mathbf{m}_{ub}} = b\int_0^l \left(\mathbf{N}^T J_2 \mathbf{H}_{,x}\right) dx,\ \underset{2\times4}{\mathbf{m}_{us}} = b\int_0^l \left(\mathbf{N}^T J_3 \mathbf{H}_{,x}\right) dx \\ &\underset{4\times4}{\mathbf{m}_{bs}} = b\int_0^l \left(\mathbf{H}^T J_1 \mathbf{H} + \mathbf{H}_{,x}^T J_4 \mathbf{H}_{,x}\right) dx \end{aligned} \tag{18}$$

Equations of motion for the beam in the context of finite element analysis can be obtained from the Hamilton's principle

$$\delta\int_{t_1}^{t_2} \left[ \sum^{n_{EB}} \left(U_e^{\mathrm{B}} - T_e\right) + \sum^{n_{EF}} U_e^{\mathrm{F}} \right] = 0 \tag{19}$$

where $n_{EB}$ and $n_{EF}$ are, respectively, the total number of elements used for the beam and foundation.

Assuming a harmonic form for the vector of nodal displacements, the Hamilton's principle leads to discrete equation of motion in the form

$$\left([\mathbf{K}] - \omega^2 [\mathbf{M}]\right)\bar{\mathbf{D}} = 0 \tag{20}$$

where $\mathbf{M}$, and $\mathbf{K}$ are, respectively, the global mass matrix and stiffness matrix; $\omega$ and $\bar{\mathbf{D}}$ are, respectively, the frequency and the eigenvector of the nodal displacements corresponding to an eigenvalue.

## 5. NUMERICAL INVESTIGATION

Numerical investigation is carried out in this section to study the effect of various parameters on the vibration of the BFGSW beam partially resting on a Pasternak foundation. Otherwise stated, a BFGSW beam formed from aluminium (Al) and alumina ($Al_2O_3$) with the following properties [16]:

- For Aluminium: $E_m = 70$ GPa, $\rho_m = 2702$ kg/m$^3$, $\nu_m = 0.3$;
- For Alumina: $E_c = 380$ GPa, $\rho_c = 3960$ kg/m$^3$, $\nu_c = 0.3$.

The following non-dimensional parameters are used for the frequencies and foundation stiffness [16]

$$\mu_i = \omega_i L\sqrt{\frac{I_{00}}{A_{100}}},\ K_0 = \frac{k_W L^2}{A_{110}},\ K_1 = \frac{k_G}{A_{110}} \tag{21}$$

where $\omega_i$ is the $i$th natural frequency, $A_{110} = E_m h$ and $I_{00} = \rho_m h$.

Three number in brackets are used herein to denote the layer thickness ratio, e.g. (1-2-1) means that the thickness ratio of the bottom layer, the core layer and the top layer is 1:2:1. Four types of boundary conditions, namely simply-supported (SS), clamped-clamped (CC), clamped-free (CF) and clamped-simply supported (CS) are considered herein.

Before computing the vibration characteristics of the beam, the accuracy and convergence of the derived formulation are necessary to verify. Since the data for the BFGSW beam partially supported by the elastic foundation are not available in the literature, the accuracy of the derived formulation is verified herewith by comparing the fundamental frequency parameters of a unidirectional FGSW beam obtained in the present work with the published data. To this end, Table 1 compares the fundamental frequency parameters of the unidirectional FGSW beam with $L/h$ = 10 fully supported on a Pasternak foundation of the present work with the result of Ref. [16]. Noting that the unidirectional FGSW beam model in Ref. [16] can be obtained from the present beam just by setting $n_x = 0$. Table 1 shows a good agreement between the frequencies of the present work with the Chebyshev collocation method based result of Ref. [16], regardless of the boundary conditions and the foundation stiffness.

The convergence of the derived formulation in evaluating the fundamental frequency parameter of the 2D-FGSW beam partially supported by the foundation is shown in Table 2 for the SS and CF beam with $\alpha_F = 0.5$, $K_0 = K_1 = 0.2$ and various values of the grading indexes and aspect ratios. The convergence, as seen from the table, is achieved by using 24 elements, regardless of the boundary conditions and the grading indexes. Because of this convergence, a mesh of 24 elements is used in all the computations reported below.

*Table 1.* Verification study for dimensionless fundamental frequency ($\bar{\omega}$) of 1D-FGSW beam (with $n_x = 0$, $n_z = 0.5$ & $L/h = 10$) fully supported by Pasternak foundation.

| $K_0, K_1$ | Source | (1-0-1) | | | (1-1-1) | | | (2-1-2) | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | SS | CC | CS | SS | CC | CS | SS | CC | CS |
| 0, 0 | Ref. [16] | 0.3530 | 0.7738 | 0.5431 | 0.3845 | 0.8434 | 0.5917 | 0.3687 | 0.8091 | 0.5676 |
| | Present | 0.3535 | 0.7792 | 0.5467 | 0.3849 | 0.8483 | 0.5952 | 0.3692 | 0.8145 | 0.5711 |
| 0.2, 0 | Ref. [16] | 0.5255 | 0.8661 | 0.6681 | 0.5418 | 0.9257 | 0.7041 | 0.5329 | 0.8959 | 0.6856 |
| | Present | 0.5258 | 0.8709 | 0.6710 | 0.5421 | 0.9301 | 0.7071 | 0.5332 | 0.9007 | 0.6885 |
| 0.2, 0.2 | Ref. [16] | 1.3310 | 1.5657 | 1.4387 | 1.3161 | 1.5837 | 1.4387 | 1.3209 | 1.5728 | 1.4368 |
| | Present | 1.3310 | 1.5752 | 1.4451 | 1.3162 | 1.5919 | 1.4439 | 1.3210 | 1.5819 | 1.4421 |

*Table 2.* Convergence of the derived formulation in evaluating frequency parameter $\mu_1$ of BFGSW beams partially supported on a Pasternak foundation with $\alpha_F$ = 0.5 and $K_0 = K_1$ = 0.2.

| $n_x$ | $n_z$ | $n_{EB}$ | SS, $L/h$ = 5 | | | CF, $L/h$ = 20 | | |
|---|---|---|---|---|---|---|---|---|
| | | | (2-1-2) | (2-1-1) | (2-2-1) | (2-1-2) | (2-1-1) | (2-2-1) |
| 0.5 | 0.5 | 10 | 1.0936 | 1.1005 | 1.1118 | 0.2038 | 0.2071 | 0.2125 |
| | | 12 | 1.0936 | 1.1005 | 1.1118 | 0.2036 | 0.2069 | 0.2123 |
| | | 14 | 1.0936 | 1.1005 | 1.1118 | 0.2035 | 0.2068 | 0.2121 |
| | | 16 | 1.0936 | 1.1005 | 1.1118 | 0.2034 | 0.2067 | 0.2120 |
| | | 18 | 1.0936 | 1.1005 | 1.1118 | 0.2033 | 0.2066 | 0.2120 |
| | | 20 | 1.0936 | 1.1005 | 1.1118 | 0.2033 | 0.2066 | 0.2119 |
| | | 22 | 1.0936 | 1.1005 | 1.1118 | 0.2033 | 0.2065 | 0.2119 |
| | | 24 | 1.0936 | 1.1005 | 1.1118 | 0.2033 | 0.2065 | 0.2119 |
| 2 | 2 | 10 | 1.0593 | 1.0614 | 1.0607 | 0.1767 | 0.1795 | 0.1830 |
| | | 12 | 1.0593 | 1.0614 | 1.0606 | 0.1764 | 0.1792 | 0.1827 |
| | | 14 | 1.0592 | 1.0613 | 1.0606 | 0.1762 | 0.1790 | 0.1825 |
| | | 16 | 1.0592 | 1.0613 | 1.0606 | 0.1761 | 0.1788 | 0.1824 |
| | | 18 | 1.0592 | 1.0613 | 1.0606 | 0.1760 | 0.1787 | 0.1823 |
| | | 20 | 1.0592 | 1.0613 | 1.0606 | 0.1759 | 0.1787 | 0.1822 |
| | | 22 | 1.0592 | 1.0613 | 1.0606 | 0.1758 | 0.1786 | 0.1822 |
| | | 24 | 1.0592 | 1.0613 | 1.0606 | 0.1758 | 0.1786 | 0.1822 |

*Table 3*. Frequency parameter ( $\mu_1$ ) of SS beam partially supported by the foundation with $K_0 = K_1$ = 0.2.

| $\alpha_F$ | $n_x$ | $n_z$ | $L/h=5$ | | | | $L/h=20$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | (1-0-1) | (2-1-2) | (2-1-1) | (2-2-1) | (1-0-1) | (2-1-2) | (2-1-1) | (2-2-1) |
| 0.2 | 0.5 | 0.5 | 0.8995 | 0.9153 | 0.9257 | 0.9438 | 0.3583 | 0.3675 | 0.3738 | 0.3841 |
| | | 1 | 0.8527 | 0.8685 | 0.8814 | 0.9026 | 0.3293 | 0.3388 | 0.3472 | 0.3600 |
| | | 1.5 | 0.8133 | 0.8178 | 0.8330 | 0.8502 | 0.3038 | 0.3049 | 0.3153 | 0.3266 |
| | 1 | 0.5 | 0.8802 | 0.8912 | 0.8987 | 0.9113 | 0.3497 | 0.3567 | 0.3616 | 0.3695 |
| | | 1 | 0.8440 | 0.8552 | 0.8647 | 0.8795 | 0.3258 | 0.3332 | 0.3398 | 0.3495 |
| | | 1.5 | 0.8133 | 0.8160 | 0.8275 | 0.8387 | 0.3046 | 0.3052 | 0.3135 | 0.3218 |
| | 5 | 0.5 | 0.8299 | 0.8317 | 0.8330 | 0.8349 | 0.3244 | 0.3266 | 0.3281 | 0.3305 |
| | | 1 | 0.8225 | 0.8246 | 0.8263 | 0.8286 | 0.3162 | 0.3186 | 0.3206 | 0.3236 |
| | | 1.5 | 0.8149 | 0.8164 | 0.8185 | 0.8204 | 0.3078 | 0.3086 | 0.3112 | 0.3138 |
| 0.5 | 0.5 | 0.5 | 1.0861 | 1.0935 | 1.1004 | 1.1117 | 0.7034 | 0.7092 | 0.7143 | 0.7211 |
| | | 1 | 1.0639 | 1.0670 | 1.0745 | 1.0853 | 0.6781 | 0.6850 | 0.6929 | 0.7026 |
| | | 1.5 | 1.0683 | 1.0531 | 1.0589 | 1.0602 | 0.6564 | 0.6529 | 0.6649 | 0.6737 |
| | 1 | 0.5 | 1.0772 | 1.0811 | 1.0858 | 1.0926 | 0.6945 | 0.6986 | 0.7026 | 0.7076 |
| | | 1 | 1.0627 | 1.0629 | 1.0679 | 1.0740 | 0.6745 | 0.6790 | 0.6850 | 0.6919 |
| | | 1.5 | 1.0705 | 1.0559 | 1.0595 | 1.0573 | 0.6582 | 0.6545 | 0.6632 | 0.6683 |
| | 5 | 0.5 | 1.0724 | 1.0713 | 1.0716 | 1.0713 | 0.6714 | 0.6721 | 0.6728 | 0.6737 |
| | | 1 | 1.0735 | 1.0711 | 1.0713 | 1.0703 | 0.6676 | 0.6682 | 0.6692 | 0.6702 |
| | | 1.5 | 1.0819 | 1.0755 | 1.0752 | 1.0716 | 0.6648 | 0.6640 | 0.6652 | 0.6657 |

The fundamental frequency parameters of the SS beam are typically given in Table 3 for different layer thickness ratios and the material grading indexes. The layer thickness ratio and the foundation supporting parameter $\alpha_F$ have significant influence on the frequency, and the frequency parameter is higher for the beam associated with a larger core thickness and a higher parameter $\alpha_F$. For $\alpha_F = 0.2$, the parameter $\mu_1$ decreases with the increase of the indexes $n_x$ and $n_z$, regardless of the boundary condition and the layer thickness ratio. The decrease of the frequency parameter by the increase of the material grading indexes can be explained by the decease of the ceramic content, as seen from Eq. (1). The decrease of $\mu_1$ by increasing $n_x$ and $n_z$, however is altered for the foundation supporting $\alpha_F$ = 0.5, and the parameter $\mu_1$ is not always increased by the increase of $n_x$.

In order to examine the effect of the supporting parameter $\alpha_F$ on the relation between frequency parameter $\mu_1$ with the indexes $n_x$ and $n_z$ in more detail, Figs. 3-5 respectively show the variation of $\mu_1$ with $n_x$ and $n_z$ of (2-1-2) SS, CC and CF beams with $L/h$ = 20 for various values of $\alpha_F$. As seen from the figures, the supporting parameter $\alpha_F$ has a significant influence on the variation of $\mu_1$ with $n_x$ and $n_z$. For $\alpha_F$ = 0 and 0.3, Fig. 3 shows that the parameter $\mu_1$ decreases with the increase of the indexes $n_x$ and $n_z$, but this tendency is not correct for $\alpha_F$ = 0.6 and 0.9. For the two larger values of $\alpha_F$, the frequency parameter is firstly decreased with the increase of $n_x$ and $n_z$, but it then increases by the increase of these two indexes. The situation is similar for the CC and CF beams, as seen from Fig. 4 and Fig. 5.

The dependence of the higher frequency parameters of the 2D-FGSW beams on the material grading indexes is shown in Figs. 6, 7, where the variation of the first four frequency parameters of the (2-1-2) SS and CC beams with the indexes $n_x$ and $n_z$ is respectively depicted for $\alpha_F$ = 0.5 and $K_0$ = $K_1$ = 0.2. The variation of the higher frequency parameters with the grading indexes is similar to that of the first frequency parameter, regardless of the boundary conditions. Among the two types of the boundary conditions, the frequency parameters of the CC beam, as expected, are higher, while that of SS beam are smaller.

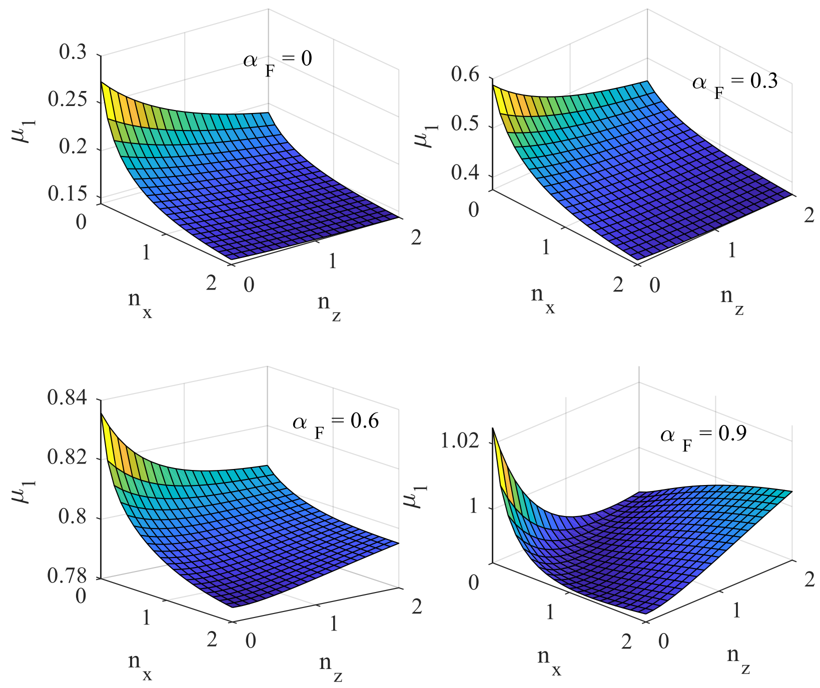


*Figure 3*. Variation of frequency parameter $\mu_1$ of (2-1-2) SS beam with grading indexes $n_x$ and $n_z$ for $L/h$ = 20, $K_0$ = $K_1$ = 0.2 and different $\alpha_F$.

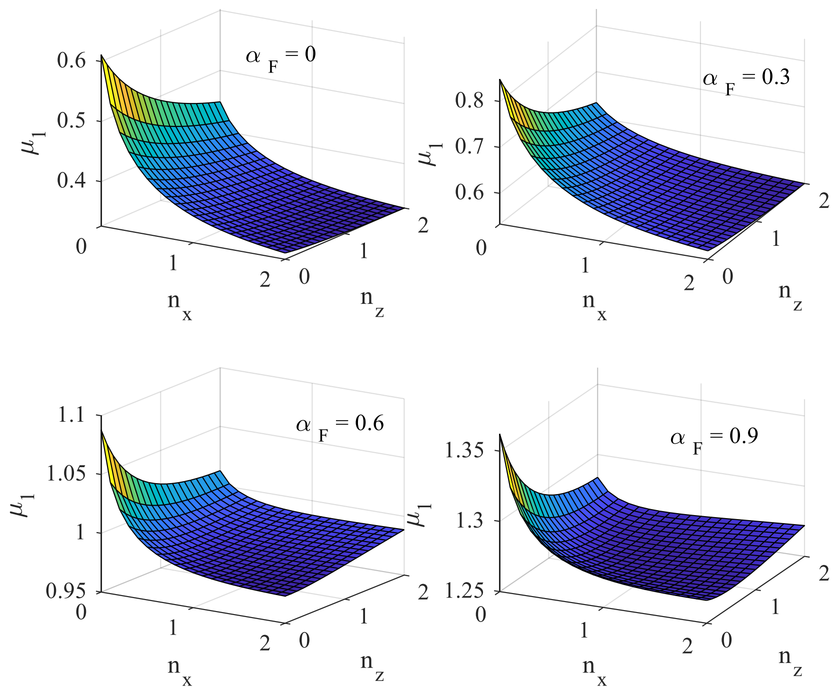


*Figure 4*. Variation of frequency parameter $\mu_1$ of (2-1-2) CC beam with grading indexes $n_x$ and $n_z$ for $L/h$ = 20, $K_0 = K_1$ = 0.2 and different $\alpha_F$.

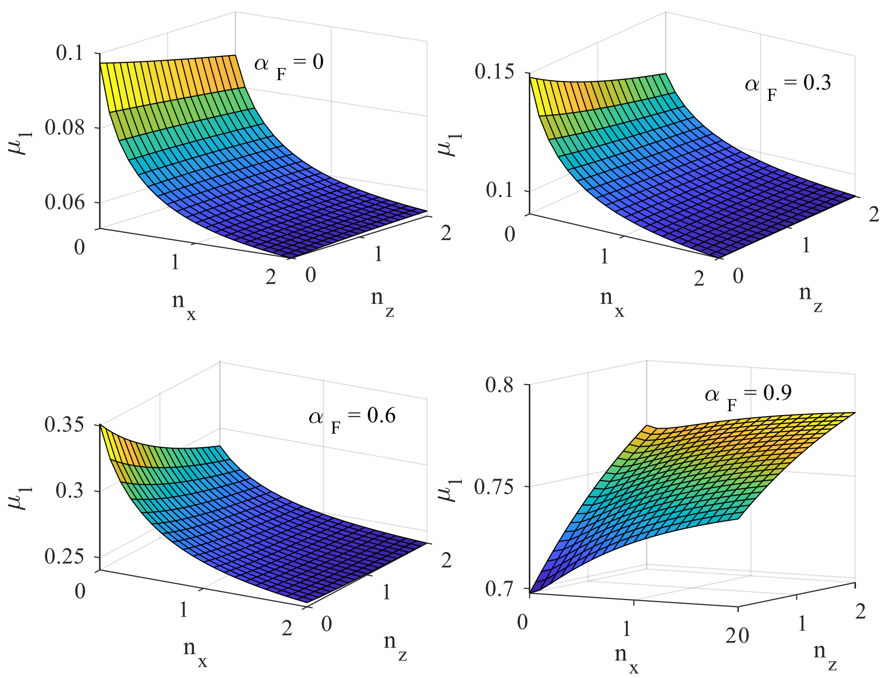


*Figure 5*. Variation of frequency parameter $\mu_1$ of (2-1-2) CF beam with grading indexes $n_x$ and $n_z$ for $L/h$ = 20, $K_0 = K_1$ = 0.2 and different $\alpha_F$.

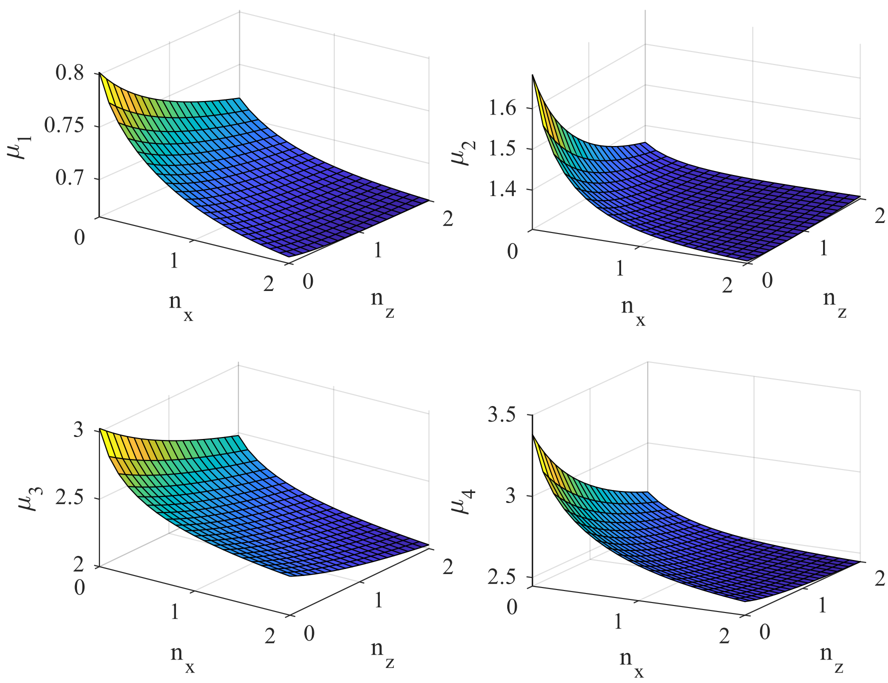

*Figure 6*. Variation of the first four frequency parameters of (2-1-2) SS beam with grading indexes $n_x$ and $n_z$ for $L/h$ = 20, $\alpha_F$ = 0.5 and $K_0 = K_1$ = 0.2.

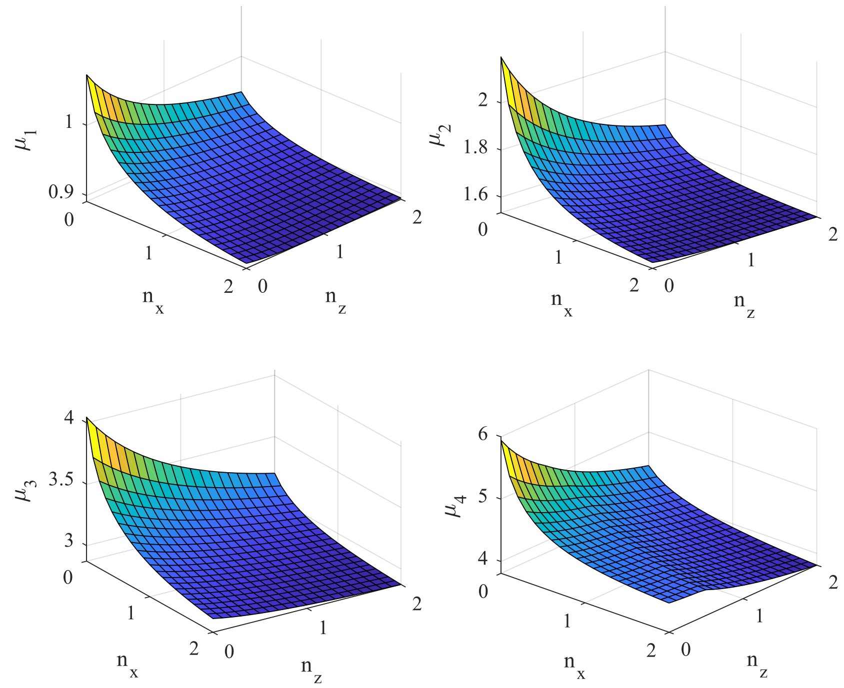

*Figure 7*. Variation of the first four frequency parameters of (2-1-2) CC beam with grading indexes $n_x$ and $n_z$ for $L/h$ = 20, $\alpha_F$ = 0.5 and $K_0 = K_1$ = 0.2.

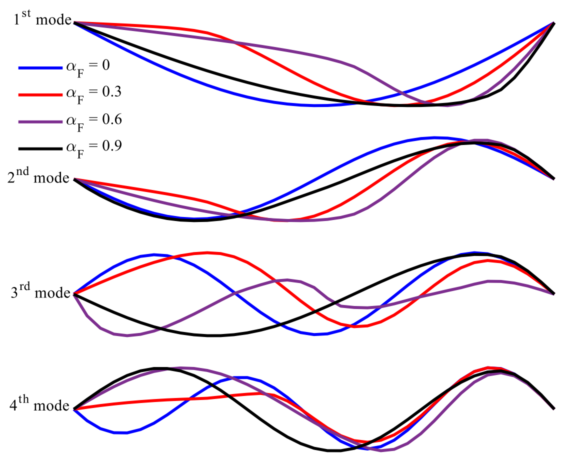


*Figure 8*. First four mode shapes for transverse displacements of a SS (2-1-2) beam for $L/h = 20$, $n_x = n_z = 0.5$, $K_0 = K_1 = 0.2$ and different foundation supporting parameter $\alpha_F$.

The first four mode shapes for the transverse displacements of the SS (2-1-2) beam are illustrated in Fig. 8 for $L/h = 20$, $n_x = n_z = 0.5$, $K_0 = K_1 = 0.2$ and different foundation supporting parameter $\alpha_F$. The effect of the partial support by the foundation is clearly seen from the figure, where the supporting part of the beam tends to stick on the foundation.

## 6. CONCLUSIONS

Free vibration of BFGSW beams partially supported on a Pasternak foundation has been investigated using a finite element formulation. The sandwich beams with an axially functionally graded core and two bidirectional functionally graded face sheets is assumed to be made from a mixture of ceramic and metal. The material properties of the face sheets are considered to vary in both the thickness and length directions by the power-law distributions, and they are estimated by Mori-Tanaka scheme. Based on a sinusoidal shear deformation, equation of motion in term of the finite element analysis has been derived, and the vibration characteristics have been evaluated for the beams with various boundary conditions. The comparison study confirmed the accuracy of the derived finite element formulation. The numerical results have revealed that, the foundation supporting parameter defined as the ratio of the supporting part to the beam length plays an important role on the vibration behaviour of the beams. The dependence of the frequency upon the material grading indexes is founded to be governed by the foundation supporting parameter. A parametric study has been carried out to show the dependence of the frequency upon the material grading indexes, the layer thickness ratio, the value of stiffness foundation, the length of supported foundation and the side-to-thickness ratio of the BFGSW beam.

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