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THE REAL-WORLD-SEMANTICS INTERPRETABILITY OF LINGUISTIC RULE BASES AND THE APPROXIMATE REASONING METHOD OF FUZZY SYSTEMS

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Abstract: The real-world-semantics interpretability concept of fuzzy systems introduced in [1] is new for the both methodology and application and is necessary to meet the demand of establishing a mathematical basis to construct computational semantics of linguistic words so that a method developed based on handling the computational semantics of linguistic terms to simulate a human method immediately handling words can produce outputs similar to the one produced by the human method. As the real world of each application problem having its own structure which is described by certain linguistic expressions, this requirement can be ensured by imposing constraints on the interpretation assigning computational objects in the appropriate computational structure to the words so that the relationships between the computational semantics in the computational structure is the image of relationships between the real-world objects described by the word-expressions. This study will discuss more clearly the concept of real-world-semantics interpretability and point out that such requirement is a challenge to the study of the interpretability of fuzzy systems, especially for approaches within the fuzzy set framework. A methodological challenge is that it requires both the computational expression representing a given linguistic fuzzy rule base and an approximate reasoning method working on this computation expression must also preserve the real-world semantics of the application problem. Fortunately, the hedge algebra (HA) based approach demonstrates the expectation that the graphical representation of the rule of fuzzy systems and the interpolation reasoning method on them are able to preserve the real-world semantics of the real-world counterpart of the given application problem.

Keywords: interpretable, fuzzy system, hedge algebra.

Classification numbers: 4.7.4, 4.8.4, 4.10.3.

1. INTRODUCTION

Fuzzy rule based systems (FRBSs) have been strongly developed in recent years due to their exceptional capabilities such as expert linguistic knowledge-based activities. They can be designed optimally based on genetic algorithms, i.e. they are constructed by using machine learning methods and techniques and especially, they are easy to understand and explain to users to interact with people in natural language. In the above features, being equipped with a linguistic knowledge basis and capable of simulating human reasoning can be considered as extremely important and is one of the main objectives of the FRBSs design. Thus, to interact with users, the interpretability of the FRBSs has attracted a lot of attention and interest from the research community in this field. Example as Alonso et al. [2], Antonelli et al. [3], Cordon [4], Gacto et al. [5], Ishibuchi and Nojima [6], Mencar et al. [7, 8], Nauck [9], Zhou và Gan [10].

The interpretability of the FRBSs has been interested since the 1990s and it is mainly based on the comprehensibility view, so the terms ‘interpretability’ and ‘comprehensiveness’ are considered synonyms. The nature of interpretable in the uncertain linguistic information environment is to ensure that the modeling and simulation of things and phenomena beyond the real world (RW) based on formal computational systems and on handing their computational semantics instead of linguistic words and sentences is sound and consistent with RW-processes. Linguistic words and rules are just symbolic strings which do not have any meaning and they only have semantics when they are given meaning by humans. Therefore, when assigning mathematical objects to the linguistic words to computationally manipulate them, it requires that we must have a formalized methodological basis to ensure that computational systems manipulate on them also has the same results as humans manipulating their respective linguistic elements.

In essence, each fuzzy system (FSyst) is a fuzzy set expression manipulated based on a calculation formalism of fuzzy set theory (such as fuzzy set algebras, reasoning methods, etc.). In this formalism, each fuzzy set is labeled by a linguistic word and they are considered as representing the computational semantics of their associated linguistic labels. Thus, each fuzzy set expression corresponds to a linguistic expression that can be read and understood by humans and it is considered as representing its corresponding linguistic expression. The interpretability problem of formalized programming languages or more broadly, of the formalized theory based on a formalized language, is establishing interpretations that assign computational objects of their respective desired computational structures to the well-formed symbolic expressions of their formalized languages so that the syntactic properties of the formalized programming languages or of the formalized theories, like the axioms and theorems of formalized theory derived by applying *syntactical rules to symbolic strings*, are preserved in the respective computational structures. As these computational structures, which are usually mathematic theories, do soundly represent the structures of their respective RW-counterparts, in nature the interpretation of symbolic expressions of a formalized language is an assignment of RW-semantics of its RW-counterpart to symbolic expressions so that the properties formulated in the formalized language are *just properties of the RW-counterpart observed by humans*.

In the hedge-algebra-based HA-approach, the word-domains of the linguistic variables are formalized as their algebraic structures, called hedge algebras (HAs), similar as for programming languages, in the studies [11, 12] as well as in this study, applying the interpretation concept it is possible to translate symbolic strings representing linguistic words to elements of their respective HAs and, then, to their respective computational quantities based on the quantification of the HAs. Thus, the interpretability of a computational representation of a word-expression is studied based on how preserving the structural semantic characteristics of the word-expression which is described by human experts in terms of their language. Thus, in the HA-approach, words are considered as elements of an HA and their qualitative semantics are defined by the order relation among them in the word-domain of the variable. Meanwhile, the computational semantics of words and of word-expressions are produced or constructed from the qualitative semantics of words in the linguistic domain based on the numerical semantics and intervallic semantics of words. This means the words are not just labels, but they play a crucial role in determining their computational semantics and linguistic expressions.

In this article, we will give clearer and deeper explanation and discussion on the concept of the real-world-semantics (RWS-) interpretability of FSysts and of word-expressions and on how to solve the RWS-interpretability problem based on the HA-approach. Following the research methodology examined in [1, 13] it is demonstrated that the method of representing the linguistic rule bases (LRBs) of FSysts and the approximate reasoning method (ARMd) based on the interpolation method is possible to preserve the real-world structural semantics of applied problem expressed in the basis of their linguistic rule knowledge.

2. THE REAL-WORLD-SEMANTICS INTERPRETABILITY

2.1. The real-world-semantics interpretability of the computational representation of the linguistic expression

In 2017, a new approach to the interpretability of FSysts which is an approach based on real world semantic interpretation, was first proposed and studied in [1] based on the real-world structural semantics of words and the semantic relation between of FSyst components and corresponding sub-structures of the real world. In particular, the RWS-approach is to study the relation among the three entities: (1) a FSyst, considered to be a formal symbolic expression; (2) computational model which is the computational image of the formal expression and (3) its corresponding real world structural semantic. The RWS- interpretability of computational expressions for each composed component of the FSyst is ensured by an interpretation assignment and is imposed by constraints that are discovered by human experts from the real world. The RWS-approach establishes a formal basis to overcome the difference in nature among the computational semantics of the components of a FSyst constructed by the designers and the RW-semantics of just the components of the FSyst, including linguistic frames of cognition of variables, LRB and ARMd. This difference exists inevitably because computational semantics are mathematical objects with distinguished specific natures defined in a mathematical structure, while linguistic semantics are RW-entities and relation among them defined and described in terms of words and linguistic sentences. This distinction exists objectively because the semantics of words and sentences point at objects or entities that exist and act objectively in the real world, while computational semantics are mathematical objects, they operate or interact each other in a mathematical structure built by humans. Whether they properly represent the real-world semantics that the linguistic expressions describe *is just the RWS-interpretability problem of their computational semantics, i.e. the computational representations of the linguistic expressions*. Thus, the RWS-interpretability is essential and therefore any formal symbolic language that exists up to now must be RWS-interpretable. For examples, mathematical theories, theoretical physics, especially human natural language, etc. are all RWS-interpretable though they are not so explicitly declared. This problem becomes necessary when in the field of fuzzy sets, the RWS-interpretability problem of the computational representations of linguistic expressions has not been taken into account and, hence, this may cause many questions [1], especially, when the word-domains of linguistic variables have not been mathematically formalized. For instance, let us consider the case human may use a numeric variable $\mathcal{V}_N$ as well as a linguistic variable $\mathcal{V}_L$ to model a RW-variable $\mathcal{V}_{RW}$, say the velocity of a car. It is known that the domain of $\mathcal{V}_N$ is linearly ordered arithmetic math-structure, but the word-domain of $\mathcal{V}_L$ is not taken into account as a math-structure, while, by the compatibility of two variables $\mathcal{V}_N$ and $\mathcal{V}_L$ and the RWS-interpretability of the human language, the word-domain of $\mathcal{V}_L$ must also be at least linearly ordered. In contrast, the order of the fuzzy sets representing the word-set of $\mathcal{V}_L$ is not take care in most studies in the fuzzy set framework, recalling that raking fuzzy sets is a hard problem. Thus, the RWS-interpretability concept is essential and very practical in studying FSysts and in simulating human ability in immediately handling fuzzy linguistic information. So, the problem of how to ensure the RWS-interpretability of the computational expressions representing word-expressions within fuzzy set theory is not only a novel problem, but also challenging in the field of fuzzy set.

To consider whether a formal theory is RWS-interpretable or not, the authors of [1] has introduced the following definition applying the concept of interpretability of a theory *S* in another theory *T* defined by Tarski et al. [14]:

Definition 1. [1] A formalized method/theory *T* formulated in a formalized language to simulate a real-world structure, denoted by W_T, is said to be RWS-interpretable if there exists a realization R_T: $W_T \rightarrow T$, which assigns real-world objects of *W* to elementary formalized elements of *T*, that can convey or preserve the essential properties of W_T. In this case, *T* is called an RWS-model of W_T or W_T is interpretable in *T*. Such a formalized method *T* is called *RWS-interpretable*. Note that, the structure W_T is a subjective concept as it depends on the observation/perception of a human user. In this sense, most of classical mathematical theories are RWS-interpretable.

The question is whether or not there is an RWS-interpretable theory to form a basic mathematical formalism to immediately manipulate linguistic words and their semantics. The studies [1, 13] point out that the theory of HAs is RWS-interpretable based on the assumption that human natural language is RWS-interpretable.

Although fuzzy set theory is strongly developed and has widespread application, methodically, it is difficult to consider it as a RWS-interpretable formalism to develop fuzzy methods to solve application problems. For example, we have the word-expression of the truth variable $\mathbb{E}$ = *"true* OR *very true"*. By the RWS-interpretability of natural language, we have: *true* OR *very true* = *very true*. However, denoting by *FS*(.) the fuzzy set expression of the word-expression ".", it can clearly be seen that, in the formalism of the standard algebra of fuzzy sets, we have $FS(\mathbb{E}) = FS(true) \cup FS(very\ true) \neq FS(very\ true)$. This means that the equality between word-expressions cannot be preserved when they are translated into the standard algebra of fuzzy sets. In other words, the computational representation of $\mathbb{E}$ in standard fuzzy set algebra does not preserve the real-world structural semantic. Methodologically, this leads to the fact that methods developed to solve application problems in the fuzzy set framework in general require a lot of experimental study to adjust parameters to achieve acceptable solutions.

2.2. Schema for constructing computational representations of linguistic expressions in FSysts

Linguistic expressions of FSysts have the following forms: linguistic rules, LRB, and word-sets of variables called linguistic frames of cognition (LFoCs). The concept of LFoCs of variables is similar as the one of Frames of Cognition in the fuzzy set framework, each of which consists of declared fuzzy sets of a variable and considered as frame of view in cognition. This declaration depends strongly on individual applications. Thus, in the linguistic information environment, the semantics of the variables depend strongly on the declared LFoCs and, so, the semantic interpretation of LFoCs is a very important problem in studying FSysts and in the fuzzy set framework, in general. As the RWS-interpretability problem of computational representations of word-expression was studied in [13] and as this problem is quite new and complex, this study aims to focus on the RWS-interpretability problem of computational representations of fuzzy LRBs and ARMd running on them.

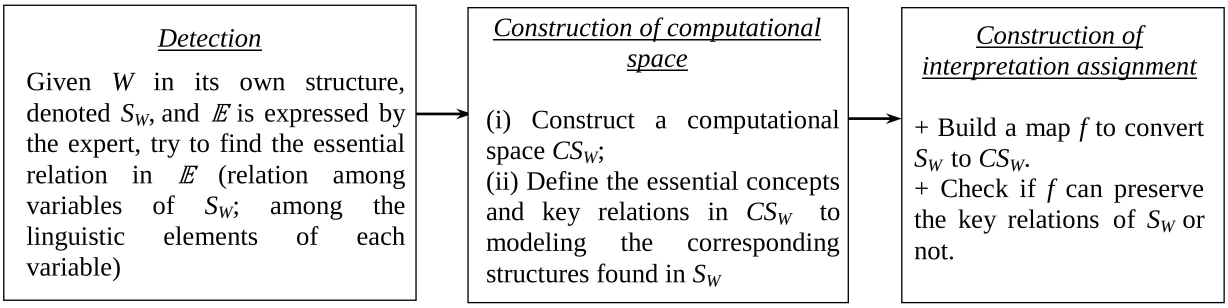

*Figure 1***.** The interpretable problem-solving schema RWS.

In order to solve the RWS-interpretability problem of FSysts properly and fully, the study should rely on the schema shown in Figure 1, of which the RWS-interpretability problem of a word-expressions and the ARMd of FSysts depends on the structural characteristics of the RW-part described by the word-expressions. A component of the FSyst, including the whole FSyst, or any method or algorithm described by linguistic sentences can also be considered a word-expression $\mathbb{E}$, that is expressed by human expert to solve the given practical problem in a real-world W in question.

In case $\mathbb{E}$ is an LFoC: Then $\mathbb{E}$ is the word-sets of the variables. On every word-domain, there are two important relations: *order relation* and *generality-specificity relation of words* [16]. However, order relation is the most essential, i.e. it should be preserved by any interpretation, while generality-specificity relation is not necessary to be preserved for all application problems. In other words, S_W structure is not described by generality-specificity relation, for example, control problems, e.g. for control problems. However, for classification problems [15], regression problems [16] or linguistic data summarization, the generality-specificity relation is very crucial.

In case $\mathbb{E}$ is the rule base: In principle, each linguistic rule represents a relation between the real-world variables corresponding to the variables occurring in this rule. Indeed, consider a simple case that the rules have m input variables and 1 output variable. Similar as in the numeric case, each rule is represented by a point in Cartesian product Π of m linguistic domains of its variables and as such, the n rules of the given rule base define n points in space Π. A non-contradictory rule base defines the output linguistic variable as a function of m input linguistic variables. Similar also as for numeric functions, between each pair of input – output variable of a linguistic function there may be a monotonic relation defined on a certain "segment" of the word-domain of the input variable. They reflect the structural characteristics of its RW-counterpart of the LRB.

It is clear that as human natural language is RWS-interpretable, structural properties of the real world can be recognized based on the inherent semantics of the words of the variables. For instance, from the given rule “If the car-engine is *strong*, it can run *fast*” one can deduce that the variable ‘car-velocity” increasingly depends on the variable ‘car-engine’ on a certain neighborhood of the linguistic point “*strong*” of the linguistic domain of ‘car-engine’. These recognized structural properties will be used to impose constraints on the established semantic interpretations that assign computational semantics, or computational objects in their respective suitably constructed computational structures, to the words appearing in the LRB. Only in such way, we may ensure that the manipulation on computational objects assigned to their respective words by the established interpretations based on the formalism of the constructed computational structures is compatible with the manipulation of the words by human expert. We will show that hedge algebras and its quantification methodology will form a formalism to solve the RWS-interpretability problem of FSysts [1].

3. THE RWS-INTERPRETABILITY OF LINGUISTIC RULE THE BASES OF FUZZY SYSTEMS AND OF THE APPROXIMATE REASONING METHOD

The RWS-interpretability of the LRBs and the ARMd is introduced and examined in [1] and further analyzed and discussed in [13]. These studies show that this problem is very important and essential for designing effective FSysts but it is complex as it is based on a high abstract interpretation concept of math-logics. In this article, we emphasize and study two features related to ARMds: (i) As it requires that ARMds are developed so that they may work on the computational representation of any given LRB, we should deal with the method to generate the computational representations of LRBs; (ii) It is necessary to introduce criteria to verify the RWS-interpretability of the both kinds og just mentioned methods based on the structural semantics properties discovered from the RW-counterpart described by the given LRBs.

As a consequence of feature (i) above, ARMd should be developed to work on the computational representation constructed by a developed method to generate computational representation of any LRB. For the criteria mentioned in feature (ii), it is clear that they depend on each application problem, because the structures of the RW-counterparts described by the LRBs of different application problems are of course different. As discussed in point 2), Section 2.2, we may rely on the monotonicity of dependence between any two input variable and output variable to impose constraints on the examination of the RWS-interpretability of the methods mentioned in features (i) and (ii). As a consequence of these two features, the RWS-interpretability of the two methods will be defined in a close relation with each other.

First of all, we study the RWS-interpretability of the method to generate the computational representation of LRBs.

3.1. The RWS-interpretability of the computational representation of LRBs

3.1.1. Challenges in studying of the RWS-interpretability of the computational representation of LRBs

The study [1] suggests that one can reveal information about dependence of any two RW-variables only if it is monotonic on a certain interval of each variable, since otherwise their dependent relation is chaotic. As the RWS-interpretability problem of LRB is related to three objects, RW-objects, math-objects and human linguistic words, to avoid confusion, we introduce notations as follows: If $\mathcal{X}$ denotes a RW-variable, then the notations $\mathcal{X}_N$ and $\mathcal{X}_L$ denote respectively the numerical variable and the linguistic one.

Consider a linguistic rule with one output and *m* input variables written in the following form:

$$(r)\ \text{IF } \mathcal{X}_{1L} \text{ is } x_1\ \&\ \dots\ \&\mathcal{X}_{mL} \text{ is } x_m\text{, THEN } \mathcal{X}_{m+1,L} \text{ is } x_{m+1} \quad (1)$$

in which each expression "$\mathcal{X}_{jL}$ is x_j" is a linguistic predicate, for $j = 1$ to $m + 1$. Similar as for analyzing a classical multi-variable function, for every rule r, one may consider m dependent relations 'IF $\mathcal{X}_{jL}$ is x_j, THEN $\mathcal{X}_{m+1,L}$ is x_{m+1}', $j = 1$ to $m + 1$, and therefore, r denotes m monotonic dependences between variables $\mathcal{X}_{m+1,L}$ and $\mathcal{X}_{jL}$ on certain interval of each respective RW-variable of the RW-counterpart.

So, the semantics of ***linguistic rules*** reflect that their RW-semantics are very important but, in the fuzzy set framework, such semantics of fuzzy rules is not taken into consideration and, therefore, there is no formalism to define computational semantics of fuzzy rules in relation with the linguistic labels of the fuzzy sets occurring in the rules. Most importantly, due to the RWS-interpretability of natural language, the above monotonic dependences can be discovered from the linguistic rules of the form (1). For example, in the field of fuzzy control there are many application problems whose LRBs describe increasingly or decreasingly monotonic RW-function of a RW-variable $\mathcal{X}_{m+1,RW}$ on the $\mathcal{X}_{j,RW}$, $j = 1$ to m, and hence so are their respective LRBs.

In approaches within the fuzzy set framework, methodology, the inherent qualitative semantics are completely ignored and words are only considered as linguistic labels assigned to the fuzzy sets designed by human expert of the FSyst to computationally represent their semantics. In such approaches, the computational semantics of a fuzzy rule base $\mathcal{RB}$ consisting of n rules in form (1), in which the words x_m's are considered as linguistic labels of the designed fuzzy sets, can be expressed by the fuzzy relation R_F defined in Cartesian product $U_1 \times \ldots \times U_{m+1}$ constructed by a certain representation method, where U_j's are the reference domains of the respective variables $\mathcal{X}_{j,L}$'s. In general, there are some computational representation methods to compute such fuzzy relation R_F. Applying the composition rule of inference introduced by Zadeh, the computational representation method $\mathcal{M}$ can transform LRB of the rules in the form (1) into a fuzzy relation as follows:

(i) Established an interpretation $I_{\mathcal{X}j}$ that maps the words of $\mathcal{X}_j$ into the designed fuzzy sets of a fussy set space CS_j, i.e. $I_{\mathcal{X}j}(x_j)$ is the fuzzy set of CS_j, with $j = 1$ to m+1. These fuzzy sets usually form a fuzzy partition of the reference domain $U_{\mathcal{X}j}$ of $\mathcal{X}_j$, $j = 1$ to m+1;

(ii) Construct a procedure P which translates connectives AND, OR appearing in $\mathcal{RB}$ and the rule themselves into fuzzy relations defined on Cartesian product $U = U_1 \times \ldots \times U_{m+1}$ in the following way:

- AND, OR: It is known that these connectives are translated respectively into the intersection and union of fuzzy sets using min "$\wedge$" and max "$\vee$" and which are pointwise defined;
- IF-THEN: the IF-THEN appearing in each rule is translated into an implication of a multi-valued logic, denoted by "$\rightarrow$", which is a binary operation $s \rightarrow t$, $s, t \in [0,1]$, that is decreasing with respect to s and increasing with respect to t;
- Then, the composition $P\circ (I_{\mathcal{X}1}, \ldots, I_{\mathcal{X}m+1})$ with functionality to convert every rule r_i of the form (1), $i = 1, \ldots, n$, of linguistic rule base $\mathcal{RB}$ into a fuzzy relation $R_F(r_i) \in \mathbb{F}(\underline{U})$, the set of all fuzzy relations defined on U, defined as follows:

$$\begin{aligned} R_F(r_i) &= P \circ (I_{\mathcal{X}1}, \ldots, I_{\mathcal{X}m+1})(r_i) \\ &= P\circ (I_{\mathcal{X}1}, \ldots, I_{\mathcal{X}m+1})[\text{IF } \mathcal{X}_{1L} \text{ is } x_{i1} \,\&\, \ldots \,\&\mathcal{X}_{mL} \text{ is } x_{im}, \text{ THEN } \mathcal{X}_{m+1,L} \text{ is } x_{i,m+1})] \\ &= I_{\mathcal{X}1}(x_{i1}) \cap \ldots \cap I_{\mathcal{X}m}(x_{im}) \rightarrow I_{\mathcal{X}m+1}(x_{i,m+1}) \end{aligned} \tag{2}$$

(iii) Finally, the rule base $\mathcal{RB}$ is represented by the fuzzy relation $R_F(\mathcal{RB})$ in $\mathbb{F}(\underline{U})$ as follows:

$$\bigcup\{ R_F(r_i): i = 1, \ldots, n\} \quad (3)$$

By (i) – (iii), note that the operations on fuzzy sets mentioned in (ii) and (iii) are pointwise defined on the reference domain the variables. We may find that it does not take advantage of any qualitative information or semantics of words, and of course it does not rely on real-world information that such linguistic expressions describe. Thus, there is no basis for formalization as a basis for the study of real-world-semantics interpretability based on the concept and schema mentioned in Section 2. To easily see the nature of the problem we consider the following rule:

If SPEED(o) = "*large*" & WEIGHT(o) = "*heavy*", Then KINETIC ENERGY(o) = "*large*" (4)

Many studies within fuzzy set theory express the above linguistic rule by the following expression:

$$FS_{SPEED;large}(s) \wedge FS_{WEIGHT;heavy}(t) \rightarrow FS_{K\_ENERGY;large}(u),\ s \in U_{SPEED},\ t \in U_{TR\_L},\ u \in U_{K\_ENERGY} \quad (5)$$

in which $FS_{\mathcal{X};x}$ denotes fuzzy set with linguistic label x of $\mathcal{X}$ and "→"denotes an implication of multi-valued logic with the truth values in [0,1]. According to [1], analyzing linguistic rule (4), we see that the variable "KINETIC ENERGY" *monotonically increasingly* depends on each of the variables "SPEED" and "WEIGHT". However, as membership functions take a value of 1 at the cores of fuzzy sets and are monotonically decreasing to 0 on both sides of their cores and so they are non-monotonic. Thus, increasing variations of the variables s, t and u do not result in an incremental variation of the values of the fuzzy set functions. So, if the word "*heavy*" is replaced by a greater word "*very heavy*" of the variable "WEIGHT", there is no basis to make sure that we also have $FS_{TR\_L;heavy}(t) \le FS_{TR\_L;very\ heavy}(t)$, $\forall t \in U_{TR\_L}$. Consequently, it is not ensured that the corresponding values of the variable "KINETIC ENERGY" also increases. In other words, there is no basis to ensure that representation (5) of rule (4) preserves the RW-semantics of the linguistic rule (4).

From the above analysis, we infer that the problem of aggregation of the semantic information of predicates in rules so that it preserves the semantics of the rules in the fuzzy environment is a challenge to examine the semantics of linguistic rules and their computational semantics. However, if we stand on a viewpoint that words of a variable being the elements of hedge algebra associated with the variable, then rule (4) can be represented by a linguistic point in Cartesian product of the linguistic domains of the variables present in the rule. So, n rules of a given linguistic rule base will be represented by m points in this Cartesian product and they define a graph in it, namely a graph of a linguistic function. Successful applications of analytical mathematics to solve application problems so far demonstrate that graphical representation of functions is a useful way to properly aggregate linguistic information of individual rule variables to preserve the structural semantics of the rule base RW-counterpart, as discussed below.

3.1.2. The RWS-interpretability of the computational representation of LRBs and ARMds

Consider a LRB $\mathcal{RB}$ consisting of n rules r_i in the form given in (1):

$$(r_i)\ \text{IF } \mathcal{X}_{1L} \text{ is } x_{i1}\ \&\ \ldots\ \&\ \mathcal{X}_{mL} \text{ is } x_{im}, \text{ THEN } \mathcal{X}_{m+1,L} \text{ is } x_{i(m+1)},\ i = 1 \ldots, n, \quad (6)$$

The question is whether or not there exists a method to produce computational representation of the LRB $\mathcal{RB}$ which is RWS-interpretable and on which one can develop an approximate reasoning method being also RWS-interpretable? In this section, we will conduct a study using the HA-approach in which the inherent order based semantics of the words and semantic structures of the domains of variables are utilized to determine their computational semantics. Because this approach establishes a formalism to deal directly with the words with their own semantics of variables, we will use the terminologies linguistic rules instead of fuzzy rules in the fuzzy set framework to emphasize their linguistic semantic features. Methodologically, in general, when the word-domains are formalized into mathematical structures, each linguistic rule in form (1) can be considered as a linguistic point in the Cartesian product space of (m+1) hedge algebras which are formalized linguistic domains of variables. Thus, every LRB in form (6) can be considered as a model of a linguistic function with m variables going through n linguistic points defined by the given LRB. On this basis we can construct a computational representation for linguistic rule base using interpretation assignments.

Firstly, we define a computational representation method of $\mathcal{RB}$ based on the concept of computational interpretation assignment for the words of variables $\mathcal{X}_{jL}$, $j = 1, \ldots, m + 1$. We denote by $I_{\mathcal{X}j}$ an interpretation assignment of computational objects of an ordered based computational space $\mathcal{CS}_j = (CS_j, \leq_j)$ associated with the variable $\mathcal{X}_{jL}$ to the words of $\mathcal{X}_{jL}$. Assume that $\mathcal{CS} = (\boldsymbol{CS}, \leq)$ is a partially ordered computational space defined on the Cartesian product of $\mathcal{CS}_j$: $\boldsymbol{CS} = CS_1 \times \ldots \times CS_{(m+1)}$ with the order relation $\leq$ defined based on the order relations of components $\leq_j$, $j = 1, \ldots, m + 1$, as usual. Then, we develop a graphical method to computationally represent the given LRB $\mathcal{RB}$ in Euclidean space $[0,1]^{m+1}$, where [0,1] is the normalized domain of the reference domain U_j of $\mathcal{X}_{jL}$ as follows.

1) The interpretation assignment of elements of hedge algebra to words of rules and a graphic representation of LRBs: As mentioned above, methodologically, every rule r_i should be considered as a symbolic expression. Now, we will assign meaning to r_i using interpretation assignment. Because every HA $\mathcal{AX}$ associated with a variable can be considered as a mathematical model of its word-domain, which is formalized in such a way that each element of $\mathcal{AX}$ can be obtained by a direct translation of a word of the word-domain. By this, for every variable $\mathcal{X}_{jL}$ occurring in r_i whose associated HA is declared to be $\mathcal{AX}_{jL} = (X_j, G_j, H_j, \leq_j)$ by specifying: (i) the names of the negative and positive primary words c^- and c^+ of the set G_j of generators; (ii) specifying the set H_j of the positive and negative hedges; and (iii) establishing a table of the relative "algebraic" signs between the declared hedges. Then, there exists a "natural" interpretation $I_{\mathcal{A}j}$: LDom($\mathcal{X}_{jL}$) $\rightarrow X_j$, where LDom($\mathcal{X}_{jL}$) is the word-set of $\mathcal{X}_{jL}$, that assigns an element of $\mathcal{AX}_{jL}$ to a word of LDom($\mathcal{X}_{jL}$).

Denote by $\mathbb{I} = (I_{\mathcal{A}1}, \ldots, I_{\mathcal{A}m+1})$ a set of natural interpretations of the words of their respective variables whose functionality is defined as follows, for all rules r_i in form (6):

$$\mathbb{I} = (I_{\mathcal{A}1}, \ldots, I_{\mathcal{A}m+1}) : r_i \rightarrow (x_{i1}, \ldots, x_{i(m+1)}) \in X_1 \times \ldots \times X_{(m+1)} \quad (7)$$

Definition 2. Let be given a LRB $\mathcal{RB}$ consists of n rules in the form (6). Assume that each variable $\mathcal{X}_{jL}$ is associated with an HA $\mathcal{AX}_{jL} = (X_j, G_j, H_j, \leq_j)$, $j = 1, \ldots, m$+1, defined as given above, and an interpretation $\boldsymbol{I}_{\mathcal{A}j}$ established for each variable $\mathcal{X}_{jL}$. Then, the set $\{(x_{i1}, \ldots, x_{i(m+1)}) : i = 1, \ldots, n\} \subseteq X_1 \times \ldots \times X_{(m+1)}\}$, denoted by $\mathcal{L}Gph_{\mathbb{I}}(\mathcal{RB})$, is called a *linguistic graphical representation* of $\mathcal{RB}$.

Proposition 1. If a LRB $\mathcal{RB}$ consisting of n rules in form (6) is consistent, i.e. two rules of $\mathcal{RB}$ have the same "IF" components, their "THEN" components are also the same, the graph of $\mathcal{RB}$ describes a functional relation.

Proof: the correctness is immediately derived from the consistency of the LRB $\mathcal{RB}$.

2) Assignment of numerical semantics to linguistic words: To construct numerical semantics for words, we need to apply the hedge algebra quantitative methodology. There are three basic quantitative semantics of the words of each variable $\mathcal{X}$, defined in close relation to each other: fuzzy measure, fuzzy interval (considered as interval semantics) and semantically quantifying mapping (SQM) of the words of variables. They are uniquely defined when the numerical values of the independent fuzzy parameters of variables are provided. The SQM-values of words are called the numerical semantics of words. In this section, however, we utilize only SQMs which are characterized by two properties that they are order isomorphisms, i.e. they must preserve the order relations among words and the images of linguistic domains of variables under these isomorphisms are dense in the reference domains of the corresponding variables (similar as the countable set of the rational numbers is dense in the real line).

For each variable $\mathcal{X}_{jL}$ and the HA $\mathcal{AX}_{jL}$ assigned to its, we define an SQM of $\mathcal{AX}_{jL}$, $f_j : X_j \rightarrow [0,1]$, $j = 1, \ldots, m + 1$, and consider the composition $I_{\mathcal{A}j} \circ f_j : \mathrm{Dom}(\mathcal{X}_{jL}) \rightarrow [0,1]$ as an *interpretation assigning numerical semantics* to the words of the variable $\mathcal{X}_{jL}$, called numeric semantic interpretation of $\mathcal{X}_{jL}$.

Definition 3. Let be give an LRB $\mathcal{RB}$ consisting of *n* rules in form (6). Assume that, for each variable $\mathcal{X}_{jL}$, the numerical semantic interpretation $I_{\mathcal{A}j} \circ f_j : \mathrm{Dom}(\mathcal{X}_{jL}) \rightarrow [0,1]$ is established, $j = 1, \ldots, m + 1$. Let $\mathbb{I} \circ \mathbb{f} = (I_{\mathcal{A}1} \circ f_1, \ldots, I_{\mathcal{A}(m+1)} \circ f_{m+1})$ denotes a vector of numeric semantic interpretations. Then, the computational image of the linguistic graph $\mathcal{L}Gph_{\mathbb{I}}(\mathcal{RB})$ of $\mathcal{RB}$ is defined as follows

$$\mathbb{I}\circ\mathbb{f}(\mathcal{L}Gph_{\mathbb{I}}(\mathcal{RB})) = \mathbb{I}\circ\mathbb{f}(\{(x_{i1}, \ldots, x_{i(m+1)}) : i = 1, \ldots, n\})$$
$$= \{(f_1(I_{\mathcal{A}1}(x_{i1})), \ldots, f_{m+1}(I_{\mathcal{A}(m+1)}(x_{i(m+1)})) : i = 1, \ldots, n\} \subseteq [0,1]^{m+1}$$

and it is called *numeric graphical representation* of $\mathcal{RB}$, denoted by $\mathcal{N}Gph_{\mathbb{I}\circ\mathbb{f}}(\mathcal{RB})$ and the method to defined its is called *graphical representation method of LRBs* (GRMd).

Due to natural language is RWS-interpretable, if the linguistic rule base $\mathcal{RB}$ aims to describe a RW-function f_W in the real-world *W*, then $\mathcal{RB}$ must also represent a linguistic function $f_{L,\mathcal{RB}}$ of linguistic variable $\mathcal{X}_{jL,(m+1)}$ on the remaining ones $\mathcal{X}_{jL}$'s, whose graph is $\mathcal{L}Gph_{\mathbb{I}}(\mathcal{RB})$. So, $\mathcal{L}Gph_{\mathbb{I}}(\mathcal{RB})$ is a model of f_W which models a RW-semantic feature of *W*. On the other hand, as f_W is RW-function of the RW-variable $\mathcal{X}_{jRW,(m+1)}$ on the remaining ones $\mathcal{X}_{jRW}$'s, applying the numerical analytical theory to model this RW-semantic feature of *W*, this numerical model must also be a numeric function of the variable $\mathcal{X}_{(m+1)N}$ depending on the remaining variables $\mathcal{X}_{jN}$'s, , denoted by $f_{N,\mathcal{RB}}$, as the numerical analytical theory is RWS-interpretable as discussed in Section 2.1.

Now, we will demonstrate that the computational representation method $\mathcal{N}Gph_{\mathbb{I}\circ\mathbb{f}}(\mathcal{RB})$ of $\mathcal{RB}$ is RWS-interpretable in the following sense: If the numeric graphical representation $\mathcal{L}Gph_{\mathbb{I}}(\mathcal{RB})$ of $\mathcal{RB}$ represent an increasing (or, decreasing) linguistic functional dependence of the variable $\mathcal{X}_{jL,(m+1)}$ on the their words of the remaining variables $\mathcal{X}_{jL}$'s, then the numeric graphical representation $\mathcal{N}Gph_{\mathbb{I}\circ\mathbb{f}}(\mathcal{RB})$ of $\mathcal{RB}$ must preserve this dependence. It can be seen that this RWS-interpretability is broader than the concept examined in [13].

Theorem 1. The GRMd to produce $\mathcal{N}Gph_{\mathbb{I}\circ\mathbb{f}}(\mathcal{RB})$ of any given LRB $\mathcal{RB}$ described in Def. 3 is RWS-interpretable.

Proof: Firstly, we need to prove that if there are two vectors of words $(u_1, \ldots, u_{i(m+1)})$ and $(v_1, \ldots, v_{(m+1)})$ of the two rules describing *increasing monotonic* relation, we have:

$(u_1, \ldots, u_{(m+1)}) \leq (v_1, \ldots, v_{(m+1)}) \Rightarrow \mathbb{I} \circ \mathbb{f}(u_1, \ldots, u_{(m+1)}) \leq \mathbb{I} \circ \mathbb{f}(v_1, \ldots, v_{(m+1)})$

Indeed, as argued in [1], the natural language, the hedge algebra theory and their SQMs of HAs are RWS-interpretable, i.e. they preserve the order-semantics of the domains of their respective variables, from the inequality in the left side, we infer

$(I_{\mathcal{A}1}(u_1), \ldots, I_{\mathcal{A}(m+1)}(u_{(m+1)})) \leq (I_{\mathcal{A}1}(v_1), \ldots, I_{\mathcal{A}(m+1)}(v_{(m+1)}))$.

As the quantitative mapping $f_1, \ldots, f_{m+1}$ is the order isomorphism, i.e. they preserve the order of the numeric semantics of the words, we obtain:

$(f_1(I_{\mathcal{A}1}(u_1)), \ldots, f_{m+1}(I_{\mathcal{A}(m+1)}(u_{(m+1)})) \leq (f_1(I_{\mathcal{A}1}(v_1)), \ldots, f_{m+1}(I_{\mathcal{A}(m+1)}(v_{(m+1)}))$.

Because the decreasing monotonicity case is demonstrated similarly, so the theorem is demonstrated.

3.2. The interpretability of the approximate reasoning method

3.2.1. The RWS-interpretability of ARMds and computational representation methods of the linguistic rule base

ARMds developed to solve application problems plays an important role to build FSysts and therefore, its interpretability is essential to ensure their performance in solving application problems, due to in the opposite case we have no formal basis to ensure that the outputs of their ARMd are compatible with the results expected by human designer. This question strongly depends on the RWS-interpretability of the constructed computational representation method, $\mathcal{M}$, to produce computational representations of LRBs as well as of ARMds running on. Any ARMd, say $\mathbb{R}$, needs to be developed to be able to work on the computational representation of $\mathcal{RB}$ and this implies that its real-world-semantics interpretability depends heavily on $\mathcal{M}$. Therefore, the RWS-interpretability of an ARMds should be defined based on the computational representation method associated with it. In [13], the authors introduced the following definition, in which $\boldsymbol{a} = (a_1, ..., a_m)$ is the input vector and $\mathbb{R}(\boldsymbol{a})$ denotes the numerical output of the vector $\boldsymbol{a}$ produced by $\mathbb{R}$.

Definition 4. [13] Assume that an ARMd $\mathbb{R}$ is developed to work on computational representations of LRBs produced by a computational representation method $\mathcal{M}$. Then, $\mathbb{R}$ is said to be RWS-interpretable if for any give LRB $\mathcal{RB}$ being increasingly monotonic to all individual input variables of $\mathcal{RB}$, $\mathbb{R}$ must satisfy the following condition:

$$(\forall \boldsymbol{a}, \boldsymbol{a}')\{[\boldsymbol{a} \preccurlyeq \boldsymbol{a}' \Rightarrow \mathbb{R}_{\mathcal{M}(\mathbb{B})}(\boldsymbol{a}) \leq \mathbb{R}_{\mathcal{M}(\mathbb{B})}(\boldsymbol{a}')] \text{ and } [\boldsymbol{a} \neq \boldsymbol{a}' \Rightarrow \mathbb{R}_{\mathcal{M}(\mathbb{B})}(\boldsymbol{a}) \neq \mathbb{R}_{\mathcal{M}(\mathbb{B})}(\boldsymbol{a}')]\} \quad (2)$$

3.2.2. Interpolative approximate reasoning method on graphical representations of LRBs

Give a LRB $\mathcal{RB}$ in form as above and a GRMd, denoted by $\mathcal{M}_{\text{Graph}}$. Then an ARMd $\mathbb{R}$ running on $\mathcal{RB}$ is stated as follows:

Approximate reasoning problem: Give a numerical vector $\boldsymbol{a}_{in} = (a_{in,1}, \ldots, a_{in,m}) \in U_{\mathcal{X}1} \times \ldots \times U_{\mathcal{X}m}$ and a linguistic rule base $\mathcal{RB}$, calculate a numerical semantic of the output corresponding to the input $\boldsymbol{a}_{in}$, denoted by $\text{Out}_{\mathcal{RB}}(\boldsymbol{a}_{in})$, based on the knowledge given by $\mathcal{RB}$.

This problem can be solved in this study by an interpolative method in Euclidean space as follows:

Interpolative method on LRB $\mathcal{RB}$: Let be given values of the fuzzy parameters of the variables present in $\mathcal{RB}$ and a graphical representation method $\mathbb{M}_{\text{Graph}}$. Then, $\mathcal{M}_{\text{Graph}}(\mathcal{RB})$ defines a grid of a surface $\boldsymbol{S}_{\mathcal{RB}}$ in Euclidean space $[0, 1]^{m+1}$. So, every (numerical) interpolative method INTMd on the surface $\boldsymbol{S}_{\mathcal{RB}}$ can be apply to define a ARMd to solve the approximate reasoning problem for the given linguistic rule knowledge base $\mathcal{RB}$.

For a given an INTMd $\mathcal{M}_{\text{Inter}}$, it is clear that, for each input vector $\boldsymbol{a}_{in}$, $\text{Out}_{\mathcal{RB}}(\boldsymbol{a}_{in})$ can be calculated by applying $\mathcal{M}_{\text{Inter}}$ on the surface $\boldsymbol{S}_{\mathcal{RB}}$, denoted by $\mathcal{M}_{\text{Inter}}(\boldsymbol{S}_{\mathcal{RB}})$, and obtain $\text{Out}_{\mathcal{RB}}(\boldsymbol{a}_{in}) = \mathcal{M}_{\text{Inter}}(\boldsymbol{S}_{\mathcal{RB}})(\boldsymbol{a}_{in})$, i.e. it is the value calculated by $\mathcal{M}_{\text{Inter}}$ on $\boldsymbol{S}_{\mathcal{RB}}$ in the Euclidean space $[0, 1]^{m+1}$.

RWS-interpretability of interpolative approximate reasoning methods

1) *In case m = 2, i.e. in Euclidean space* $[0, 1]^3$

Table 1. Simple FRB for the first stage actuator.

x_2 \ $\dot{x}_2$	*S*	*W*	*l*
S	*S*	*S*	*W*
W	*Ls*	*W*	*Ll*
l	*W*	*PS*	*l*

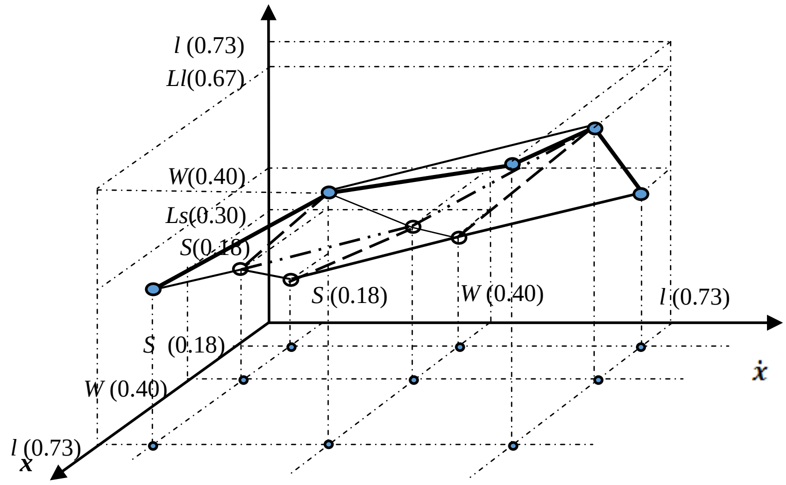

Figure 2. Numerical graphical representation of LRB passing through 9

In this case we can apply the linear interpolative ARMd. In case that the LRB has two inputs, we have a linear interpolative approximate reasoning method on surface in $[0, 1]^3$. For example, the LRB $\mathcal{RB}$ given in Table 1 with 9 linguistic rules defines a surface $\boldsymbol{S}_{\mathcal{RB}}$ as represented in Figure 2. Then, the interpolative ARMd is developed based on the triangular sections and denoted by $\text{Li}\Delta_{\boldsymbol{P}}$, where $\boldsymbol{P}$ is a set of three points of the numeric graph representation $\mathcal{N}Gph_{\mathbb{I}\circ\mathbb{f}}(\mathcal{RB})$ defining the section, e.g. the section in Fig. 2 whose linguistic vertices are (*l*, *W*, *l*), (*l*, *l*, *l*) and (*l*, *W*, *Ll*). This interpolative method is called the LiΔ-method, which is extended from the method studied in the work [3] but it RWS-interpretability is still not examined, and is described as follows:

- For each input vector $\boldsymbol{a}_{in} = (a_1, a_2)$, define the smallest rectangle, whose three vertices are denoted by P_k, $k = 1, 2, 3$, in the coordinate plane $x \times y$ containing point (a_1, a_2), including when it lies on the edge of the triangle, so that any its two vertices always have a common coordinate.
- Establish the section whose projection on the coordinate plane $x \times y$ is the above defined triangle: Denote by $\boldsymbol{S}_{\mathcal{RB}}(P_k)$, k = 1, 2, 3, the points in $[0,1]^3$ lying on the surface $\boldsymbol{S}_{\mathcal{RB}}$ whose projections on the plane $x \times y$ are the points P_k, $k = 1, 2, 3$ and establish the plane equation going through these points, denoted by $z = EQ_{(S_{\mathcal{RB}}(P1), S_{\mathcal{RB}}(P2), S_{\mathcal{RB}}(P3))}(x, y)$.
- Calculate the output by equality $Out(a_{in}) = EQ_{(S\mathcal{RB}(P1), S\mathcal{RB}(P2), S\mathcal{RB}(P3))}(a_1, a_2)$.

We can easily demonstrate the correctness of the following theorem:

Theorem 2. F^2LLX^2 The linearly interpolative LiΔ-method, denoted by Δ-$\mathcal{M}$, is RWS-interpretable.

Proof: Assuming that LRB $\mathcal{RB}$ describes an increasing linguistic function, as this equation is linear it is easy to prove that the inequality $(a_1, b_1) \leq (a_2, b_2)$ implies that $\Delta\text{-}\mathcal{M}(\boldsymbol{S}_{\mathcal{RB}})(a_1, b_1) \leq \Delta\text{-}\mathcal{M}(\boldsymbol{S}_{\mathcal{RB}})(a_2, b_2)$.

2) *In case* $m > 2$

There are many interpolative methods with the number of dimensions $n > 3$ but they are in general very complicated when n is large. The approximate reasoning method applied to LRBs with the number of variable $n \geq 3$ is developed based on reducing the number of dimensions from n to 2. In this case, we can use an aggregation operator usually used in fuzzy set theory to convert approximate reasoning problems in $m + 1$ dimensional space to two-dimensional one.

Assume that the LRB $\mathcal{RB}$ consists of n rules r_i in form (1), i.e.:

r_i : IF $\mathcal{X}_{1L}$ is $x_{1,i}$ & … & $\mathcal{X}_{mL}$ is $x_{m,i}$, THEN $\mathcal{X}_{m+1,L}$ is $x_{m+1,i}$, $i = 1, \ldots, n$ (*)

Step 1) Apply the numeric graphical representation method of $\mathcal{RB}$ we obtain a grid $Grid_{m+1}(\mathcal{RB})$ of the graph $\mathcal{N}Gph_{\mathbb{F}\circ\mathcal{f}}(\mathcal{RB})$ in space $[0, 1]^{m+1}$:

$Grid_{m+1}(\mathcal{RB}) = \{(SQM_1(r_i|_{\mathcal{X}_1}), \ldots, SQM_{m+1}(r_i|_{\mathcal{X}_{m+1}}) : i = 1, \ldots, n\} \subseteq [0, 1]^{m+1}$

in which if $\boldsymbol{a}$ is a vector of $[0, 1]^{m+1}$, the symbol $\boldsymbol{a}|_{\mathcal{X}_j}$ is its component corresponding to variable $\mathcal{X}_j$.

Step 2) Aggregate the m first coordinates of the vectors in $Grid_{m+1}(\mathcal{RB})$ using a selected aggregation operator, denoted by $\mathcal{g}$, we obtain a grid which approximates a curve in $[0, 1]^2$:

$Grid_2(\mathcal{RB}) = \{(\mathcal{g}[SQM_1(r_i|_{\mathcal{X}_1}), \ldots, SQM_m(r_i|_{\mathcal{X}_m})], SQM_{m+1}(r_i|_{\mathcal{X}_{m+1}})): i = 1, \ldots, n\} \subseteq [0, 1]^2$

Step 3) Select an interpolative method on the obtained grid $Grid_2(\mathcal{RB})$, denoted by $IntM_2$ whose inputs are numerical singleton values. Then, for each numerical input vector $\boldsymbol{a}_{in} = (a_{in,1}, \ldots, a_{in,m}) \in U_{\mathcal{X}1} \times \ldots \times U_{\mathcal{X}m}$, the output value in $U_{\mathcal{X}m+1}$ is calculated by the $IntM_2$ method and the aggregation operator $\mathcal{g}$ as follows:

$Out(\boldsymbol{a}_{in}) = IntM_{2\mathcal{RB}}(\mathcal{g}(a_{in,1}, \ldots, a_{in,m}))$.

Theorem 3. Let be given a LRB $\mathcal{RB}$ and assume that the aggregation operator used is a weighted average with weight vector $\boldsymbol{w} = (w_1, \ldots, w_m)$ corresponding to m antecedent variables of $\mathcal{RB}$, denoted by $\mathcal{g}_w$. Then, the linear interpolation using $\mathcal{g}_w$, denoted by $Li\_IntM_{2,w}$ is RWS-interpretable.

Proof: Assume that $\mathcal{RB}$ is a LRB represented by the graph $\mathcal{N}Gph_{\mathbb{I}\circ\mathcal{F}}(\mathcal{RB})$ with the grid

$\text{Grid}_2(\mathcal{RB}) = \{(g_w[\text{SQM}_1(x_{1,i}), \ldots, \text{SQM}_m(x_{m,i})], \text{SQM}_{m+1}(x_{m+1,i})): i = 1, \ldots, n\}$.

Due to $\mathcal{RB}$ is increasingly monotonic and assume that there are two rules r_i and $r_{i'}$ in form (*) whose linguistic vectors created by the words in their antecedent parts, denoted by $\boldsymbol{x}(r_i) = (x_{1,i}, \ldots, x_{m,i})$ and $\boldsymbol{x}(r_{i'}) = (x_{1,i'}, \ldots, x_{m,i'})$, satisfy the condition that $\boldsymbol{x}(r_i) \leq \boldsymbol{x}(r_{i'})$, i.e. $x_{j,i} \leq x_{j,i'}$, for $j = 1, \ldots, m$, implies $r_i|\mathcal{X}_{m+1} = x_{i,m+1} \leq r_{i'}|\mathcal{X}_{m+1} = x_{i',m+1}$. As SQM_j are order isomorphisms, we have $\text{SQM}_j(x_{j,i}) \leq \text{SQM}_j(x_{j,i'})$, $j = 1, \ldots, m+1$, and therefore we obtain $g_w(x(r_i)) \leq g_w(x(r_{i'}))$.

Consider two input vectors $\boldsymbol{a}_{in} = (a_{in,1}, \ldots, a_{in,m}) \leq \boldsymbol{b}_{in} = (b_{in,1}, \ldots, b_{in,m})$. Then, similarly as above, we have $g_w(a_{in,1}, \ldots, a_{in,m}) \leq g_w(b_{in,1}, \ldots, b_{in,m})$. There are two cases:

Case 1: There exists a smallest interval $[g_w(x(r_{j1})), g_w(x(r_{j2}))]$ containing the both values $g_w(a_{in,1}, \ldots, a_{in,m})$ and $g_w(b_{in,1}, \ldots, b_{in,m})$ computed from the two given inputs. As $g_w(\boldsymbol{x}(r_{j1})) < g_w(\boldsymbol{x}(r_{j2}))$, the two linear interpolation values of the two input vectors, $\text{Out}(\boldsymbol{a}_{in}) = IntM_{2\mathcal{RB}}(g_w(a_{in,1}, \ldots, a_{in,m}))$ and $\text{Out}(\boldsymbol{b}_{in}) = IntM_{2\mathcal{RB}}(g_w(b_{in,1}, \ldots, b_{in,m}))$, which both lie on the interpolation line connecting two points $(g_w(\boldsymbol{x}(r_{j1})), \text{SQM}_{m+1}(r_{j1}|\mathcal{X}_{m+1}))$ and $(g_w(\boldsymbol{x}(r_{j2})), \text{SQM}_{m+1}(r_{j2}|\mathcal{X}_{m+1}))$.

As $\text{SQM}_{m+1}(r_{j1}|\mathcal{X}_{m+1}) = \text{SQM}_{m+1}(x_{j1,m+1}) < \text{SQM}_{m+1}(r_{j1}|\mathcal{X}_{m+1}) = \text{SQM}_{m+1}(x_{j2,m+1})$ and $g_w(a_{in,1}, \ldots, a_{in,m}) \leq g_w(b_{in,1}, \ldots, b_{in,m})$, we must have $L\_IntM_{2,w}(g_w(a_{in,1}, \ldots, a_{in,m})) < L\_IntM_{2,w}(g_w(b_{in,1}, \ldots, b_{in,m}))$. I.e. the linear interpolative approximate reasoning method $L\_IntM_{2,w}$ preserves the increasing monotonicity of the linguistic rule base $\mathcal{RB}$.

Case 2: The two values $g_w(a_{in,1}, \ldots, a_{in,m})$ and $g_w(b_{in,1}, \ldots, b_{in,m})$ lie on different intervals $I_1 = [g_w(\boldsymbol{x}(r_{j1})), g_w(\boldsymbol{x}(r_{j1*}))]$ and $I_2 = [g_w(\boldsymbol{x}(r_{j2})), g_w(\boldsymbol{x}(r_{j2*}))]$ created by the adjacent horizon coordinates of the grid $\text{Grid}_2(\mathcal{RB})$ in $[0, 1]^2$. Assume that $g_w(a_{in,1}, \ldots, a_{in,m}) \in I_1$ and $g_w(b_{in,1}, \ldots, b_{in,m}) \in I_2$, we infer $I_1 < I_2$ and due to increasing monotonicity of $\mathcal{RB}$, we also have $\text{SQM}_{m+1}(r_{j1*}|\mathcal{X}_{m+1}) \leq \text{SQM}_{m+1}(r_{j2}|\mathcal{X}_{m+1})$, where $\text{SQM}_{m+1}(r_{j1*}|\mathcal{X}_{m+1})$ and $\text{SQM}_{m+1}(r_{j2}|\mathcal{X}_{m+1})$ are two values of $\text{Grid}_2(\mathcal{RB})$, the first of which is the right end-point of I_1 and and the other is the left end-point of I_2. Also as $\mathcal{RB}$ is increasing, we infer that $L\_IntM_{2,w}(g_w(a_{in,1}, \ldots, a_{in,m})) < L\_IntM_{2,w}(g_w(b_{in,1}, \ldots, b_{in,m}))$. I.e. the linear interpolative approximate reasoning method $Li\_IntM_{2,w}$ also preserves the increasing monotonicity of the linguistic rule base $\mathcal{RB}$ in this case. The theorem is proved.

4. CONCLUSIONS

On the basis of more specific formalized analysis on the RWS-interpretability of basic components in fuzzy systems, especially of the composition of linguistic rule base and approximate reasoning methods running on them, the study has solved the following main issues:

It is pointed out that the study of interpretability is essential to ensure that the manipulation, calculation or reasoning in a formalism of a theory or a methodology to draw a conclusion/action must be compatible and appropriate to the RW-semantics of their respective RW-counterparts when they interact with them. However, this is also a challenging problem, e.g. the methodologies within the fuzzy set framework are in general not RWS-interpretable. Therefore, there is no formal basis to ensure that the fuzzy representations of linguistic rule bases and fuzzy reasoning methods on them constructed in the fuzzy set framework are RWS-interpretable.

After analyzing the aggregation/synthesis of the semantic information of composed elements of a linguistic rule by the aggregation operators within the fuzzy set, such as t-norm, s-norm and implication to show that it is hard to have a formal basis to ensure that they can preserve the RW-semantics of linguistic rules, the study proposes a computational representation method of linguistic rule base by graphs in Euclidean space. The article has demonstrated that the proposed graphical representation method is RWS-interpretable.

It is argued that approximate reasoning method is one of key distinguished component of fuzzy systems and its RWS-interpretability problem must be defined and solved in a closed relation with the RWS-interpretability of linguistic rule bases. It is demonstrated that there exists an RWS-interpretable approximate reasoning method working on the above graphical representations produced by the proposed computational representation method.

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