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KINEMATIC AND DYNAMIC ANALYSIS OF MULTIBODY SYSTEMS USING THE KRONECKER PRODUCT

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Abstract. Using the Kronecker product, a much larger size matrix can be formed from two matrix operands; therefore, the capability of matrix algebra in analyzing the kinematics and dynamics of multibody systems are extended. This paper employs Khang's definition of the partial derivative of a matrix with respect to a vector and the Kronecker product to define translational and rotational Hessian matrices. With these definitions, the generalized velocities in the expression of a linear acceleration or an angular acceleration are collected into a quadratic term. The relations of Jacobian and Hessian matrices in relative motion are then established. A new matrix form of Lagrange's equations showing clearly the quadratic term of generalized velocities is also introduced.

Keywords: Jacobian matrix, Hessian matrix, Kronecker product, velocity-free Coriolis matrix, matrix form of Lagrange's equations.

Classification numbers: 5.4.1

1. INTRODUCTION

Matrix operations are commonly used in the field of multibody system due to the convenience of writing generalized formulas. However, basic operations such as matrix multiplication and addition are not enough in certain cases. For instance, while rotational and translational Jacobian matrices are popular, their partial derivatives, Hessian matrices, are defined differently by different authors [1, 2] and are much less common. Another example is that the Coriolis/centripetal matrix is usually calculated using Christoffel symbols [3] instead of matrix operations.

Using the Kronecker product, research by Khang [4, 5] presents a consistent definition of the partial derivative of a matrix with respect to a vector and its properties. Khang's work does not bring about any new physics but the convenience of using pure matrix notation while establishing equations of motion of multibody system. Equations establishment is usually not visible in publications so that it is hard to tell how much interest the work has drawn.

Nevertheless, a few research groups, aside from Khang's, clearly state that they adopt his work [6, 7]. To the main author's point of view, the most notable citation is Taghirad's book [8] because it may effectively introduce the use of Kronecker product to researchers and students who are new to this field so that they will more probably use this method while experienced ones may prefer the method they are already familiar with. Khang's own book serves the same purpose for Vietnamese readers [9].

Seeing Khang's work to be potentially subject to development, this paper seeks to extend it to achieve better matrix formulations in kinematic and dynamic analysis of multibody system.

2. MATRIX ALGEBRA WITH KRONECKER PRODUCT AND SOME OTHER MATRIX OPERATORS

2.1. Kronecker product and partial derivative of a matrix with respect to a vector

Definition 1. Kronecker product of two matrices

The Kronecker product of two matrices $\mathbf{A}_{m\times n} = [a_{ij}]$ and $\mathbf{B}_{p\times q}$ is an $mp \times nq$ matrix given by [10, 11, 12]

$$\mathbf{A}\otimes\mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \dots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \dots & a_{mn}\mathbf{B} \end{bmatrix}. \tag{1}$$

Some of the important properties of the Kronecker product are [10, 11, 12]

$$(\mathbf{A}\otimes\mathbf{B})\otimes\mathbf{C} = \mathbf{A}\otimes(\mathbf{B}\otimes\mathbf{C}), \tag{2}$$

$$(\mathbf{A}\otimes\mathbf{B})^T = \mathbf{A}^T\otimes\mathbf{B}^T, \tag{3}$$

and if there exist matrix products $\mathbf{AC}$ and $\mathbf{BD}$, we have

$$(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D}) = (\mathbf{AC})\otimes(\mathbf{BD}). \tag{4}$$

Definition 2. Partial derivative of a matrix with respect to a vector

There are many ways to define the partial derivative of a matrix with respect to a vector. Here the definition by [4, 5] is used. The partial derivative of an $r\times s$ matrix $\mathbf{A}(\mathbf{x})$ which is a matrix function of an $n\times 1$ vector $\mathbf{x}$ with respect to vector $\mathbf{x}$ is an $r\times sn$ matrix given as

$$\frac{\partial\mathbf{A}(\mathbf{x})}{\partial\mathbf{x}} = \begin{bmatrix} \dfrac{\partial\mathbf{a}_1}{\partial\mathbf{x}} & \dfrac{\partial\mathbf{a}_2}{\partial\mathbf{x}} & \dots & \dfrac{\partial\mathbf{a}_s}{\partial\mathbf{x}} \end{bmatrix} \tag{5}$$

where $\mathbf{a}_i$ is the i-th column of matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_s \end{bmatrix}. \tag{6}$$

Definition 3. Vec operator of a matrix

The vec operator of matrix $\mathbf{A}$ in (6) is given as [11, 12]

$$vec(\mathbf{A}) = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_s \end{bmatrix}. \tag{7}$$

With this operator, all the elements in $\mathbf{A}$ are rearranged to form a vector.

Theorem 1. The derivative with respect to time of an $r \times s$ matrix function $\mathbf{A}(\mathbf{x})$ where $\mathbf{x}(t)$ is an $n \times 1$ vector function of time t satisfies [4, 5]

$$\dot{\mathbf{A}}(\mathbf{x}) = \frac{d\mathbf{A}(\mathbf{x})}{dt} = \frac{\partial \mathbf{A}(\mathbf{x})}{\partial \mathbf{x}} (\mathbf{E}_s \otimes \dot{\mathbf{x}}) \tag{8}$$

where $\mathbf{E}_s$ is the $s \times s$ identity matrix.

Theorem 2. The partial derivative with respect to an $n \times 1$ vector $\mathbf{x}$ of a matrix product $\mathbf{A}(\mathbf{x})\mathbf{B}(\mathbf{x})$ satisfies [4, 5]

$$\frac{\partial (\mathbf{A}(\mathbf{x})\mathbf{B}(\mathbf{x}))}{\partial \mathbf{x}} = \frac{\partial \mathbf{A}(\mathbf{x})}{\partial \mathbf{x}} (\mathbf{B}(\mathbf{x}) \otimes \mathbf{E}_n) + \mathbf{A}(\mathbf{x}) \frac{\partial \mathbf{B}(\mathbf{x})}{\partial \mathbf{x}} . \tag{9}$$

Theorem 3. The vec operator of a matrix product $\mathbf{AXB}$ satisfies [11, 12]

$$vec(\mathbf{AXB}) = (\mathbf{B}^T \otimes \mathbf{A}) vec(\mathbf{X}) . \tag{10}$$

Using theorems 1 and 2 and properties of the Kronecker product, the following theorems are also proved

a)
$$\frac{d(\mathbf{J}_{r\times n}(\mathbf{q}_{n\times 1})\dot{\mathbf{q}})}{dt} = \mathbf{J}(\mathbf{q})\ddot{\mathbf{q}} + \frac{\partial \mathbf{J}(\mathbf{q})}{\partial \mathbf{q}} (\dot{\mathbf{q}} \otimes \dot{\mathbf{q}}) . \tag{11}$$

Proof.

$$\frac{d(\mathbf{J}(\mathbf{q})\dot{\mathbf{q}})}{dt} = \mathbf{J}(\mathbf{q})\ddot{\mathbf{q}} + \frac{d(\mathbf{J}(\mathbf{q}))}{dt}\dot{\mathbf{q}} = \mathbf{J}(\mathbf{q})\ddot{\mathbf{q}} + \frac{\partial \mathbf{J}(\mathbf{q})}{\partial \mathbf{q}} (\dot{\mathbf{q}} \otimes \mathbf{E}_n)(\mathbf{E}_1 \otimes \dot{\mathbf{q}})$$

$$= \mathbf{J}(\mathbf{q})\ddot{\mathbf{q}} + \frac{\partial \mathbf{J}(\mathbf{q})}{\partial \mathbf{q}} (\dot{\mathbf{q}}\mathbf{E}_1) \otimes (\mathbf{E}_n \dot{\mathbf{q}}) = \mathbf{J}(\mathbf{q})\ddot{\mathbf{q}} + \frac{\partial \mathbf{J}(\mathbf{q})}{\partial \mathbf{q}} (\dot{\mathbf{q}} \otimes \dot{\mathbf{q}}) .$$

b)
$$(\mathbf{E}_p \otimes \mathbf{x}_{n\times 1})\mathbf{A}_{p\times m}\mathbf{d}_{m\times 1} = (\mathbf{A} \otimes \mathbf{E}_n)(\mathbf{d} \otimes \mathbf{x}) . \tag{12}$$

Proof.

$$(\mathbf{E}_p \otimes \mathbf{x})\mathbf{A}\mathbf{d} = (\mathbf{E}_p \otimes \mathbf{x})(\mathbf{A}\mathbf{d} \otimes \mathbf{E}_1) = (\mathbf{E}_p \mathbf{A}\mathbf{d}) \otimes (\mathbf{x}\mathbf{E}_1) = (\mathbf{A}\mathbf{d}) \otimes (\mathbf{E}_n \mathbf{x}) = (\mathbf{A} \otimes \mathbf{E}_n)(\mathbf{d} \otimes \mathbf{x}) .$$

c)
$$\mathbf{d}_{p\times 1} \otimes \mathbf{x}_{n\times 1} = (\mathbf{d} \otimes \mathbf{E}_n)\mathbf{x} . \tag{13}$$

Proof.

$$\mathbf{d}_{p\times 1} \otimes \mathbf{x}_{n\times 1} = (\mathbf{d}\mathbf{E}_1) \otimes (\mathbf{E}_n \mathbf{x}) = (\mathbf{d} \otimes \mathbf{E}_n)(\mathbf{E}_1 \otimes \mathbf{x}) = (\mathbf{d} \otimes \mathbf{E}_n)\mathbf{x} .$$

d)
$$(\mathbf{E}_p \otimes \mathbf{x}_{n\times 1})\mathbf{A}_{p\times rm}(\mathbf{E}_r \otimes \mathbf{y}_{m\times 1})\mathbf{d}_{r\times 1} = (\mathbf{A} \otimes \mathbf{E}_n)(\mathbf{d} \otimes \mathbf{E}_{nm})(\mathbf{y} \otimes \mathbf{x}) . \tag{14}$$

Proof.

$$(\mathbf{E}_p \otimes \mathbf{x})\mathbf{A}(\mathbf{E}_r \otimes \mathbf{y})\mathbf{d} = (\mathbf{E}_p \otimes \mathbf{x})\mathbf{A}(\mathbf{E}_r \otimes \mathbf{y})\mathbf{E}_r\mathbf{d} = (\mathbf{E}_p \otimes \mathbf{x})\mathbf{A}(\mathbf{E}_r \otimes \mathbf{E}_m)(\mathbf{d} \otimes \mathbf{y})$$
$$= (\mathbf{E}_p \otimes \mathbf{x})\mathbf{A}(\mathbf{d} \otimes \mathbf{y}) = (\mathbf{A} \otimes \mathbf{E}_n)(\mathbf{d} \otimes \mathbf{y} \otimes \mathbf{x}) = (\mathbf{A} \otimes \mathbf{E}_n)(\mathbf{d} \otimes \mathbf{E}_{nm})(\mathbf{y} \otimes \mathbf{x})$$

in which (12) and (13) are also used.

2.2. Skew-symmetric matrix associated to cross product and its generalization

To present the cross product of two 3×1 vectors in the form of the matrix product, the skew-symmetric matrix of vector

$$\mathbf{a} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}^T$$

is defined as [9, 13]

$$\tilde{\mathbf{a}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}. \tag{15}$$

We can also define the block skew-symmetric matrix of a 3-row matrix

$$\mathbf{A} = \begin{bmatrix} \boldsymbol{\alpha}_1^T \\ \boldsymbol{\alpha}_2^T \\ \boldsymbol{\alpha}_3^T \end{bmatrix}$$

as

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{0}^T & -\boldsymbol{\alpha}_3^T & \boldsymbol{\alpha}_2^T \\ \boldsymbol{\alpha}_3^T & \mathbf{0}^T & -\boldsymbol{\alpha}_1^T \\ -\boldsymbol{\alpha}_2^T & \boldsymbol{\alpha}_1^T & \mathbf{0}^T \end{bmatrix}. \tag{16}$$

With this definition, we have

$$\mathbf{A}\mathbf{a} = \tilde{\mathbf{A}}(\mathbf{E}_3 \otimes \mathbf{a}) \tag{17}$$

and

$$\tilde{\mathbf{a}}\mathbf{A}_{3\times m} = -\tilde{\mathbf{A}}(\mathbf{a} \otimes \mathbf{E}_m)\,. \tag{18}$$

3. TRANSLATIONAL AND ROTATIONAL JACOBIAN AND HESSIAN MATRICES

3.1. Translational and rotational Jacobian matrices

Consider two frames $(a): O_a x_a y_a z_a$ and $(b): O_b x_b y_b z_b$, the linear velocity of frame (b) with respect to frame (a) is determined in the form of an algebraic vector ${}^a\mathbf{v}_b$ as

$${}^a\mathbf{v}_b^{(a)} = {}^a\dot{\mathbf{d}}_b^{(a)} \tag{19}$$

where ${}^a\mathbf{d}_b$ is the algebraic form of $\overrightarrow{O_aO_b}$. The right superscripts indicate the frames on which the vectors are based.

The angular velocity of frame (b) with respect to frame (a) ${}^{a}\boldsymbol{\omega}_{b}$ can be calculate as follows [9, 13]

$$ {}^{a}\tilde{\boldsymbol{\omega}}_{b}^{(a)} = \dot{\mathbf{A}}_{b}^{(a)}\mathbf{A}_{b}^{(a)T} \tag{20}$$

or

$$ {}^{a}\tilde{\boldsymbol{\omega}}_{b}^{(b)} = \mathbf{A}_{b}^{(a)T}\dot{\mathbf{A}}_{b}^{(a)} \tag{21}$$

where $\mathbf{A}_{b}^{(a)}$ is the direction cosine matrix of frame (b) with respect to frame (a) and is determined as

$$\mathbf{A}_{b}^{(a)} = [\mathbf{x}_{b}^{(a)}, \mathbf{y}_{b}^{(a)}, \mathbf{z}_{b}^{(a)}] \tag{22}$$

with $\mathbf{x}_{b}^{(a)}$, $\mathbf{y}_{b}^{(a)}$ and $\mathbf{z}_{b}^{(a)}$ are the unit vectors of the axes of frame (b) written in frame (a).

Note that

$$\mathbf{A}_{b}^{(a)T} = \mathbf{A}_{a}^{(b)} \tag{23}$$

and

$$\mathbf{A}_{a}^{(b)}\mathbf{u}^{(a)} = \mathbf{u}^{(b)} \tag{24}$$

with $\mathbf{u}^{(\cdot)}$ is the algebraic form of an arbitrary vector.

Now suppose that the position and direction of frame (b) with respect to frame (a) is determined by a vector of variables $\mathbf{q}$

$$\mathbf{q} = \begin{matrix} q_1 & q_2 & \dots & q_n \end{matrix}^{T}. \tag{25}$$

It means

$$ {}^{a}\mathbf{d}_{b}^{(a)} = {}^{a}\mathbf{d}_{b}^{(a)}(\mathbf{q}) \tag{26}$$

and

$$\mathbf{A}_{b}^{(a)} = \mathbf{A}_{b}^{(a)}(\mathbf{q}). \tag{27}$$

Taking derivative of (26) with respect to time and noting (19) yield

$$ {}^{a}\mathbf{v}_{b}^{(a)} = \frac{\partial\, {}^{a}\mathbf{d}_{b}^{(a)}(\mathbf{q})}{\partial \mathbf{q}}\dot{\mathbf{q}}. \tag{28}$$

With the introduction of the translational Jacobian matrix ${}^{a}\mathbf{J}_{T_b}^{(a)}$ of frame (b) with respect to frame (a)

$$ {}^{a}\mathbf{J}_{T_b}^{(a)}(\mathbf{q}) = \frac{\partial\, {}^{a}\mathbf{d}_{b}^{(a)}(\mathbf{q})}{\partial \mathbf{q}}, \tag{29}$$

equation (28) can be rewritten as

$$ {}^{a}\mathbf{v}_{b}^{(a)} = {}^{a}\mathbf{J}_{T_b}^{(a)}\dot{\mathbf{q}}. \tag{30}$$

Denoting

$$ {}^{a}\mathbf{J}_{T_b}^{(b)}(\mathbf{q}) = \mathbf{A}_a^{(b)}(\mathbf{q})\frac{\partial\, {}^{a}\mathbf{d}_b^{(a)}(\mathbf{q})}{\partial \mathbf{q}} = \mathbf{A}_a^{(b)}(\mathbf{q})\, {}^{a}\mathbf{J}_{T_b}^{(a)}(\mathbf{q}), \tag{31} $$

we also have

$$ {}^{a}\mathbf{v}_b^{(b)} = {}^{a}\mathbf{J}_{T_b}^{(b)}\dot{\mathbf{q}}. \tag{32} $$

In general, one can write

$$ {}^{a}\mathbf{v}_b^{(k)} = {}^{a}\mathbf{J}_{T_b}^{(k)}\dot{\mathbf{q}} \tag{33} $$

and

$$ {}^{a}\mathbf{J}_{T_b}^{(k)} = \mathbf{A}_l^{(k)}\, {}^{a}\mathbf{J}_{T_b}^{(l)} \tag{34} $$

but it should be noted that ${}^{a}\mathbf{J}_{T_b}^{(.)}$ in this case may depend on variables other than those in $\mathbf{q}$ and (34) is only valid if the elements in $\dot{\mathbf{q}}$ are linearly independent.

Equation (20) can be rewritten as

$$ {}^{a}\tilde{\boldsymbol{\omega}}_b^{(a)} = \dot{\mathbf{A}}_a^{(b)T}\mathbf{A}_a^{(b)} = \begin{bmatrix} 0 & \dot{\mathbf{x}}_a^{(b)T}\mathbf{y}_a^{(b)} & \dot{\mathbf{x}}_a^{(b)T}\mathbf{z}_a^{(b)} \\ \dot{\mathbf{y}}_a^{(b)T}\mathbf{x}_a^{(b)} & 0 & \dot{\mathbf{y}}_a^{(b)T}\mathbf{z}_a^{(b)} \\ \dot{\mathbf{z}}_a^{(b)T}\mathbf{x}_a^{(b)} & \dot{\mathbf{z}}_a^{(b)T}\mathbf{y}_a^{(b)} & 0 \end{bmatrix}. \tag{35} $$

Hence,

$$ {}^{a}\boldsymbol{\omega}_b^{(a)} = \begin{bmatrix} \dot{\mathbf{z}}_a^{(b)T}\mathbf{y}_a^{(b)} \\ \dot{\mathbf{x}}_a^{(b)T}\mathbf{z}_a^{(b)} \\ \dot{\mathbf{y}}_a^{(b)T}\mathbf{x}_a^{(b)} \end{bmatrix} = \begin{bmatrix} \mathbf{y}_a^{(b)T}\dot{\mathbf{z}}_a^{(b)} \\ \mathbf{z}_a^{(b)T}\dot{\mathbf{x}}_a^{(b)} \\ \mathbf{x}_a^{(b)T}\dot{\mathbf{y}}_a^{(b)} \end{bmatrix} = \begin{bmatrix} \mathbf{y}_a^{(b)T}\dfrac{\partial \mathbf{z}_a^{(b)}}{\partial \mathbf{q}} \\ \mathbf{z}_a^{(b)T}\dfrac{\partial \mathbf{x}_a^{(b)}}{\partial \mathbf{q}} \\ \mathbf{x}_a^{(b)T}\dfrac{\partial \mathbf{y}_a^{(b)}}{\partial \mathbf{q}} \end{bmatrix}\dot{\mathbf{q}}. \tag{36} $$

Now denote the rotational Jacobian matrix of frame (b) with respect to frame (a)

$$ {}^{a}\mathbf{J}_{R_b}^{(a)}(\mathbf{q}) = \begin{bmatrix} \mathbf{y}_a^{(b)T}\dfrac{\partial \mathbf{z}_a^{(b)}}{\partial \mathbf{q}} \\ \mathbf{z}_a^{(b)T}\dfrac{\partial \mathbf{x}_a^{(b)}}{\partial \mathbf{q}} \\ \mathbf{x}_a^{(b)T}\dfrac{\partial \mathbf{y}_a^{(b)}}{\partial \mathbf{q}} \end{bmatrix}. \tag{37} $$

Equation (36) is rewritten as

$$ {}^{a}\boldsymbol{\omega}_b^{(a)} = {}^{a}\mathbf{J}_{R_b}^{(a)}\dot{\mathbf{q}}. \tag{38} $$

Similarly, from (21) we have

$$ {}^{a}\boldsymbol{\omega}_b^{(b)} = {}^{a}\mathbf{J}_{R_b}^{(b)}\dot{\mathbf{q}} \tag{39} $$

where

$$ {}^{a}\mathbf{J}_{R_b}^{(b)}(\mathbf{q}) = \begin{bmatrix} \mathbf{z}_b^{(a)T} \dfrac{\partial \mathbf{y}_b^{(a)}}{\partial \mathbf{q}} \\ \mathbf{x}_b^{(a)T} \dfrac{\partial \mathbf{z}_b^{(a)}}{\partial \mathbf{q}} \\ \mathbf{y}_b^{(a)T} \dfrac{\partial \mathbf{x}_b^{(a)}}{\partial \mathbf{q}} \end{bmatrix}. \tag{40} $$

Similar to the case of translational Jacobian matrix, one can write

$$ {}^{a}\boldsymbol{\omega}_b^{(k)} = {}^{a}\mathbf{J}_{R_b}^{(k)}\dot{\mathbf{q}} \tag{41} $$

and, if the elements in $\dot{\mathbf{q}}$ are linearly independent,

$$ {}^{a}\mathbf{J}_{R_b}^{(k)} = \mathbf{A}_l^{(k)}\,{}^{a}\mathbf{J}_{R_b}^{(l)}. \tag{42} $$

It is important in dynamics to determine not only the motion of the frame origin but also the velocity of the center of mass of a rigid body attached to that frame. Denote $\mathbf{p}_b^{(.)}$ as the algebraic vector of $\overrightarrow{O_b G_b}$, we have

$$ \begin{aligned} {}^{a}\mathbf{v}_{G_b}^{(a)} &= \frac{d}{dt}({}^{a}\mathbf{d}_b^{(a)} + \mathbf{p}_b^{(a)}) = \frac{d}{dt}({}^{a}\mathbf{d}_b^{(a)} + \mathbf{A}_b^{(a)}\mathbf{p}_b^{(b)}) = {}^{a}\mathbf{v}_b^{(a)} + \dot{\mathbf{A}}_b^{(a)}\mathbf{p}_b^{(b)} = {}^{a}\mathbf{v}_b^{(a)} + \dot{\mathbf{A}}_b^{(a)}\mathbf{A}_b^{(a)T}\mathbf{A}_b^{(a)}\mathbf{p}_b^{(b)} \\ &= {}^{a}\mathbf{v}_b^{(a)} + {}^{a}\tilde{\boldsymbol{\omega}}_b^{(a)}\mathbf{p}_b^{(a)} = {}^{a}\mathbf{J}_{T_b}^{(a)}\dot{\mathbf{q}} - \tilde{\mathbf{p}}_b^{(a)}\,{}^{a}\boldsymbol{\omega}_b^{(a)} = {}^{a}\mathbf{J}_{T_b}^{(a)}\dot{\mathbf{q}} - \tilde{\mathbf{p}}_b^{(a)}\,{}^{a}\mathbf{J}_{R_b}^{(a)}\dot{\mathbf{q}} \\ &= ({}^{a}\mathbf{J}_{T_b}^{(a)} - \tilde{\mathbf{p}}_b^{(a)}\,{}^{a}\mathbf{J}_{R_b}^{(a)})\dot{\mathbf{q}} \end{aligned} \tag{43} $$

or

$$ {}^{a}\mathbf{v}_{G_b}^{(a)} = {}^{a}\mathbf{J}_{TG_b}^{(a)}\dot{\mathbf{q}} \tag{44} $$

with ${}^{a}\mathbf{J}_{TG_b}^{(a)}$ is the translational Jacobian matrix of point G_b on frame (b) with respect to frame (a)

$$ {}^{a}\mathbf{J}_{TG_b}^{(a)} = {}^{a}\mathbf{J}_{T_b}^{(a)} - \tilde{\mathbf{p}}_b^{(a)}\,{}^{a}\mathbf{J}_{R_b}^{(a)} = {}^{a}\mathbf{J}_{T_b}^{(a)} + {}^{a}\tilde{\mathbf{J}}_{R_b}^{(a)}(\mathbf{p}_b^{(a)} \otimes \mathbf{E}_n). \tag{45} $$

Similarly, we have

$$ {}^{a}\mathbf{v}_{G_b}^{(b)} = {}^{a}\mathbf{J}_{TG_b}^{(b)}\dot{\mathbf{q}} \tag{46} $$

where

$$ {}^{a}\mathbf{J}_{TG_b}^{(b)} = {}^{a}\mathbf{J}_{T_b}^{(b)} - \tilde{\mathbf{p}}_b^{(b)}\,{}^{a}\mathbf{J}_{R_b}^{(b)}. \tag{47} $$

Base-changing rule is also valid in this case

$$ {}^{a}\mathbf{J}_{TG_b}^{(b)} = \mathbf{A}_a^{(b)}\,{}^{a}\mathbf{J}_{TG_b}^{(a)}. \tag{48} $$

Equations (44)-(48) are applicable to any point fixed on frame (b).

3.2. Translational and rotational Hessian matrices

Taking derivative of (30) with respect to time and noting (11) yield

$$ {}^{a}\dot{\mathbf{v}}_b^{(a)} = {}^{a}\mathbf{J}_{T_b}^{(a)}\ddot{\mathbf{q}} + \frac{\partial\, {}^{a}\mathbf{J}_{T_b}^{(a)}}{\partial \mathbf{q}}(\dot{\mathbf{q}} \otimes \dot{\mathbf{q}}) . \tag{49} $$

The left-hand side of (49) is the linear acceleration ${}^{a}\mathbf{a}_b^{(a)}$ of frame (b) with respect to frame (a):

$$ {}^{a}\mathbf{a}_b^{(a)} = {}^{a}\dot{\mathbf{v}}_b^{(a)} . \tag{50} $$

We now introduce the translational Hessian matrix ${}^{a}\mathbf{H}_{T_b}^{(a)}$ of frame (b) with respect to frame (a)

$$ {}^{a}\mathbf{H}_{T_b}^{(a)}(\mathbf{q}) = \frac{\partial\, {}^{a}\mathbf{J}_{T_b}^{(a)}(\mathbf{q})}{\partial \mathbf{q}} = \frac{\partial^2\, {}^{a}\mathbf{d}_b^{(a)}(\mathbf{q})}{\partial \mathbf{q}^2} \tag{51} $$

and rewrite (49) as

$$ {}^{a}\mathbf{a}_b^{(a)} = {}^{a}\mathbf{J}_{T_b}^{(a)}\ddot{\mathbf{q}} + {}^{a}\mathbf{H}_{T_b}^{(a)}(\dot{\mathbf{q}} \otimes \dot{\mathbf{q}}) . \tag{52} $$

However, note that (51) is not the only matrix ${}^{a}\mathbf{H}_{T_b}^{(a)}$ satisfying (52) because the elements in $(\dot{\mathbf{q}} \otimes \dot{\mathbf{q}})$ are not linearly independent.

It should also be noted that in mathematics, the Hessian matrix might be understood to be a square matrix with elements being second derivatives of a scalar function [14], which is not applicable here because the position of a frame is a vector function.

Similarly, denoting

$$ {}^{a}\mathbf{H}_{T_b}^{(b)}(\mathbf{q}) = \frac{\partial\, {}^{a}\mathbf{J}_{T_b}^{(b)}(\mathbf{q})}{\partial \mathbf{q}} , \tag{53} $$

we have

$$ {}^{a}\dot{\mathbf{v}}_b^{(b)} = {}^{a}\mathbf{J}_{T_b}^{(b)}\ddot{\mathbf{q}} + {}^{a}\mathbf{H}_{T_b}^{(b)}(\dot{\mathbf{q}} \otimes \dot{\mathbf{q}}) . \tag{54} $$

However, it should be noted that

$$ {}^{a}\dot{\mathbf{v}}_b^{(b)} \neq {}^{a}\mathbf{a}_b^{(b)} \tag{55} $$

and

$$ {}^{a}\mathbf{H}_{T_b}^{(b)}(\mathbf{q}) \neq \mathbf{A}_a^{(b)}\, {}^{a}\mathbf{H}_{T_b}^{(a)}(\mathbf{q}) . \tag{56} $$

The two Hessian matrices are related by the following expression

$$ {}^{a}\mathbf{H}_{T_b}^{(a)} = \frac{\partial}{\partial \mathbf{q}}(\mathbf{A}_b^{(a)}\, {}^{a}\mathbf{J}_{T_b}^{(b)}) = \mathbf{A}_b^{(a)}\, {}^{a}\mathbf{H}_{T_b}^{(b)} + \frac{\partial \mathbf{A}_b^{(a)}}{\partial \mathbf{q}}({}^{a}\mathbf{J}_{T_b}^{(b)}(\mathbf{q}) \otimes \mathbf{E}_n) . \tag{57} $$

Denoting

$$ {}^{a}\mathbf{H}_{T_b}^{*(b)} = \mathbf{A}_a^{(b)}\, {}^{a}\mathbf{H}_{T_b}^{(a)} , \tag{58} $$

we have

$$ {}^{a}\mathbf{a}_b^{(b)} = {}^{a}\mathbf{J}_{T_b}^{(b)}\ddot{\mathbf{q}} + {}^{a}\mathbf{H}_{T_b}^{*(b)}(\dot{\mathbf{q}} \otimes \dot{\mathbf{q}}) . \tag{59} $$

Similarly, we define the rotational Hessian matrices

$$
{}^{a}\mathbf{H}_{R_b}^{(a)}(\mathbf{q}) = \frac{\partial\, {}^{a}\mathbf{J}_{R_b}^{(a)}(\mathbf{q})}{\partial \mathbf{q}} \tag{60}
$$

and

$$
{}^{a}\mathbf{H}_{R_b}^{(b)}(\mathbf{q}) = \frac{\partial\, {}^{a}\mathbf{J}_{R_b}^{(b)}(\mathbf{q})}{\partial \mathbf{q}}\,. \tag{61}
$$

Taking derivative of (38) and (39) with respect to time yields

$$
{}^{a}\boldsymbol{\alpha}_{b}^{(a)} = {}^{a}\dot{\boldsymbol{\omega}}_{b}^{(a)} = {}^{a}\mathbf{J}_{R_b}^{(a)}\ddot{\mathbf{q}} + \frac{\partial\, {}^{a}\mathbf{J}_{R_b}^{(a)}}{\partial \mathbf{q}}(\dot{\mathbf{q}}\otimes\dot{\mathbf{q}}) = {}^{a}\mathbf{J}_{R_b}^{(a)}\ddot{\mathbf{q}} + {}^{a}\mathbf{H}_{R_b}^{(a)}(\dot{\mathbf{q}}\otimes\dot{\mathbf{q}}) \tag{62}
$$

and

$$
{}^{a}\dot{\boldsymbol{\omega}}_{b}^{(b)} = {}^{a}\mathbf{J}_{R_b}^{(b)}\ddot{\mathbf{q}} + \frac{\partial\, {}^{a}\mathbf{J}_{R_b}^{(b)}}{\partial \mathbf{q}}(\dot{\mathbf{q}}\otimes\dot{\mathbf{q}}) = {}^{a}\mathbf{J}_{R_b}^{(b)}\ddot{\mathbf{q}} + {}^{a}\mathbf{H}_{R_b}^{(b)}(\dot{\mathbf{q}}\otimes\dot{\mathbf{q}})\,. \tag{63}
$$

Unlike the case of translation, we have the simple relation of angular acceleration

$$
{}^{a}\dot{\boldsymbol{\omega}}_{b}^{(b)} = {}^{a}\boldsymbol{\alpha}_{b}^{(b)}\,. \tag{64}
$$

This can be proved as follows

$$
\begin{aligned}
{}^{a}\dot{\boldsymbol{\omega}}_{b}^{(b)} &= \frac{d(\mathbf{A}_{a}^{(b)}\,{}^{a}\boldsymbol{\omega}_{b}^{(a)})}{dt} = \mathbf{A}_{a}^{(b)}\,{}^{a}\dot{\boldsymbol{\omega}}_{b}^{(a)} + \dot{\mathbf{A}}_{a}^{(b)}\,{}^{a}\boldsymbol{\omega}_{b}^{(a)} = \mathbf{A}_{a}^{(b)}\,{}^{a}\boldsymbol{\alpha}_{b}^{(a)} + (\mathbf{A}_{b}^{(a)}\dot{\mathbf{A}}_{b}^{(a)})^{T}\,\mathbf{A}_{a}^{(b)}\,{}^{a}\boldsymbol{\omega}_{b}^{(a)} \\
&= {}^{a}\boldsymbol{\alpha}_{b}^{(b)} - {}^{a}\tilde{\boldsymbol{\omega}}_{b}^{(b)}\,{}^{a}\boldsymbol{\omega}_{b}^{(b)} = {}^{a}\boldsymbol{\alpha}_{b}^{(b)}\,.
\end{aligned}
$$

As a consequence, from (62)-(64) one can write

$$
{}^{a}\mathbf{H}_{R_b}^{(b)}(\dot{\mathbf{q}}\otimes\dot{\mathbf{q}}) = \mathbf{A}_{a}^{(b)}\,{}^{a}\mathbf{H}_{R_b}^{(a)}(\dot{\mathbf{q}}\otimes\dot{\mathbf{q}})\,. \tag{65}
$$

It should be note that we cannot simply eliminate $(\dot{\mathbf{q}}\otimes\dot{\mathbf{q}})$ from both sides of (65) because this vector includes linearly dependent elements. Nevertheless, in practice, the Hessian matrices always go with $(\dot{\mathbf{q}}\otimes\dot{\mathbf{q}})$ so when calculating Hessian matrices, one does not have to care much about this problem.

Deriving (44) with respect to time, one obtains

$$
{}^{a}\mathbf{a}_{G_b}^{(a)} = {}^{a}\mathbf{J}_{TG_b}^{(a)}\ddot{\mathbf{q}} + {}^{a}\mathbf{H}_{TG_b}^{(a)}(\dot{\mathbf{q}}\otimes\dot{\mathbf{q}}) \tag{66}
$$

where

$$
{}^{a}\mathbf{H}_{TG_b}^{(a)} = \frac{\partial\, {}^{a}\mathbf{J}_{TG_b}^{(a)}}{\partial \mathbf{q}}\,. \tag{67}
$$

Equations (52) and (62) are practical forms to express accelerations and angular accelerations as functions of generalized coordinates and their time derivatives. In these forms, the generalized coordinates $\mathbf{q}$ lie only in the Jacobian and Hessian matrices and these matrices are treated as coefficients for $\ddot{\mathbf{q}}$ and $(\dot{\mathbf{q}}\otimes\dot{\mathbf{q}})$ respectively. Hence, these forms are capable of displaying two important characteristics of accelerations and angular accelerations: they are linear functions of generalized accelerations and are quadratic functions of generalized velocities.

3.3. Jacobian and Hessian matrices in relative motion

Consider the third frame $(c): O_c x_c y_c z_c$. The velocities of this frame with respect to frame (b) are

$$ {}^{b}\mathbf{v}_{c}^{(b)} = {}^{b}\mathbf{J}_{T_c}^{(b)}\dot{\mathbf{q}}, \tag{68}$$

$$ {}^{b}\boldsymbol{\omega}_{c}^{(b)} = {}^{b}\mathbf{J}_{R_c}^{(b)}\dot{\mathbf{q}}, \tag{69}$$

$$ {}^{b}\mathbf{v}_{G_c}^{(b)} = {}^{b}\mathbf{J}_{TG_c}^{(b)}\dot{\mathbf{q}}. \tag{70}$$

By deriving the position relations

$$ {}^{a}\mathbf{d}_{c}^{(a)} = {}^{a}\mathbf{d}_{b}^{(a)} + \mathbf{A}_{b}^{(a)}\,{}^{b}\mathbf{d}_{c}^{(b)}, \tag{71}$$

$$ \mathbf{A}_{c}^{(a)} = \mathbf{A}_{b}^{(a)}\mathbf{A}_{c}^{(b)}, \tag{72}$$

$$ {}^{a}\mathbf{r}_{G_c}^{(a)} = {}^{a}\mathbf{d}_{c}^{(a)} + \mathbf{A}_{b}^{(a)}\mathbf{A}_{c}^{(b)}\mathbf{p}_{G_c}^{(c)} \tag{73}$$

with respect to time, we obtain

$$ {}^{a}\mathbf{v}_{c}^{(a)} = {}^{a}\mathbf{v}_{b}^{(a)} + {}^{a}\tilde{\boldsymbol{\omega}}_{b}^{(a)}\,{}^{b}\mathbf{d}_{c}^{(a)} + {}^{b}\mathbf{v}_{c}^{(a)}, \tag{74}$$

$$ {}^{a}\boldsymbol{\omega}_{c}^{(a)} = {}^{a}\boldsymbol{\omega}_{b}^{(a)} + {}^{b}\boldsymbol{\omega}_{c}^{(a)}, \tag{75}$$

$$ {}^{a}\mathbf{v}_{G_c}^{(a)} = {}^{a}\mathbf{v}_{c}^{(a)} + {}^{a}\tilde{\boldsymbol{\omega}}_{c}^{(a)}\mathbf{p}_{G_c}^{(a)}. \tag{76}$$

Thus,

$$ {}^{a}\mathbf{J}_{T_c}^{(a)} = {}^{a}\mathbf{J}_{T_b}^{(a)} - {}^{b}\tilde{\mathbf{d}}_{c}^{(a)}\,{}^{a}\mathbf{J}_{R_b}^{(a)} + {}^{b}\mathbf{J}_{T_c}^{(a)}, \tag{77}$$

$$ {}^{a}\mathbf{J}_{R_c}^{(a)} = {}^{a}\mathbf{J}_{R_b}^{(a)} + {}^{b}\mathbf{J}_{R_c}^{(a)}, \tag{78}$$

$$ {}^{a}\mathbf{J}_{TG_c}^{(a)} = {}^{a}\mathbf{J}_{T_c}^{(a)} - \tilde{\mathbf{p}}_{G_c}^{(a)}\,{}^{a}\mathbf{J}_{R_c}^{(a)}. \tag{79}$$

Rewriting (77)-(79) in frame (c) yields

$$ {}^{a}\mathbf{J}_{T_c}^{(c)} = \mathbf{A}_{b}^{(c)}\,{}^{a}\mathbf{J}_{T_b}^{(b)} - \mathbf{A}_{b}^{(c)}\,{}^{b}\tilde{\mathbf{d}}_{c}^{(b)}\,{}^{a}\mathbf{J}_{R_b}^{(b)} + {}^{b}\mathbf{J}_{T_c}^{(c)}, \tag{80}$$

$$ {}^{a}\mathbf{J}_{R_c}^{(c)} = \mathbf{A}_{b}^{(c)}\,{}^{a}\mathbf{J}_{R_b}^{(b)} + {}^{b}\mathbf{J}_{R_c}^{(c)}, \tag{81}$$

$$ {}^{a}\mathbf{J}_{TG_c}^{(c)} = {}^{a}\mathbf{J}_{T_c}^{(c)} - \tilde{\mathbf{p}}_{G_c}^{(c)}\,{}^{a}\mathbf{J}_{R_c}^{(c)}. \tag{82}$$

Continuing to take derivative of (74)-(76) with respect to time yields

$$ \begin{aligned} {}^{a}\mathbf{a}_{c}^{(a)} &= {}^{a}\mathbf{a}_{b}^{(a)} + {}^{a}\tilde{\boldsymbol{\alpha}}_{b}^{(a)}\,{}^{b}\mathbf{d}_{c}^{(a)} + {}^{a}\tilde{\boldsymbol{\omega}}_{b}^{(a)}\,{}^{b}\dot{\mathbf{d}}_{c}^{(a)} + {}^{b}\dot{\mathbf{v}}_{c}^{(a)} \\ &= {}^{a}\mathbf{a}_{b}^{(a)} + {}^{a}\tilde{\boldsymbol{\alpha}}_{b}^{(a)}\,{}^{b}\mathbf{d}_{c}^{(a)} + 2\,{}^{a}\tilde{\boldsymbol{\omega}}_{b}^{(a)}\,{}^{b}\mathbf{v}_{c}^{(a)} + {}^{a}\tilde{\boldsymbol{\omega}}_{b}^{(a)}\,{}^{a}\tilde{\boldsymbol{\omega}}_{b}^{(a)}\,{}^{b}\mathbf{d}_{c}^{(a)} + {}^{b}\mathbf{a}_{c}^{(a)}, \end{aligned} \tag{83}$$

$$ {}^{a}\boldsymbol{\alpha}_{c}^{(a)} = {}^{a}\boldsymbol{\alpha}_{b}^{(a)} + {}^{b}\boldsymbol{\alpha}_{c}^{(a)} + \dot{\mathbf{A}}_{b}^{(a)}\mathbf{A}_{a}^{(b)}\mathbf{A}_{b}^{(a)}\,{}^{b}\boldsymbol{\omega}_{c}^{(b)} = {}^{a}\boldsymbol{\alpha}_{b}^{(a)} + {}^{b}\boldsymbol{\alpha}_{c}^{(a)} + {}^{a}\tilde{\boldsymbol{\omega}}_{b}^{(a)}\,{}^{b}\boldsymbol{\omega}_{c}^{(a)}, \tag{84}$$

$$ {}^{a}\mathbf{a}_{G_c}^{(a)} = {}^{a}\mathbf{a}_{c}^{(a)} + {}^{a}\tilde{\boldsymbol{\alpha}}_{c}^{(a)}\mathbf{p}_{G_c}^{(a)} + {}^{a}\tilde{\boldsymbol{\omega}}_{c}^{(a)}\,{}^{a}\tilde{\boldsymbol{\omega}}_{c}^{(a)}\mathbf{p}_{G_c}^{(a)}. \tag{85}$$

Therefore, using (11)-(14) and (16), we have

$$ \begin{aligned} &{}^{a}\mathbf{H}_{T_c}^{(a)}(\dot{\mathbf{q}}\otimes\dot{\mathbf{q}}) \\ &= ({}^{a}\mathbf{H}_{T_b}^{(a)} - {}^{b}\tilde{\mathbf{d}}_{c}^{(a)}\,{}^{a}\mathbf{H}_{R_b}^{(a)} + 2\,{}^{a}\tilde{\mathbf{J}}_{R_b}^{(a)}({}^{b}\mathbf{J}_{T_c}^{(a)}\otimes\mathbf{E}_n) + {}^{a}\tilde{\mathbf{J}}_{R_b}^{(a)}({}^{a}\tilde{\mathbf{J}}_{R_b}^{(a)}\otimes\mathbf{E}_n)({}^{b}\mathbf{d}_{c}^{(a)}\otimes\mathbf{E}_{nn}) + \mathbf{A}_{b}^{(a)}\,{}^{b}\mathbf{H}_{T_c}^{(b)})(\dot{\mathbf{q}}\otimes\dot{\mathbf{q}}), \end{aligned} \tag{86}$$

$$ {}^{a}\mathbf{H}_{R_c}^{(a)}(\dot{\mathbf{q}}\otimes\dot{\mathbf{q}}) = ({}^{a}\mathbf{H}_{R_b}^{(a)} + \mathbf{A}_b^{(a)}\,{}^{b}\mathbf{H}_{R_c}^{(b)} + {}^{a}\tilde{\mathbf{J}}_{R_b}^{(a)}({}^{b}\mathbf{J}_{R_c}^{(a)}\otimes\mathbf{E}_n))(\dot{\mathbf{q}}\otimes\dot{\mathbf{q}}), \tag{87} $$

$$ {}^{a}\mathbf{H}_{TG_c}^{(a)}(\dot{\mathbf{q}}\otimes\dot{\mathbf{q}}) = ({}^{a}\mathbf{H}_{T_c}^{(a)} - \tilde{\mathbf{p}}_{G_c}^{(a)}\,{}^{a}\mathbf{H}_{R_c}^{(a)} + {}^{a}\tilde{\mathbf{J}}_{R_c}^{(a)}({}^{a}\tilde{\mathbf{J}}_{R_c}^{(a)}\otimes\mathbf{E}_n)(\mathbf{p}_{G_c}^{(a)}\otimes\mathbf{E}_{nn}))(\dot{\mathbf{q}}\otimes\dot{\mathbf{q}}). \tag{88} $$

Rewriting (86)-(88) in frame (c) yields

$$ \begin{aligned} {}^{a}\mathbf{H}_{T_c}^{*(c)} &= \mathbf{A}_b^{(c)}\,{}^{a}\mathbf{H}_{T_b}^{*(b)} - {}^{b}\tilde{\mathbf{d}}_c^{(c)}\mathbf{A}_b^{(c)}\,{}^{a}\mathbf{H}_{R_b}^{(b)} + 2\mathbf{A}_b^{(c)}\,{}^{a}\mathbf{J}_{R_b}^{(b)}({}^{b}\mathbf{J}_{T_c}^{(c)}\otimes\mathbf{E}_n) \\ &\quad + \mathbf{A}_b^{(c)}\,{}^{a}\mathbf{J}_{R_b}^{(b)}(\mathbf{A}_b^{(c)}\,{}^{a}\mathbf{J}_{R_b}^{(b)}\otimes\mathbf{E}_n)({}^{b}\mathbf{d}_c^{(c)}\otimes\mathbf{E}_{nn}) + {}^{b}\mathbf{H}_{T_c}^{*(c)} \end{aligned}, \tag{89} $$

$$ {}^{a}\mathbf{H}_{R_c}^{(c)}(\dot{\mathbf{q}}\otimes\dot{\mathbf{q}}) = (\mathbf{A}_b^{(c)}\,{}^{a}\mathbf{H}_{R_b}^{(b)} + {}^{b}\mathbf{H}_{R_c}^{(c)} + \mathbf{A}_b^{(c)}\,{}^{a}\mathbf{J}_{R_b}^{(b)}({}^{b}\mathbf{J}_{R_c}^{(c)}\otimes\mathbf{E}_n))(\dot{\mathbf{q}}\otimes\dot{\mathbf{q}}), \tag{90} $$

$$ {}^{a}\mathbf{H}_{TG_c}^{*(c)}(\dot{\mathbf{q}}\otimes\dot{\mathbf{q}}) = ({}^{a}\mathbf{H}_{T_c}^{*(c)} - \tilde{\mathbf{p}}_{G_c}^{(c)}\,{}^{a}\mathbf{H}_{R_c}^{(c)} + {}^{a}\tilde{\mathbf{J}}_{R_c}^{(c)}({}^{a}\tilde{\mathbf{J}}_{R_c}^{(c)}\otimes\mathbf{E}_n)(\mathbf{p}_{G_c}^{(c)}\otimes\mathbf{E}_{nn}))(\dot{\mathbf{q}}\otimes\dot{\mathbf{q}}). \tag{91} $$

It can be seen that these matrix relations are analogous to the known vector relations for relative motion.

4. A NEW MATRIX FORM OF LAGRANGE'S EQUATIONS

The general form of Lagrange's equations of second kind for a n-DOF serial multibody is written as

$$ \frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\mathbf{q}}}\right)^T - \left(\frac{\partial T}{\partial\mathbf{q}}\right)^T = \mathbf{f}, \tag{92} $$

in which $\mathbf{q}$ is a $n\times 1$ vector containing generalized independent coordinates, $\mathbf{f}$ is a $n\times 1$ vector containing generalized force, and scalar T is the kinetic energy of the whole system which is usually expressed as

$$ T = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}(\mathbf{q})\dot{\mathbf{q}} \tag{93} $$

or with $vec(.)$ function as

$$ T = \frac{1}{2}(\dot{\mathbf{q}}^T\otimes\dot{\mathbf{q}}^T)vec(\mathbf{M}(\mathbf{q})) \tag{94} $$

where the inertia matrix is determined as follows

$$ \mathbf{M}(\mathbf{q}) = \sum_{i=1}^{n}(m_i\mathbf{J}_{TG_i}^T\mathbf{J}_{TG_i} + \mathbf{J}_{R_i}^T\mathbf{I}_{G_i}\mathbf{J}_{R_i}). \tag{95} $$

where m_i and $\mathbf{I}_{G_i}^{(.)}$ are the mass and the matrix of inertia tensor of the i-th body, respectively. In (95), the superscripts are omitted for the sake of simplicity. The left superscripts are all zeros while the right superscripts should be the same for all the matrices that are multiplied to each other.

Substituting (93) into (92), one obtains [3, 5]

$$ \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} = \mathbf{f} \tag{96} $$

where

$$\mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) = \frac{\partial \mathbf{M}(\mathbf{q})}{\partial \mathbf{q}}(\mathbf{E}_n \otimes \dot{\mathbf{q}}) - \frac{1}{2}\left(\frac{\partial \mathbf{M}(\mathbf{q})}{\partial \mathbf{q}}(\dot{\mathbf{q}} \otimes \mathbf{E}_n)\right)^T \tag{97}$$

is a new form of the Coriolis/centripetal matrix, which is usually calculated by Christoffel symbols [9]. Matrix equation (96) does not explicitly show that the second term is a quadratic function of $\dot{\mathbf{q}}$. To derive an equation with a quadratic expression of $\dot{\mathbf{q}}$, the derivatives of T are calculated as follows

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right)^T = \frac{d}{dt}(\mathbf{M}(\mathbf{q})\dot{\mathbf{q}}) = \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \frac{\partial \mathbf{M}(\mathbf{q})}{\partial \mathbf{q}}(\mathbf{E}_n \otimes \dot{\mathbf{q}})\dot{\mathbf{q}} = \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \frac{\partial \mathbf{M}(\mathbf{q})}{\partial \mathbf{q}}(\dot{\mathbf{q}} \otimes \dot{\mathbf{q}}), \tag{98}$$

$$\left(\frac{\partial T}{\partial \mathbf{q}}\right)^T = \frac{1}{2}\left((\dot{\mathbf{q}}^T \otimes \dot{\mathbf{q}}^T)\frac{\partial vec(\mathbf{M})}{\partial \mathbf{q}}\right)^T = \frac{1}{2}\left(\frac{\partial vec(\mathbf{M})}{\partial \mathbf{q}}\right)^T(\dot{\mathbf{q}} \otimes \dot{\mathbf{q}}). \tag{99}$$

Now (92) can be rewritten as

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}^*(\mathbf{q})(\dot{\mathbf{q}} \otimes \dot{\mathbf{q}}) = \mathbf{f} \tag{100}$$

where the velocity-free Coriolis/centripetal matrix is given as

$$\mathbf{C}^*(\mathbf{q}) = \frac{\partial \mathbf{M}(\mathbf{q})}{\partial \mathbf{q}} - \frac{1}{2}\left(\frac{\partial vec(\mathbf{M})}{\partial \mathbf{q}}\right)^T. \tag{101}$$

5. APPLIED EXAMPLE

Consider a stacker used in mining field (Fig. 1). Its schematic diagram with Denavit-Hartenberg coordinate systems is shown in Fig. 2 and the symbolic kinematic parameters are given in Table 1 and kinetic parameters in Table 2. Here the kinetic parameters are simplified to make it simple when comparing the considered form of Lagrange's equations with the conventional forms.

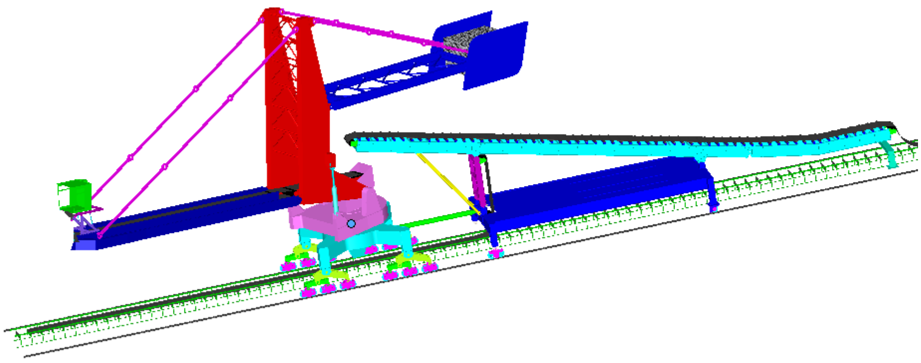

Figure 1. The stacker used in mining field (without bucket grab).

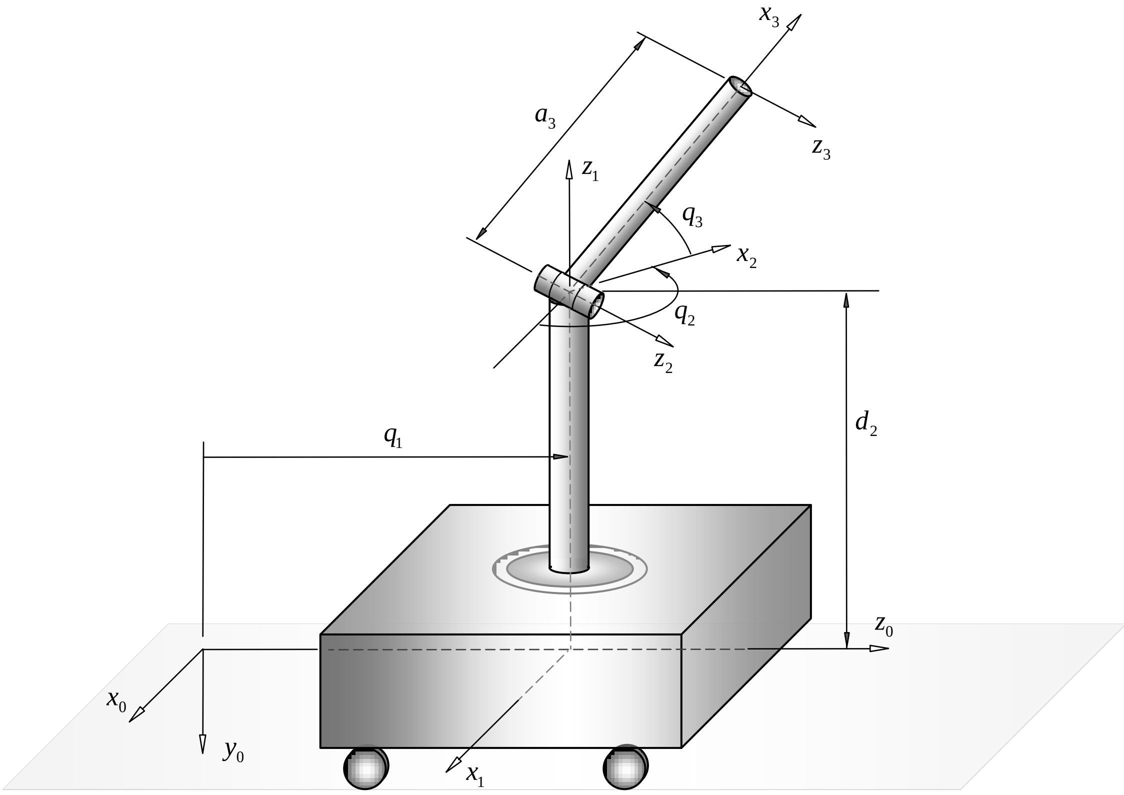

Figure 2. The stacker's schematic diagram with Denavit-Hartenberg coordinate systems.

Table 1. Denavit-Hartenberg parameters of the stacker.

Parameter / *Body*	d_i	θ_i	a_i	α_i
1	q_1	0	0	$\pi / 2$
2	d_2	q_2	0	$\pi / 2$
3	0	q_3	a_3	0

Table 2. Kinetic parameters of the stacker.

Parameter / *Body*	I_{ixx}	I_{iyy}	I_{izz}	I_{ixy}	I_{ixz}	I_{iyz}	$\mathbf{p}_{G_i}^{(i)}$
2	I_{2x}	I_{2y}	I_{2z}	0	0	0	$\left[0, y_{G_2}, 0\right]^T$
3	I_{3x}	I_{3y}	I_{3z}	0	0	0	$a_3 - l_3, 0, 0\ ^T$

5.1. Kinematic analysis

The kinematic of frame (3) can be characterized by its Jacobian and Hessian matrices as follow:

$$ {}^{0}\mathbf{J}_{T_3}^{(0)} = \begin{bmatrix} 0 & -a_3 \sin q_2 \cos q_3 & -a_3 \cos q_2 \sin q_3 \\ 0 & 0 & -a_3 \cos q_3 \\ 1 & a_3 \cos q_2 \cos q_3 & -a_3 \sin q_2 \sin q_3 \end{bmatrix}, \tag{102} $$

$$ {}^{0}\mathbf{H}_{T_3}^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & -a_3 \cos q_2 \cos q_3 & a_3 \sin q_2 \sin q_3 & 0 & a_3 \sin q_2 \sin q_3 & -a_3 \cos q_2 \cos q_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 \sin q \\ 0 & 0 & 0 & 0 & -a_3 \sin q_2 \cos q_3 & -a_3 \cos q_2 \sin q_3 & 0 & -a_3 \cos q_2 \sin q_3 & -a_3 \sin q_2 \cos q_3 \end{bmatrix}, \tag{103} $$

$$ {}^{0}\mathbf{J}_{R_3}^{(0)} = \begin{bmatrix} 0 & 0 & \sin q_2 \\ 0 & -1 & 0 \\ 0 & 0 & -\cos q_2 \end{bmatrix}, \tag{104} $$

$$ {}^{0}\mathbf{H}_{R_3}^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos q_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sin q_2 & 0 \end{bmatrix}. \tag{105} $$

Here, the Hessian matrices are computed by using (51) and (60). The calculations can be performed manually or automatically. Note that the Kronecker product is already built in common technical software such as *kron* in Matlab and *KroneckerProduct* in Mathematica/Wolfram Alpha. By storing the matrices in (102)-(105), a computer program can easily compute all the velocities and accelerations needed for a kinematic analysis using (30), (38), (44), (52), (62), and (66).

5.2. Dynamic analysis

The mass matrix and velocity-free Coriolis/centripetal matrix are

$$ \mathbf{M} = \begin{bmatrix} m_1 + m_2 + m_3 & m_3 l_3 C_2 C_3 & -m_3 l_3 S_2 S_3 \\ m_3 l_3 C_2 C_3 & I_{2y} + (m_3 l_3^2 + I_y) c_3^2 + I_x s_3^2 & 0 \\ -m_3 l_3 S_2 S_3 & 0 & m_3 l_3^2 I_{3z} \end{bmatrix}, \tag{106} $$

$$ \mathbf{C}^* = \begin{bmatrix} 0 & 0 & 0 & 0 & -m_3 l_3 S_2 C_3 & -m_3 l_3 C_2 S_3 & 0 & -m_3 l_3 C_2 S_3 & -m_3 l_3 S_2 C_3 \\ 0 & \frac{-m_3 l_3 S_2 C_3}{2} & \frac{-m_3 l_3 C_2 S_3}{2} & \frac{m_3 l_3 S_2 C_3}{2} & 0 & -2(m_3 l_3^2 + I_{3y} - I_{3x}) S_3 C_3 & \frac{m_3 l_3 C_2 S_3}{2} & 0 & 0 \\ 0 & \frac{-m_3 l_3 C_2 S_3}{2} & \frac{-m_3 l_3 S_2 C_3}{2} & \frac{m_3 l_3 C_2 S_3}{2} & (m_3 l_3^2 + I_{3y} - I_{3x}) S_3 C_3 & 0 & \frac{m_3 l_3 S_2 C_3}{2} & 0 & 0 \end{bmatrix}. \tag{107} $$

With these matrices, the equations of motion can be obtained by (100). The result is confirmed by comparing with equations obtained with conventional methods.

6. CONCLUSION

Based on Kronecker product and Khang's definition of the partial derivative of a matrix with respect to a vector, this paper introduced a theory for a kind of matrix algebra that can handle kinematic and dynamic analysis of a general multibody system.

The presented Jacobian and Hessian matrices allow one to write accelerations as a sum of two terms: one term depends linearly on generalized accelerations and one depends quadratically on generalized velocities. This kind of expressions has its own advantage over other ones: the relations between accelerations can be written in terms of generalized coordinates without the appearance of generalized velocities or generalized accelerations, which is proved through the analysis of relative motion. The separation of generalized coordinates and its time derivatives may also give more insights in the characteristics of the system when the Jacobian and Hessian matrices are further analyzed.

Similarly, in the new form of Lagrange’s equations, generalized coordinates, generalized velocities, and generalized accelerations are collected compactly into different terms, which can be easily computed symbolically with the help of technical software.

Since what was presented in this paper is a general theory, it is hard to compare it in terms of efficiency with existing methods that are specialized for a specific class of problems. In future work, this theory will be developed into methods and computational programs. At that point, comparisons can be used to determine which method is the most effective in a certain case.

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