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ANALYZING AND OPTIMIZING OF A PFLUGER COLUMN

TRAN DUC TRUNG, BUI HAI LE, CAO QUOC HUONG

ABSTRACT

The optimal shape of a Pfluger column is determined by using Pontryagin's maximum principle (PMP). The governing equation of the problem is reduced to a boundary-value problem for a single second order nonlinear differential equation. The results of the analysis problem are obtained by Spectral method. Necessary conditions for the maximum value of the first eigenvalue corresponding to given column volume are established to determine the optimal distribution of cross-sectional area along the column axis.

Keywords: optimal shape; Pontryagin's maximum principle.

1. INTRODUCTION

The problem of determining the shape of a column that is the strongest against buckling is an important engineering one. The PMP has been widely used in finding out the optimal shape of the above-mentioned problem.

Tran and Nguyen [12] used the PMP to study the optimal shape of a column loaded by an axially concentrated force. Szymczak [11] considered the problem of extreme critical conservative loads of torsional buckling for axially compressed thin walled columns with variable, within given limits, bisymmetric I cross-section basing on the PMP. Atanackovic and Simic [4] determined the optimal shape of a Pfluger column using the PMP, numerical integration and Ritz method. Glavardanov and Atanackovic [9] formulated and solved the problem of determining the shape of an elastic rod stable against buckling and having minimal volume, the rod was loaded by a concentrated force and a couple at its ends, the PMP was used to determine the optimal shape of the rod. Atanackovic and Novakovic [3] used the PMP to determine the optimal shape of an elastic compressed column on elastic, Winkler type foundation. The optimality conditions for the case of bimodal optimization were derived. The optimal cross-sectional area function was determined from the solution of a nonlinear boundary value problem. Jelicic and Atanackovic [10] determined the shape of the lightest rotating column that is stable against buckling, positioned in a constant gravity field, oriented along the column axis. The optimality conditions were derived by using the PMP. Optimal cross-sectional area was obtained from the solution of a non-linear boundary value problem. Atanackovic [2] used the PMP to determine the shape of the strongest column positioned in a constant gravity field, simply supported at the lower end and clamped at upper end (with the possibility of axial sliding). It was shown that the cross-sectional area function is determined from the solution of a nonlinear boundary value problem. Braun [5] presented the optimal shape of a compressed rotating rod which maintains stability against buckling. In the rod modeling, extensibility along the rod axis and shear stress were taken into account. Using the PMP, the optimization problem is formulated with a fourth order boundary value problem. The optimally shaped compressed rotating (fixed-free) rod has a finite cross-sectional area on the free end.

In this paper we determine the optimal shape of a Pfluger column – a simply supported column loaded by uniformly distributed follower type of load (see Atanackovic and Simic [4]). Such load has the direction of the tangent to the column axis in any configuration and does not have a potential, i.e., it is a non-conservative load. The results of the analysis problem are obtained by Spectral method.

PMP allows estimating the maximum value of the Hamiltonian function that satisfies the Hamiltonian adjoint equations instead of solving the minimum objective functions directly. An analogy between adjoint variables and original variables holds for some cases. This is an advantageous condition to determine the maximum value of the Hamiltonian function.

Although PMP have been investigated, the objective function is still implicit, the sign of the analogy coefficient k is indirectly determined and the upper and lower values of the control variable are unbounded. The present work suggests a method of supposition to determine k directly and exactly. The Maier functional, which depends on state variables in fixed locations, is used as the objective function from a multicriteria optimization viewpoint. The bounded values are set up for the control variable.

The present paper is organized as follows: following the introduction section is presented formulation of the problem, optimization problem is considered in section 3, results and discussion are given in section 4, and final remarks are summarized in section 5.

2. FORMULATION OF THE PROBLEM

The formulation of the problem is established basing on Atanackovic and Simic [4] and Atanackovic [1]:

Consider a column shown in Fig. 1. The column is simply supported at both ends with end C movable. The axis of the column is initially straight and the column is loaded by uniformly distributed follower type of load of constant intensity q_0. We shall assume that the column axis has length L and that it is inextensible.

Let x-B-y be a Cartesian coordinate system with the origin at the point B and with the x axis oriented along the column axis in the undeformed state. The equilibrium equations could now be derived

$$\frac{dH}{dS} = -q_x\,; \quad \frac{dV}{dS} = -q_y\,; \quad \frac{dM}{dS} = -V\cos\theta + H\sin\theta \tag{2.1}$$

where H and V are components of the resultant force (a force representing the influence of the part $(S, L]$ on the part $[0, S)$ of the column) along the x and y axis, respectively, M is the bending moment and θ is the angle between the tangent to the column axis and x axis. Also in (2.1) q_x and q_y are components of the distributed forces along the x and y axis respectively. Since the distributed force is tangent to the column axis we have

$$q_x = -q_0\cos\theta\,; \quad q_y = -q_0\sin\theta \tag{2.2}$$

To the system (2.1) we adjoin the following geometrical

$$\frac{dx}{dS} = \cos\theta\,; \quad \frac{dy}{dS} = \sin\theta \tag{2.3}$$

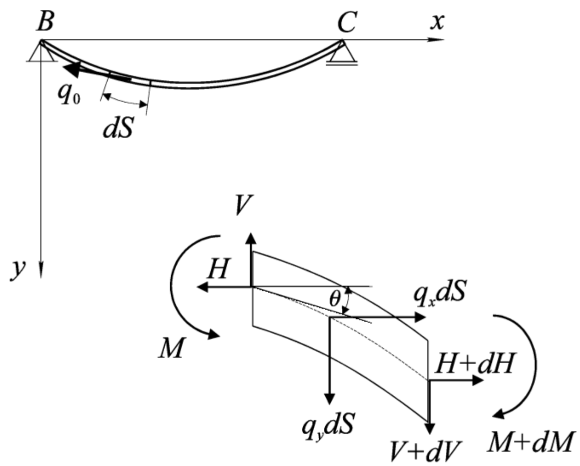

Figure 1. Coordinate system and load configuration

and constitutive relation

$$\frac{d\theta}{dS} = \frac{M}{EI}. \tag{2.4}$$

In (2.3) and (2.4) we use x and y to denote coordinates of an arbitrary point of the column axis and EI to denote the bending rigidity. The boundary conditions corresponding to the column shown in Fig. 1 are

$$x(0) = 0; \quad y(0) = 0; \quad M(0) = 0; \quad y(L) = 0; \quad M(L) = 0; \quad H(L) = 0. \tag{2.5}$$

The system (2.1)–(2.5) possesses a trivial solution in which column axis remains straight, i.e.,

$$H^0(S) = -\, q_0(L-S); \quad V^0(S) = S; \quad M^0(S) = 0; \quad x^0(S) = S; \quad y^0(S) = 0; \quad \theta^0(S) = 0. \tag{2.6}$$

In order to formulate the minimum volume problem for the column we take the cross-sectional area $A(S)$ and the second moment of inertia $I(S)$ of the cross-section in the form

$$A(S) = A_0 a(S); \quad I(S) = I_0 a^2(S) \tag{2.7}$$

where A_0 and I_0 are constants (having dimensions of area and second moment of inertia, respectively) and $a(S)$ is cross-sectional area function. For the case of a column with circular cross section we have the connection between A_0 and I_0 given by $I_0 = (1/4\pi)\, A_0^2$. Let $\Delta H, \ldots, \Delta\theta$ be the perturbations of $H, \ldots, \theta$ defined by

$$H = H^0 + \Delta H; \; V = V^0 + \Delta V; \; M = M^0 + \Delta M; \; x = x^0 + \Delta x; \quad y = y^0 + \Delta y; \; \theta = \theta^0 + \Delta\theta. \tag{2.8}$$

Then, by introducing the following dimensionless quantities

$$h = \frac{\Delta H L^2}{EI_0}; \; v = \frac{\Delta V L^2}{EI_0}; \; m = \frac{\Delta M L}{EI_0}; \; \xi = \frac{\Delta x}{L}; \; \eta = \frac{\Delta y}{L}; \; t = \frac{S}{L}; \; \lambda = \frac{q_0 L^3}{EI_0} \tag{2.9}$$

and by substituting (2.7) in (2.1)–(2.5) we arrive to the following nonlinear system of equations describing nontrivial configuration of the column

$$\begin{aligned}
\dot{h} &= -\lambda(1-\cos\theta);\\
\dot{v} &= -\lambda\sin\theta;\\
\dot{m} &= -v\cos\theta + [-\lambda(1-t)+h]\sin\theta;\\
\dot{\xi} &= 1-\cos\theta;\\
\dot{\eta} &= \sin\theta;\\
\dot{\theta} &= \frac{m}{a^2}.
\end{aligned} \tag{2.10}$$

where $(\dot{\bullet}) = d(\bullet)/dt$. The boundary conditions corresponding to (2.10) are

$$\xi(0) = 0; \quad \eta(0) = 0; \quad m(0) = 0; \quad \eta(1) = 0; \quad m(1) = 0; \quad h(1) = 0. \tag{2.11}$$

Note that the system (2.10)–(2.11) has the solution $h(t) = 0,\ldots,\ \theta(t) = 0$ for all values of λ. Next we linearize (2.10) to obtain

$$\begin{aligned}
\dot{h} &= 0;\\
\dot{v} &= -\lambda\theta;\\
\dot{m} &= -v + -\lambda(1-t)\theta;\\
\dot{\xi} &= 0;\\
\dot{\eta} &= \theta;\\
\dot{\theta} &= \frac{m}{a^2}.
\end{aligned} \tag{2.12}$$

By using boundary conditions (2.11) in (2.12) we conclude that $h(t) = \xi(t) = 0$ and the rest of Eqs (2.12) could be reduced to

$$\ddot{m} + \frac{\lambda}{a^2}(1-t)m = 0 \tag{2.13}$$

subject to:

$$m(0) = m(1) = 0. \tag{2.14}$$

The system (2.13) – (2.14) constitutes a spectral problem.

3. OPTIMIZATION PROBLEM

To determine the optimal shape of the column, we will use the PMP (Geering [8]). Let us write optimization problem as: find out $a(t)$, $a_{\min} \le a(t) \le a_{\max}$, $t \in [0, 1]$, satisfies the objective function

$$G = -(1-k_{\lambda J})\lambda_1 + k_{\lambda J} J = \min. \tag{2.15}$$

where λ_1 is the first dimensionless eigenvalue, $k_{\lambda J}$ is non-negative weight, $k_{\lambda J} \in [0, 1]$, the dimensionless volume of the column J is defined as

$$J = \int_0^1 a(t)dt \tag{2.16}$$

The state differential equations are

$$\dot{x}_1 = x_2\,; \quad \dot{x}_2 = -\frac{\lambda_1}{a^2}(1-t)x_1\,. \tag{2.17}$$

subject to

$$x_1(0) = x_2(1) = 0. \tag{2.18}$$

Proposition: *with the above-mentioned suppositions, Eqs. (2.15)–(2.18), the Hamiltonian function H is maximized, and the analogy coefficient k between adjoint variables and original variables is positive, where:*

$$H = \frac{1}{k}\left[-x_2^2 - \frac{\lambda_1}{a^2}(1-t)x_1^2\right] - k_{\lambda J}a = \text{max (in } a\text{)}. \tag{2.19}$$

Proof.

The first eigenvalue λ_1 is here considered as a state variable. It means that the role of λ_1 is equivalent to those of x_1 and x_2 in the state differential equations (2.17). The volume of the column J is also a state variable. So, the state equations (2.17) can be rewritten in the form

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{\lambda_1}{a^2}(1-t)x_1 \\ \dot{\lambda}_1 = 0 \\ \dot{J} = a \end{cases} \tag{2.20}$$

The objective function can be rewritten in term of the Maier's one:

$$G = -(1-k_{\lambda J})\lambda_1(1) + k_{\lambda J}J(1) = \min\,. \tag{2.21}$$

From the Eqs. (2.20) the Hamiltonian function H can be established in the form as follows

$$H = p_{x1}x_2 + p_{x2}\left[-\frac{\lambda_1}{a^2}(1-t)x_1\right] + p_{\lambda 1}\dot{\lambda}_1 + p_J a\,, \quad \dot{\lambda} = 0\,. \tag{2.22}$$

The adjoint equations can be expressed in the following form:

$$\dot{p}_{x1} = -\frac{\partial H}{\partial x_1} = \frac{\lambda_1}{a^2}(1-t)p_{x2} \tag{2.23a}$$

$$\dot{p}_{x2} = -\frac{\partial H}{\partial x_2} = -p_{x1} \tag{2.23b}$$

$$\dot{p}_{\lambda 1} = -\frac{\partial H}{\partial \lambda_1} = \frac{1}{a^2}(1-t)x_1 p_{x2} \tag{2.23c}$$

$$\dot{p}_J = -\frac{\partial H}{\partial J} = 0 \qquad (2.23d)$$

The conjugate variables $p_{x1}, p_{x2}, p_{\lambda 1}, p_J$ are determined from the expression:

$$\sum_{i=1}^{n} p_i(1)\delta x_i(1) - \sum_{i=1}^{n} p_i(0)\delta x_i(0) + \delta G = 0 . \qquad (2.24)$$

Thus

$$p_{x1}(1)\delta x_1(1) + p_{x2}(1)\delta x_2(1) + p_{\lambda 1}(1)\delta\lambda_1(1) + p_J(1)\delta J(1)$$
$$-p_{x1}(0)\delta x_1(0) - p_{x2}(0)\delta x_2(0) - p_{\lambda 1}(0)\delta\lambda_1(0) - p_J(0)\delta J(0) - (1-k_{\lambda J})\delta\lambda_1(1) + k_{\lambda J}\delta J(1) = 0 \qquad (2.25)$$

or

$$p_{x1}(1)\delta x_1(1) + p_{x2}(1)\delta x_2(1) + \left[p_{\lambda 1}(1) - (1-k_{\lambda J})\right]\delta\lambda_1(1) + \left[p_J(1) + k_{\lambda J}\right]\delta J(1)$$
$$-p_{x1}(0)\delta x_1(0) - p_{x2}(0)\delta x_2(0) - p_{\lambda 1}(0)\delta\lambda_1(0) - p_J(0)\delta J(0) = 0 . \qquad (2.26)$$

Hence

$$p_{x2}(1) = p_{x2}(0) = 0; p_{\lambda 1}(1) = 1 - k_{\lambda J}; p_{\lambda 1}(0) = 0; p_J(1) = -k_{\lambda J}; p_J(0) = 0 \qquad (2.27)$$

assigning

$$p_{x1} = -x_{2H}; p_{x2} = x_{1H} \qquad (2.28)$$

We obtain

$$\dot{x}_{1H} = x_{2H} ;\ \dot{x}_{2H} = -\frac{\lambda_1}{a^2}(1-t)x_{1H} . \qquad (2.29)$$

$$\text{subject to } x_{1H}(1) = x_{1H}(0) = 0 . \qquad (2.30)$$

It is seen that Eqs. (2.17) are similar in form as ones of (2.29) and the boundary conditions (2.18) are also similar in form as the conditions (2.30). As a result, we reached the following conclusion: the same analogy between the adjoint variables and the original variables holds, or

$$kx_{1H} = x_1; kx_{2H} = x_2 . \qquad (2.31)$$

The sign of *k* can be determined by integrating the Eq. (2.23c) with appropriate conditions in Eq. (2.27):

$$\int_0^1 \dot{p}_{\lambda 1} dt = p_{\lambda 1}(1) - p_{\lambda 1}(0) = 1 - k_{\lambda J} = \frac{1}{k}\int_0^1 \frac{(1-t)x_1^2}{a^2} dt > 0 \qquad (2.32)$$

Thus, the sign of the analogy coefficient *k* is larger than zero for the case of maximizing λ_1. It was demonstrated by considering the first eigenvalue λ_1 as a state variable. The Hamiltonian function (2.22) will be maximized if:

$$H = \frac{1}{k}\left[-x_2^2 - \frac{\lambda_1}{a^2}(1-t)x_1^2\right] - k_{\lambda J}a = \max \ \text{(in } a\text{)}. \qquad (2.19)$$

Thus, basing on the PMP in optimal control for above-mentioned system's first eigenvalue, the obtained optimal necessary conditions consist of: the state equations (2.17), the boundary *conditions (2.18), the control variable a(t) ∈ [a_{min}, a_{max}] and the maximum conidition of the Hamiltonian function (2.19).*

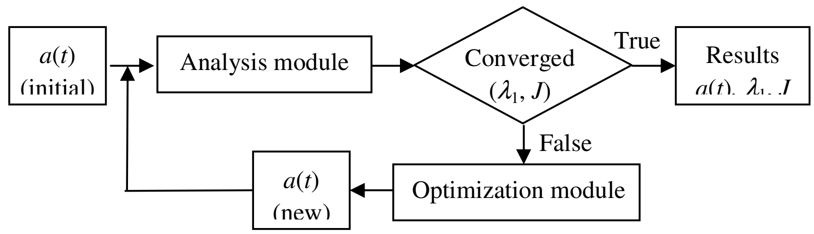

Figure 2. The general algorithm used in the present work

From the multicriteria optimization viewpoint, the Pareto front between the criterion (λ_1, J) is build basing on the Definition 6 in Coello Coello *et al*. [6] (page 10): *A solution* **x** ∈ *Ω is said to be Pareto-optimal with respect to Ω if and only if there is no* **x'** ∈ *Ω for which* **v** = *F*(**x'**)=(f_1(**x'**),...,f_k(**x'**)) *dominates* ***u*** = *F*(**x**)=(f_1(**x**),...,f_k(**x**)). *The phrase* ***Pareto-optimal*** *is meant with respect to the entire decision variable space unless otherwise specified*. In words, this definition says that **x*** is Pareto-optimal if there exists no feasible vector **x** which would decrease some criterion without causing a simultaneous increase in at least one other criterion (assuming minimization).

4. RESULTS AND DISCUSSION

4.1. Validation of the model

In order to verify results obtained in the present work, the model in Atanackovic and Simic [4] is studied for both validation analysis and optimization problems.

4.1.1. Analysis problem

The first eigenvalue of the studied column with constant circular cross-section was shown in Table 1.

Table 1. The first eigenvalue of the studied column

Methods	The first eigenvalue λ_1	
	$a(t) = 1$	$a(t) = 0.81051$
Present	18.957240	12.453513
Atanackovic and Simic [4]	18.956266	12.452807

4.1.2. Optimization problem

We take $J = 0.81051$, $0 < a(t) < \infty$. The aim of this section is to determine the column's optimal shape (variable circular cross-section) and maximum value of λ_1 according to above input data. The results are shown in Table 2 and Fig. 3.

Table 2. The maximum value of λ_1

Methods	
Present	18.950876
Atanackovic and Simic [4]	18.956266

Via sections 4.1.1 and 4.1.2, it is evident that the results of the authors, those of Atanackovic and Simic [4] are in good agreement (see Atanackovic and Simic [4] to compare the column's optimal shape).

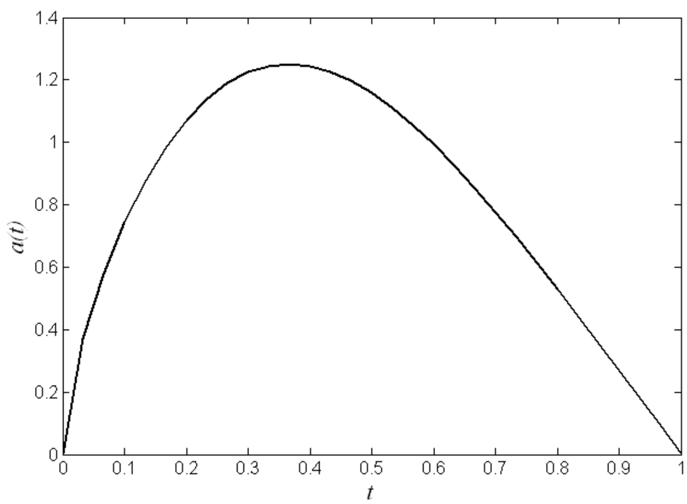

Figure 3. The column's optimal shape

4.2. Results and discussion for the optimization problem of the authors

The content of the problem consists in finding out the changing rule of the circular cross-section $a(t) \in [a_{\min}, a_{\max}]$, $t \in [0, 1]$ which satisfies the state differential equations (2.17); maximizing the first eigenvalue λ_1; the total volume J of the column is given. We take $a_{\min} = 0.9$; $a_{\max} = 1.1$. Thus, $J \in [0.9, 1.1]$.

4.2.1. Optimization problem with above-mentioned input data

Table 3. The maximum values of λ_1 corresponding to five cases of J in the section 4.2.1

Notation	J	λ_1
Case 1a	1.100	22.937693
Case 2a	1.050	22.817520
Case 3a	1.000	21.570576
Case 4a	0.950	18.705883
Case 5a	0.900	15.354985

The results shown in Table 3 and Fig. 4 are the maximum values of λ_1, the column's optimal shape configurations corresponding to five cases of *J*.

The Pareto front or trade-off curve which includes the set of points that bounds the bottom of the feasible region is shown in Fig. 5.

4.2.2. Optimization problem with above-mentioned input data and an additional constraint

The additional constraint in this section is that $a(t) = 1$, $t \in [0.1, 0.2]$. It means that the distribution of the cross-sectional area along the column axis is discontinuous.

The results described in the Table 4, Figs. 5 & 6 are the maximum values of λ_1, the column's optimal shape configurations corresponding to five cases of *J* and the Pareto front.

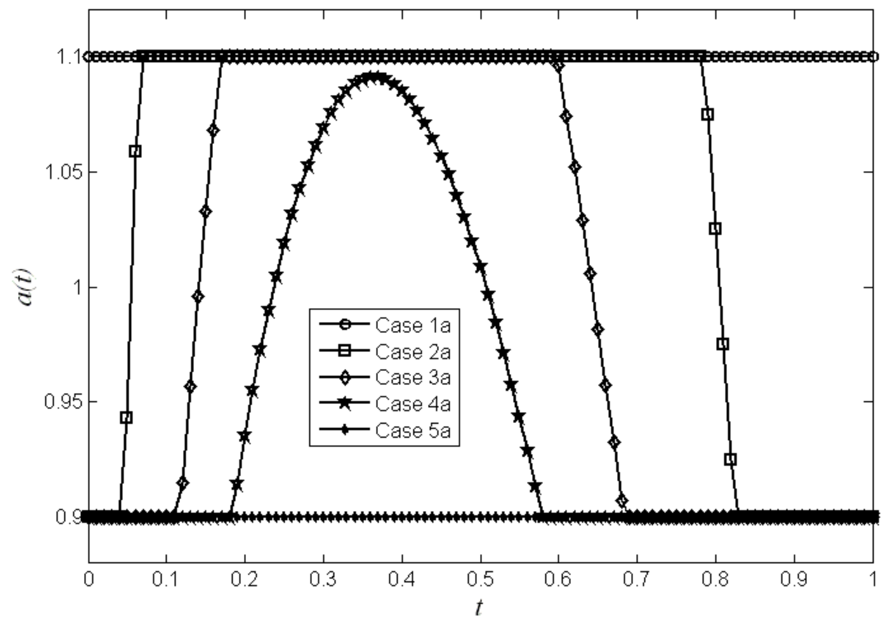

Figure 4. The column's optimal shape configurations corresponding to five cases of *J* in the section 4.2.1

Table 4. The maximum values of λ_1 corresponding to five cases of *J* in the section 4.2.2

Notation	J	λ_1
Case 1b	1.089	22.431435
Case 2b	1.050	22.386033
Case 3b	1	21.487410
Case 4b	0.950	18.427398
Case 5b	0.911	15.661187

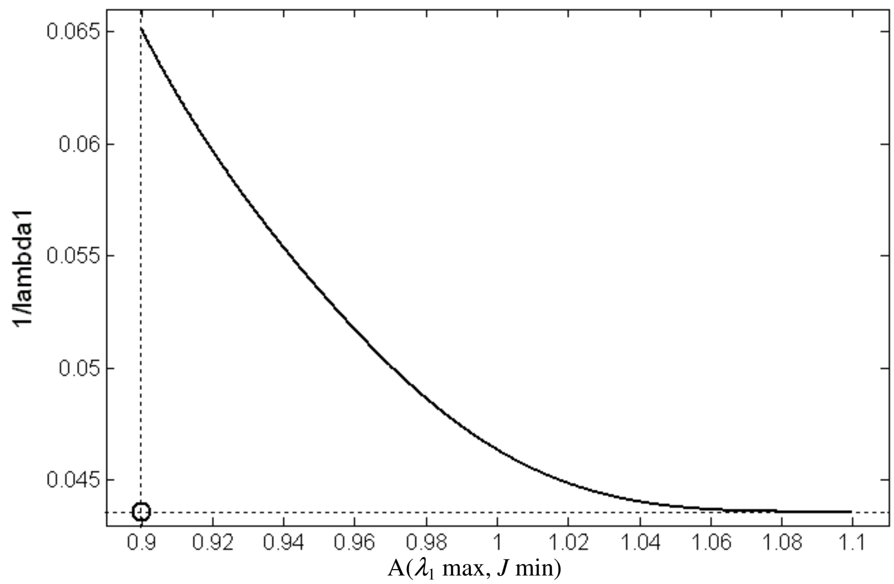

Figure 5. The Pareto front of the optimization problem in the section 4.2.1

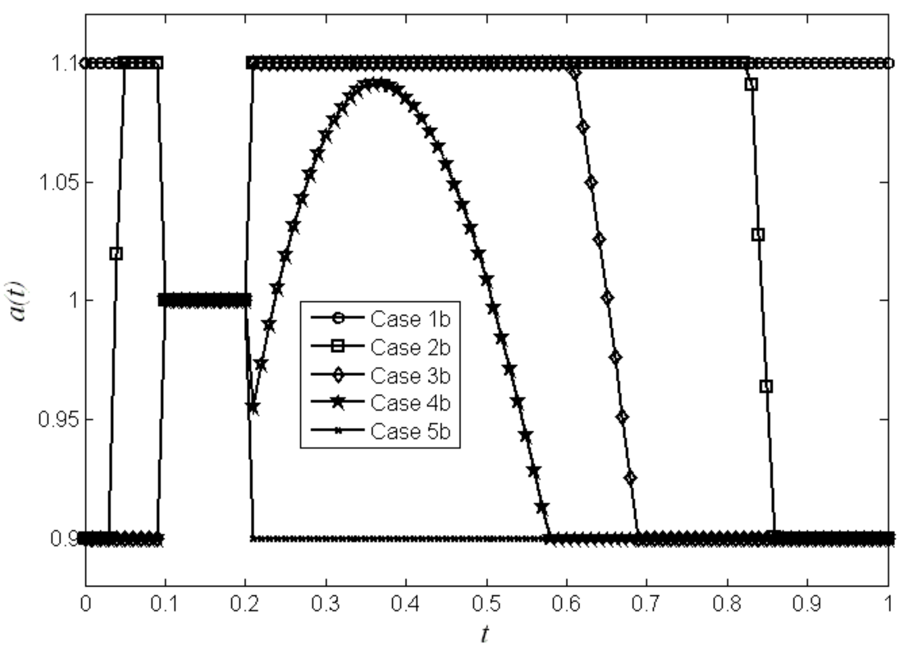

Figure 6. The column's optimal shape configurations in the section 4.2.2

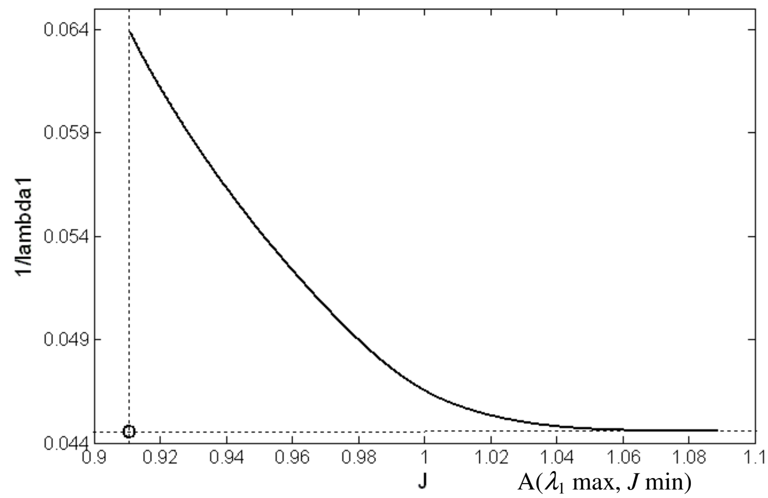

Figure 7. The Pareto front of the optimization problem in the section 4.2.2

4.2.3. Discussion

From the Table 3&4, we can see that the maximum value of the first eigenvalue λ_1 is directly proportional to the value of the column's volume J. It is a sensible relation.

The results shown in Fig. 4 & 6 are the column's optimal shape configurations corresponding to five cases of J and two cases of constraints. So, the optimization problem could be solved for both continuous and discontinuous control variables.

The Pareto front represents the possible trade-off among different objectives (λ_1, J). From the Fig. 5 & 7, we reached the following conclusion: we never have a situation in which all the objectives can be in a best possible way satisfied simultaneously (point A).

5. CONCLUSION

This paper mentioned about the optimal design of a Pfluger column. The objective function used in this research is the Maier functional from a multicriteria optimization viewpoint (λ_1, J). The sign of the analogy coefficient k between the adjoint and the original variables was determined exactly. The optimal necessary conditions for the objective function (2.15) were established. The results can be applied to determine the shape of a column that is the strongest against buckling under some given conditions. Using an optimal control principle – PMP – shows that we can control the value of the Pfluger column's first eigenvalue with the bounded and unbounded control variables.

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TÓM TẮT

Biên dạng tối ưu của cột Pfluger được xác định nhờ nguyên lí cực đại Pontryagin. Phương trình chủ đạo của bài toán được rút gọn thành một bài toán giá trị biên của phương trình vi phân bậc hai phi tuyến. Kết quả của bài toán phân tích nhận được nhờ phương pháp Spectral. Điều kiện cần đối với trị riêng thứ nhất cực đại được thiết lập để xác định phân bố tối ưu của diện tích mặt cắt ngang dọc theo trục của cột.

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