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Study on the accuracy of the numerical modeling of the groundwater movement due to spatial and temporal discretization

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ABSTRACT

Groundwater (GW) modeling has become popular in Vietnam since the last quarter of century. However not always a due attention on the accuracy of the modeling has being paid, making the modeling results be doubted and of negligible use in many cases, especially in academic works. A groundwater modeling program by finite element (FE) method (FEM) by linear shape functions has been compiled to consider the modeling accuracy due to spatial and temporal discretization. Sufficient groundwater boundary conditions in combination with aquifer parameters have been selected for needed groundwater well analytical solution in order to make assessment of the accuracy of the FE model results. Within the range of FE element size (h_{xy}) of 15m÷25m and time step 0.25day÷1.25day with backward time scheme, the water level (WL) obtained by FEM in a very narrow zone around the pumping well (30m for case of time step Δt=0.25day and 230m for Δt=1.25day) is insignificantly smaller than the true WL, while for the remaining area the WL by FEM is greater than the true WL, but not greater then 7.6cm. In all cases of different element sizes, the smaller time step, the more accurate WL results, however the improvement is hard to be more than 1cm. Some recommendations on aquifer parameter calibration, aquifer parameter determination by numerical modeling from monitoring WL and pumping test data, GW modeling for foundation seepage deformation have been proposed for required accuracy of the GW modeling for practical water resources and engineering purposes.

Keywords: Finite element, Shape function, Weighting function, Element size, Time Step.

Introduction

We know that the groundwater (GW) movement is described by a partial differential equation to represent the water level (WL) at any point in the aquifer domain though aquifer parameters with determined boundary conditions at all times. Meanwhile, in numerical modeling (including finite element (FE) modeling) the model domain is divided into a certain number of grids (elements) and modeling time into given time steps. So a difference between the model outputs and the actual values is inevitable. However, what is the extent of this difference, or more exactly what is the error of the solution by numerical model in compare to the accurate values? What the error is and whether the error is acceptable for the actual practical requirements? How the error depends on the size of the element and time step?

The paper presents some research results that help clarify the above issues through the analysis of numerical model and analytical model results. The study is demonstrated by FE method (FEM) using linear shape functions, what is also entirely consistent with the finite difference method since

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the finite difference method is a specific case of a FEM, and while error order of the two methods is the same as the FEM uses linear shape functions (Zienkiewicz and Morgan, 1983).

2. General on the GW modeling accuracy

The movement of GW in confined aquifer is described by the following equation (Bear and Verruijt, 1987; Bear, 1979):

$$\frac{\partial}{\partial x}\left(T\frac{\partial \phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(T\frac{\partial \phi}{\partial y}\right)+R-P=S\frac{\partial \phi}{\partial t} \text{ in domain } \Omega \quad (1)$$

or in a compact form:

$$\nabla'\cdot(T\nabla'\phi+R-P=S\frac{\partial \phi}{\partial t} \quad or: \ \frac{\partial}{\partial x_i}(T_{ij}\frac{\partial \phi}{\partial x_j})+R-P-S\frac{\partial \phi}{\partial t}=0 \quad (2)$$

in which: R - is areal recharge to the aquifer, P - is areal discharge from the aquifer, i=1,2,3 j=1,2,3 and $x_1=x$, $x_2=y$, and $x_3=z$ for 3-dimention xyz. For simplicity we shall consider in 2 horizontal dimensions xy.

In the FEM formulation, the hydraulic pressure ϕ is approximated as ϕ^* for the shape function N_m as:

$$\phi^* \cong \phi(x,y,t)=\sum_{m=1}^{N}\phi_m(t)N_m(x_1,x_2) \quad (3)$$

in which: N - is number of nodes, N_m - is shape function (or also called trial function) for node m, and ϕ_m - is the WL pressure at node m at time t.

The error of the WL pressure is:

$$\varepsilon=\frac{\partial}{\partial x_i}\left(T_{ij}\frac{\partial \phi}{\partial x_j}\right)+R-P-S\frac{\partial \phi}{\partial t} \quad (4)$$

With introducing weighting functions $W_l(x,y)$ (l= 1, 2,..., N) and giving them zero value at the Neumann boundary nodes of prescribed WL pressure. The FEM formulation is to find the solution that the error over the whole domain Ω is 0:

$$\int_\Omega \varepsilon \, W_l dV=0 \text{ or:}$$

$$\int_\Omega [\frac{\partial}{\partial x_i}(T_{ij}\frac{\partial \phi}{\partial x_j})+R-P-S\frac{\partial \phi}{\partial t}]W_l dV=0; \quad l=1,2,...,N \quad (5)$$

Applying Green lemma to (5) gives:

$$\int_{\partial\Omega} T_{ij}\frac{\partial \phi}{\partial x_i}n_{x_i}W_l dS-\int_\Omega [\frac{\partial}{\partial x_i}(T_{ij}\frac{\partial \phi}{\partial x_j})+R-P-S\frac{\partial \phi}{\partial t}]W_l dV=0; \ l=1,2,...,N \quad (6)$$

Using the Galerkin approach, that is the weighting function W_l is the same as the trial function N_l, (6) becomes:

$$\int_\Omega T_{ij}\frac{\partial \phi}{\partial x_i}\frac{\partial W_l}{\partial x_i}dV-\int_\Omega (R-P-S\frac{\partial \phi}{\partial t})W_l dV=\int_{\partial\Omega} T_{ij}\frac{\partial \phi}{\partial x_i}n_{x_i}W_l dS; \ l=1,2,...,N \quad (7)$$

Putting ϕ in (3) into (7) gives:

$$\sum_{m=1}^{N}\left\{\int_\Omega T_{ij}\frac{\partial \phi_m}{\partial x_i}\frac{\partial W_l}{\partial x_i}dV-\int_\Omega (R-P-S\frac{\partial \phi_m}{\partial t})W_l dV-\int_{\partial\Omega} T_{ij}\frac{\partial \phi_m}{\partial x_i}n_{x_i}W_l dS\right\}=0; \quad (8)$$
$$l=1,2,...,N$$

(8) may be written in matrix form:

$$[A]\{\Phi\}+[B]\left\{\frac{d\Phi}{dt}\right\}=\{F\} \quad (9)$$

where:

$$[A]=\sum_{e=1}^{M}[A]^e; \quad [B]=\sum_{e=1}^{M}[B]^e; \quad \{F\}=\sum_{e=1}^{M}\{F\}^e \quad (10)$$

in which: M is the number of elements.

The components of matrices in(10) is:

$$A^e_{l,m}=\int_{\Omega^e} T_{ij}\frac{\partial N_l}{\partial x_i}\frac{\partial N_m}{\partial x_j}dV; \quad B^e_{lm}=\int_{\Omega^e} SN_l N_m dV; \quad (11)$$
$$F^e_l=-\int_{\Omega^e}(R-P)N_l dV-\int_{\Omega^e} T_{ij}\frac{\partial \varphi_l}{\partial x_i}n_{x_i}N_l dS$$

Using the linear shape functions with the maximal size of the element's sides of h (the side size along x and y directions) will give ϕ of error order $O(h^2)$ and the error order in time is also $O(\Delta t^2_n)$ for central time difference scheme, and order $O(\Delta t_n)$ for forward and backward time difference scheme (Zienkiewicz and Morgan, 1983).

There are existing three types of error in numerical modeling:

- The most important one is the error due to discretization: (1) the incompleteness of partial differential equation; (2) the un-satisfaction of boundary conditions; (3) the utilization of trial functions in the approximation process;

- The round-off error in the calculations;

- The error of the mathematical model.

The second error due to round-off is minimized at the present for the powerful computer technology, and may be negligible for not large system of equations to be solved. The third error due to mathematical model is out of the scope of the study since any numerical model must be built based on a given mathematical model, which is considered as an exact model. Within this paper, the accuracy due to spatial and temporal discretization is considered.

3. Analysis of the accuracy of FE modeling of confined GW aquifer

The study on error of groundwater movement numerical models is conducted through an analysis and comparison between the results of the analytical model (truly accurate) with the results of FE model for a confined GW aquifer with a pumping well.

As we know that the larger the model domain, the larger number of elements and the longer time of boundary influence, so it is needed to select not very large model domain to completely eliminate the round-off error in solution of system of equations of the numerical model. Also, the greater hydraulic conductivity and/or the smaller the storability the greater radius of pumping influence so that the analytical analysis may not be applicable for the infinite or semi infinite requirement of the aquifer (i.e. the infinite Neumann-kind boundary) etc. Therefore, it had been selected the aquifer hydraulic conductivity, thickness, storability, analysis time, the model domain sizes and the location of the pumping well of such figures that the analysis time is shorter than the pumping influence time to the boundary.

3.1. Modeling domain and time limits for error comparison

The confined aquifer distribution area is selected to be a circle of radius *R* with a pumping well of a constant pumping rate located at the circle center (Figure 1).

For accuracy assessment, the results of FE model are to be compared with analytical results. The analytical analysis can only be applied for the following certain flow conditions (Driscoll, 1987):

- Laminar flow condition and uniform flow over the entire aquifer thickness;

- The aquifer has constant hydraulic parameters over the entire domain: hydraulic conductivity, thickness, storability;

- Infinite or semi-infinite domain of aquifer distribution of a simple configuration;

- The boundary condition is either constant head or constant flow over the entire time domain.

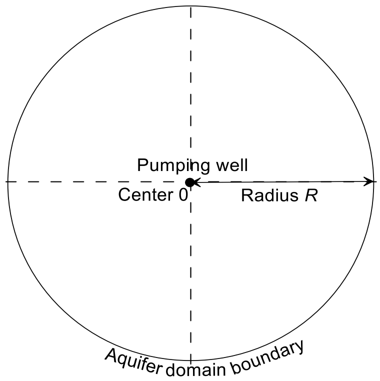

Figure 1. Aquifer domain and pumping well

Therefore, in order to compare the results of FE model with analytical results, the time must be smaller the time when the pumping has influence to the boundary. The following aquifer parameters are used in the model: the thickness is *b*=10m, hydraulic conductivity is *K*=5m/day, storability is *S*=0.001. The well constant pumping rate is 100m^3/day. The initial WL pressure is 10m (it is assume that the top of the aquifer is at elevation 0m, and therefore the WL drawdown has to be not greater than 10m). The radius of the aquifer domain is 1000m. In this case the radius of influence of the pumping is (Polubarinova-Kochina, 1977):

$$R_{\inf} = 1.5\sqrt{\frac{Kb}{S}t} \quad (12)$$

In order for R_{inf}<1000m the time needs to be:

$$t = \frac{R^2 S}{2.25Kb} \Rightarrow t < \frac{1000^2 \times 0.001}{2.25 \times 5 \times 10} \Rightarrow t < 8.9\text{(day)}$$

3.2. Used element sizes

Within this work, the FE mesh consists of either squares or isosceles right triangles in the boundary. The FE mesh generation (node's coordinates, node numberings and element numberings) and FE GW movement programming have been adopted from the research study No. NCCB-ĐHUD.2012-G/04 (Nguyen Van Hoang, 2014-2016). The sizes of square's side and the isosceles right triangle's two sides are to be used 15m, 20m and 25m. Figure 2 presents FE mesh of element size 20m and Figure 3 presents a part ofthe FE mesh with nodes' and elements' numberings. Similarly is for FE mesh of element's size of 25m. For the element size of 15m, the entire circle domains contains a very large number of nodes for which PC with 3GB of memory may hardly deal with the matrix sizes. Therefore, with the symmetrical configuration of GW movement, only the first quarter of the circle domain is used for FE modeling as shown in Figure 2.

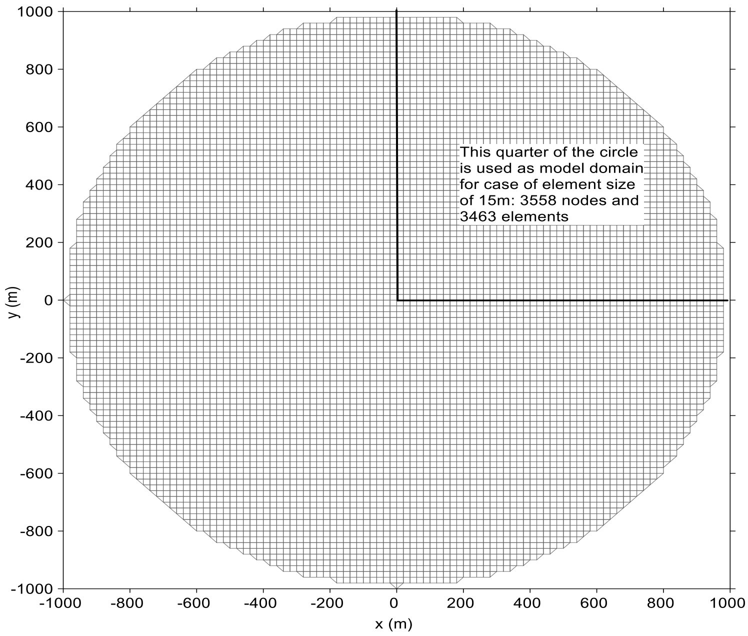

Figure 2. 20-m element size FE mesh: 7843 nodes and 7760 elements

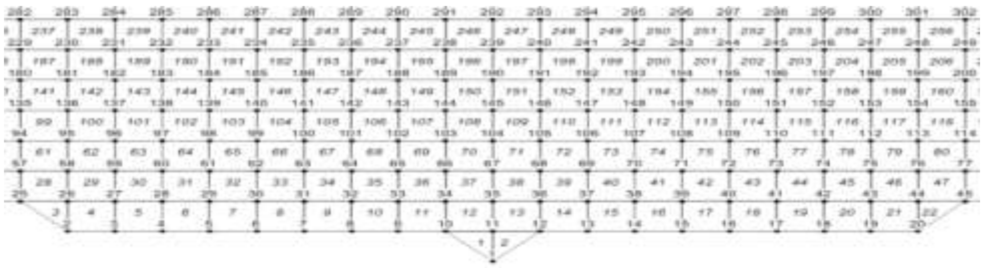

Figure 3. A part of 20m-element size FE mesh: node and element numbering (in the element center)

3.3. Time steps

For each case of element size, the time steps *Δt* in the FE modeling of 0.25, 0.5, 0.75, 1.00 and 1.25 days are used. A program of calculation of WL by this method (Driscoll, 1987) had been made. The WL at every nodes of the three FEmeshes had been determined by this program for all the times in concern.

4. Results

4.1. Presentation of the model results

From the modeling results, various presentations such as WL contour lines, WL profiles through particular lines, hydraulic gradient, water velocity etc. can be additionally determined and drawn which are useful for GW solute transport and geotechnical filtration deformation analyses.

Figure 4 illustrates the WL contour lines for the time of 5days since the time when pumping started. While Figure 5 presents the correct use of GW movement symmetrical nature and it would contribute to faster modeling process and accuracy improvement thanks to mesh refinement.

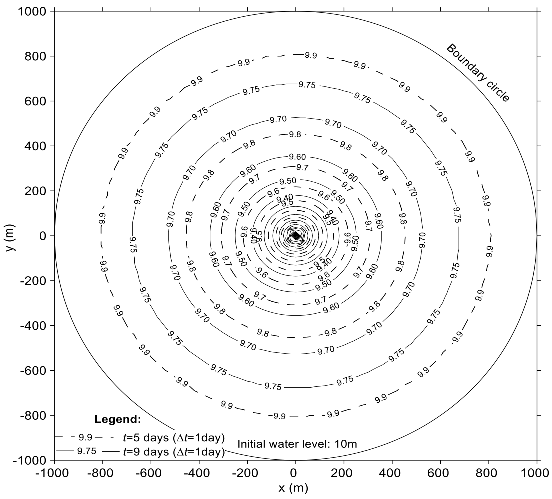

Figure 4. WL at the times of 5 days and 9 days: an entire circle model domain

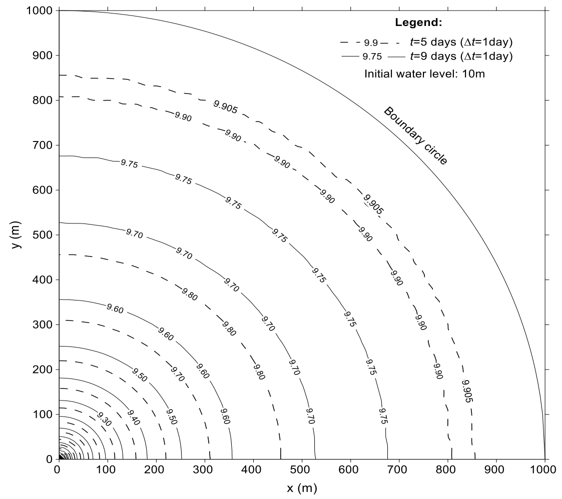

Figure 5. WL at the times of 5 days and 9 days: a quarter of circle model domain

4.2. Accuracy assessment

The modeling WL accuracy may be accessed through difference between the analytical and numerical model WL or the ratio between the difference between analytical and numerical WL and the analytical WL (hereafter referred as relative error). For practical orientation tasks, the accuracy assessment may be made for particular interested and sensitive locations in the model domain such as the WL drawdown in the narrow zone around the pumping well, hydraulic gradient in the adjacent area to the polluted (including salinized) GW areas, areas of sensitivity to infiltration soil deformation etc.

For the pumping well, the node representing the pumping well does not reflect the true physics of a physical well with a given radius, so the WL determine be FE model in this node definitely is out of satisfactory accuracy ranges. Therefore, the analysis of the error and accuracy shall not include this node's values into the comparison.

The difference between the analytical and numerical model WL shall not be used with absolute value (since its is very often important to know where the WL is over estimated or underestimated for particular geotechnical tasks) for the case of Δt=0.25 days is presented in Figure 6 (for model with element size of 20m) and 7 (for model with element size of 15m).

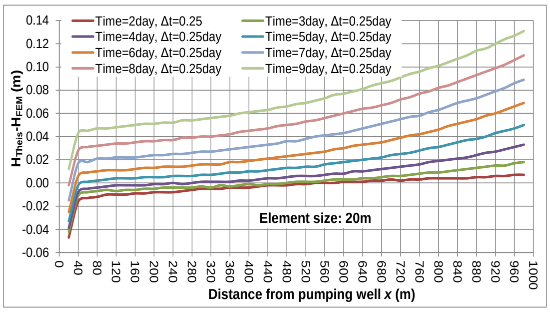

Figure 6. Difference between the analytical and numerical model WL: element size 20m

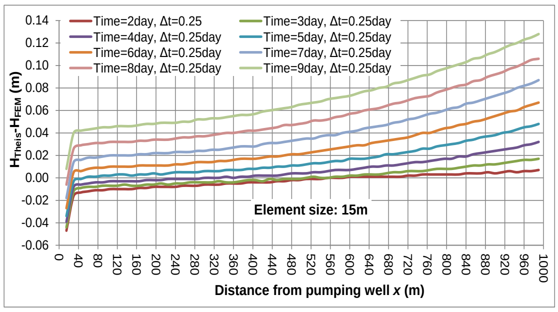

Figure 7. Difference between the analytical and numerical model WL: element size 15m

The relative error can be used in such cases for that the magnitude of the differences between the exact and numerical WL does not play important role, for example GW numerical modeling for resources evaluation. Figure 8 and 9 presents these relative errors.

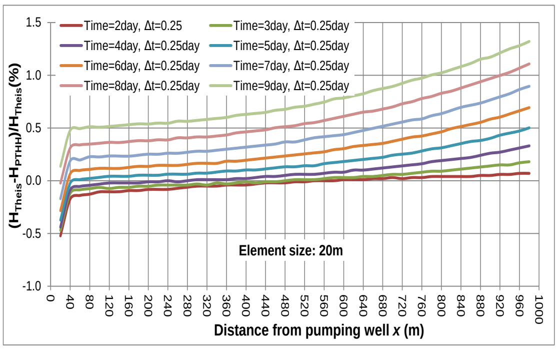

Figure 8. Relative WL error: element size 20m

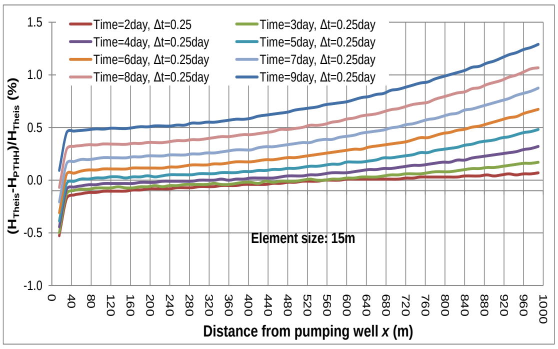

Figure 9. Relative WL error: element size 15m

During the time from the pumping beginning to 9 days, the difference between exact WL and FEM WL is from -0.05m to +0.12m. These WL errors are presented in the form of contour lines in Figure 10 and 11 for element size 20m and 15m respectively.

The mean error of minimal, average and maximal WL errors over the entire model domain for the three cases of element sizes 25m, 20m and 15m are summarized in Table 1 and Figure 12. As it was described above, the error is an order of squared element size and of time step, the errors for time step 0.25day and 1.25day were drawn versus the squared ratio of element sizes to element size of 20m in Figure 13. Figure 13 shows that there is little change in errors as the element sizes change from 25m to 15m.

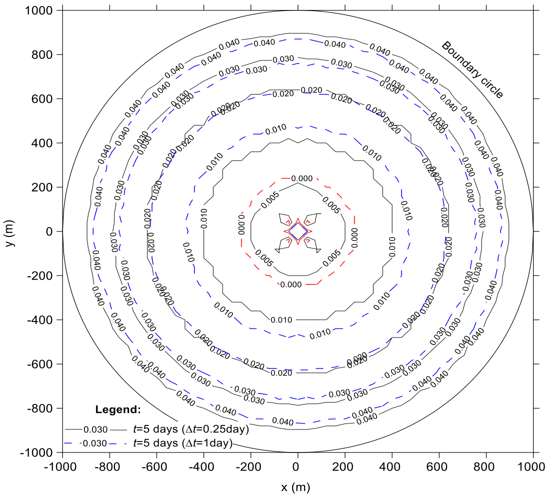

Figure 10. WL error at time 5day, Δ*t*=0.25day and Δ*t*=1day: element size 20m

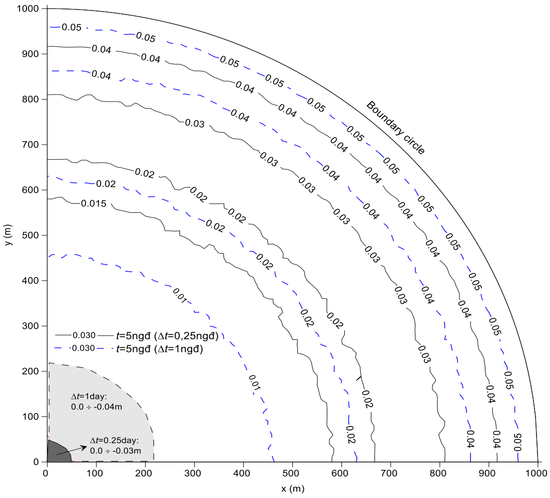

Figure 11. WL error at time 5day, Δ*t*=0.25day and Δ*t*=1day: element size 15m

Table 1. Mean WL errors of WL minimal, average and maximal errors

Time step *Δt* (day)	0.25	0.50	0.75	1.00	1.25
Element size: h_{xy}=15m					
Mean (Min)	-0.027	-0.031	-0.037	-0.038	-0.038
Mean (Max)	0.065	0.066	0.067	0.071	0.071
Mean (Avg.)	0.038	0.039	0.038	0.041	0.041
Element size: h_{xy}=20m					
Mean (Min)	-0.028	-0.031	-0.038	-0.037	-0.037
Mean (Max)	0.064	0.066	0.065	0.071	0.071
Mean (Avg.)	0.037	0.039	0.036	0.041	0.041
Element size: h_{xy}=25m					
Mean (Min)	-0.027	-0.031	-0.037	-0.038	-0.038
Mean (Max)	0.065	0.066	0.067	0.071	0.076
Mean (Avg.)	0.038	0.039	0.038	0.041	0.044

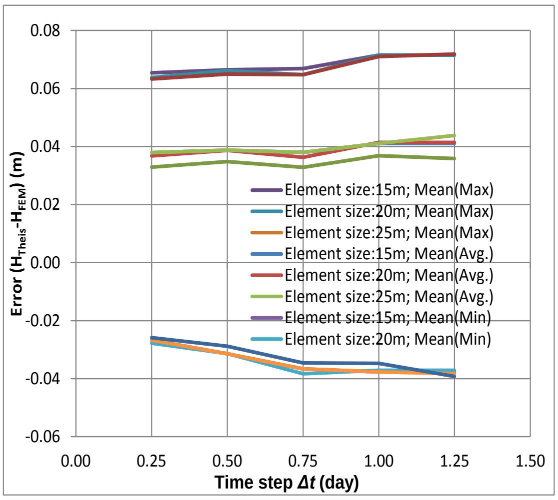

Figure 12. Variation of errors with different element seize and time step

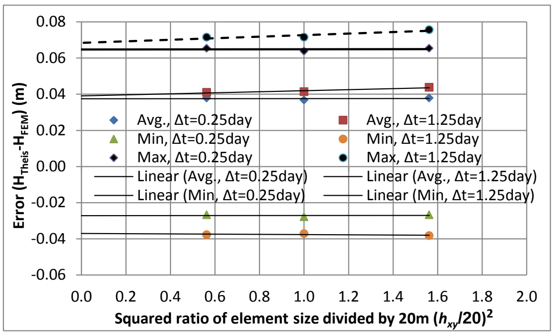

Figure 13. Variation of errors with different element seize and time step

5. Concluding remarks and recommendations

From the above described analysis on the WL errors determined by the FE model for a number of cases of models with different element sizes and time steps the following concluding remarks can be drawn:

- For the narrow zone around the pumping well, the FEM WL is smaller than the true WL values, but the difference is not greater than 4cm; the smaller time step, the greater the precision, and this zone has a radius of 230m (time step 1day) down to about 30 meters (time step 0.25day);

- Outside this narrow negligible area around pumping well where FEM WL is underestimated, in the most remaining model domain the FEM WL is greater than the true WL, but the difference does not exceed 7.6cm;

- In all cases, the smaller time step the higher precision, but the higher accuracy value isnegligibly small(the highest accuracy had improve about not greater than 1cm);

- In all cases of element sizes from 15m to 25m and time steps from 0.25 days to 1.25 days, the average error over the entire model domain is of less than 4cm.

Since there is existing spatial error patterns as above-mentioned, it can draw some following conclusions and recommendations for practical problems:

- It needs pay special attention to the use of numerical models in analysis of hydrogeological parameters according to the results of WL monitoring and pumping experiment data, i.e., the location in the pumping wells in the model domain model because at different locations in model domain there are different errors;

- The model is capable of determining WL at a high precision, which satisfies most practical problems of groundwater hydrodynamics, groundwater extraction forecasting, calculation of hydrogeological parameters etc.;

- The above results have shown that in the model domain, there is pattern of overestimated and underestimated WL than the actual values. This means that the model calibration (hydrogeological parameters estimation) in the field of numerical GW modeling is a sensitive issue which could lead to the opposite results if it only merely considers the difference of observations and numerical WL without paying attention on the numerical error pattern;

- In the application of groundwater movement numerical modeling for different geotechnical problems such as seepage deformation, land subsidence due to GW abstraction etc., it is required to pay much attention to the accuracy of the simulation results at the concerned engineering structures and sensitive seepage geotechnical soil conditions.

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