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CONTINUOUS REGULARIZATION METHOD FOR ILL-POSED OPERATOR EQUATIONS OF HAMMERSTEIN TYPE

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Abstract. The aim of this paper is to study a method of approximating a solution of the operator equation of Hammerstein type $x + F_2F_1(x) = f$ on the base of constructing a system of differential equations of the first order, where $F_i, i = 1, 2$, are the continuous monotone operators in real Hilbert space H. Then this method is considered in connection with finite-dimentional approximations for H.

Tóm tắt. Mục đích của bài báo là nghiên cứu một phương pháp xấp xỉ nghiệm của phương trình toán tử loại Hammerstein $x + F_2F_1(x) = f$ dựa trên việc xây dựng hệ phương trình vi phân cấp một, ở đây các toán tử $F_i,\ i = 1, 2$, là đơn điệu và liên tục trong không gian Hilbert H. Sau đó, phương pháp này được xét liên kết với việc xấp xỉ hữu hạn chiều của H.

1. INTRODUCTION

Let H be a real Hilbert space with norm and scalar product denoted by $\|.\|$ and $\langle x^*, x\rangle$, respectively. Let $F_i, i = 1, 2$, be monotone, in general nonlinear, bounded (i.e. image of any bounded subset is bounded) and continuous operators.

Our main aim of this paper is to study a stable method of finding an approximative solution for the equation of Hammerstein type

$$x + F_2F_1(x) = f,\ f \ \in\ \mathcal{R}(I + F_2F_1), \tag{1.1}$$

where I and $\mathcal{R}(A)$ denote the identity operator in H and the range of the operator A, respectively. Note that the solution set of (1.1), denoted by S_0, is closed convex (see [1]).

Usually instead of $F_i, i = 1, 2$, and f we know their monotone continuous approximations F_i^h and f_δ such that

$$\|F_1^h(x) - F_1(x)\| \leqslant hg(\|x\|),$$
$$\|F_2^h(x) - F_2(x)\| \leqslant hg(\|x\|) \quad \forall x \in H,$$

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where $g(t)$ is a real nonnegative, non-decreasing, bounded function (the image of a bounded set is bounded), and $\|f_\delta - f\| \leqslant \delta$. Without additional conditions for the operators F_i such as the strongly monotone property, equation (1.1) is ill-posed. For example, consider the case $H = \mathbf{E}^2$, the Euclidean space, and

$$F_1 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \quad x = (x_1, x_2).$$

It is easy to verify that $\langle F_1 x, x\rangle = x_1^2 \geqslant 0$, and $\langle F_2 x, x\rangle = x_2^2 \geqslant 0 \forall x \in \mathbf{E}^2$. It means that $F_i, i = 1, 2$, are monotone. Equation (1.1) has the form $0x_1 = f_1, \quad 2x_1 = f_2$ with $f = (f_1, f_2)$. Obviously, this system of equations has a unique solution when $f = (0, f_2)$ for arbitrary f_2. When $f_\delta = (f_1^\delta, f_2)$ with $f_1^\delta \neq 0$ equation (1.1) in this case does not have solution. So, equation (1.1) with the monotone operators $F_1, i = 1, 2$, in general is ill-posed.

To solve (1.1) we need use stable methods. One of the those is the operator equation

$$x + F_{2,\alpha}^h F_{1,\alpha}^h(x) = f_\delta \tag{1.2}$$

(see [1], [5]), where $F_{i,\alpha}^h = F_i^h + \alpha I$, $\alpha > 0$ is the small parameter of regularization. For every $\alpha > 0$ equation (1.2) has a unique solution $x_\alpha^{h,\delta}$, and the sequence $\{x_\alpha^{h,\delta}\}$ converges to a solution x_0 of satisfying

$$\|x_0\|^2 + \|x_0^*\|^2 = \min_{x\in S_0}\left(\|x\|^2 + \|F_1(x)\|^2\right), x_0^* = F_1(x_0),$$

as $(h+\delta)/\alpha, \alpha \to 0$. Moreover, this solution $x_\alpha^{h,\delta}$, for every fixed $\alpha > 0$, depends continuously on $F_i^h, i = 1, 2$ and f_δ.

Recently, the use of differential equations for regularizing ill-posed convex optimization and nonlinear monotone problems is intensively investigated (see [6]-[14] and references therein), because by discretiting them one can obtain much different iterative processes. In this paper, this idea is developed for non-monotone, in general, Hammerstein equation, i.e., we find a strong differentiable function $u(t) : [t_0, +\infty) \to H, t_0 \geqslant 0$, which is a solution of some differetial equation such that

$$\lim_{t\to+\infty} u(t) = x_0. \tag{1.3}$$

In Section 2, we give a system of differential equations with the solution $u(t), u^*(t)$ where $u(t)$ satisfies (1.3). The Galerkin approximations $u_n(t)$ for $u(t)$ with the property

$$\lim_{n,t\to+\infty} u_n(t) = x_0,$$

are considered in Section 3.

Above and below, the symbols $\rightharpoonup$ and $\to$ denote the weak convergence and convergence in the norm, respectively.

2. THE INFINITE-DIMENSIONAL CONTINUOUS REGULARIZATION

Consider the system of differential equations

$$\begin{aligned} &\frac{du(t)}{dt} + \gamma(t)\left[F_1^{h(t)}(u(t)) + \alpha(t)u(t) - u^*(t)\right] = \theta, \\ &\frac{du^*(t)}{dt} + \gamma(t)\left[F_2^{h(t)}(u^*(t)) + \alpha(t)u^*(t) + u(t) - f(t)\right] = \theta, \\ &u(t_0) = u_0, u^*(t_0) = u_0^*, t \geqslant t_0 \geqslant 0, \end{aligned} \tag{2.1}$$

where u_0, u_0^* are the fixed elements in H, θ denotes the zero element, $h = h(t), \alpha = \alpha(t) > 0, t \geqslant 0$, $\alpha(t)$ is a convex decreasing differentiable function, $\gamma(t)$ is a nondecreasing positive and differentiable function such that

$$\begin{gathered}\lim_{t\to+\infty} \alpha(t) = \lim_{t\to+\infty} h(t) = 0,\\ \lim_{t\to+\infty} \frac{h(t)}{\alpha(t)} = \lim_{t\to+\infty} \frac{\alpha'(t)}{\alpha^2(t)\gamma(t)} = \lim_{t\to+\infty} \frac{\gamma'(t)}{\alpha(t)\gamma^2(t)} = 0.\end{gathered} \tag{2.2}$$

In order to prove that $\lim_{t\to+\infty} u(t) = x_0$, we study the system of differential equations

$$\begin{aligned}&\frac{dy(t,\tau)}{dt} + \gamma(t)\Big[F_1(y(t,\tau)) + \alpha(\tau)y(t,\tau) - y^*(t,\tau)\Big] = \theta,\\ &\frac{dy^*(t,\tau)}{dt} + \gamma(t)\Big[F_2(y^*(t,\tau)) + \alpha(\tau)y^*(t,\tau) + y(t,\tau) - f\Big] = \theta,\\ &y(t_0,\tau) = u_0, y^*(t_0,\tau) = u_0^*, \forall t \geqslant t_0\end{aligned} \tag{2.3}$$

depending on the parameter $\tau \geqslant t_0$.

We have a result.

Theorem 2.1. *Assume that the following conditions hold:*

(i) problems (2.1) *and* (2.3) *possess solutions in the class* $C^1[t_0, +\infty)$ *for any* $u_0, u_0^* \in H$ *with* $\|u(t)\|, \|u^*(t)\| \leqslant d_1, d_1 > 0, t \geqslant t_0$.

(ii) the functions $\alpha(t), h(t)$ *and* $\gamma(t)$ *satisfy the above conditions.*

Then, $\lim_{\tau\to+\infty} u(\tau) = x_0$.

Proof. Set

$$\begin{aligned}\tilde r(t,\tau) &= \tilde r_1(t,\tau) + \tilde r_2(t,\tau),\\ \tilde r_1(t,\tau) &= \|y(t,\tau) - x_\alpha(\tau)\|^2,\\ \tilde r_2(t,\tau) &= \|y^*(t,\tau) - x_\alpha^*(\tau)\|^2,\end{aligned}$$

where $(x_\alpha(\tau), x_\alpha^*(\tau)), x_\alpha^*(\tau) = F_1(x_\alpha(\tau))$, is the unique solution of the system of operator equations

$$\begin{aligned}&F_1(x_\alpha(\tau)) + \alpha(\tau)x_\alpha(\tau) - x_\alpha^*(\tau) = \theta,\\ &F_2(x_\alpha^*(\tau)) + \alpha(\tau)x_\alpha^*(\tau) + x_\alpha(\tau) - f = \theta,\end{aligned} \tag{2.4}$$

and $\lim_{\tau\to+\infty} x_\alpha(\tau) = x_0$ (see [1]). Since F_1 is continuous, then $x_0^* = \lim_{\tau\to+\infty} x_\alpha^*(\tau)$. Now, from (2.3) and (2.4) it follows

$$\begin{aligned}&\langle\frac{d(y(t,\tau) - x_\alpha(\tau))}{dt}, y(t,\tau) - x_\alpha(\tau)\rangle + \gamma(t)\Big[\langle F_1(y(t,\tau)) - F_1(x_\alpha(\tau)),\\ &y(t,\tau) - x_\alpha(\tau)\rangle + \alpha(\tau)\tilde r_1(t,\tau) + \langle x_\alpha^*(\tau) - y^*(t,\tau), y(t,\tau) - x_\alpha(\tau)\rangle\Big] = 0,\\ &\langle\frac{d(y^*(t,\tau) - x_\alpha^*(\tau))}{dt}, y^*(t,\tau) - x_\alpha^*(\tau)\rangle + \gamma(t)\Big[\langle F_2(y^*(t,\tau)) - F_2(x_\alpha^*(\tau)),\\ &y^*(t,\tau) - x_\alpha^*(\tau)\rangle + \alpha(\tau)\tilde r_2(t,\tau) + \langle y(t,\tau) - x_\alpha(\tau), y^*(t,\tau) - x_\alpha^*(\tau)\rangle\Big] = 0.\end{aligned}$$

Substituting the two last equalities and using the relation

$$\frac{d\|x(t)\|^2}{dt} = 2\langle\frac{dx(t)}{dt}, x(t)\rangle$$

and the monotone property of $F_i, i = 1, 2$, we have got

$$\frac{d\tilde r(t,\tau)}{dt} + 2\gamma(t)\alpha(\tau)\tilde r(t,\tau) \leqslant 0.$$

Hence,

$$\tilde{r}(t,\tau) \leqslant \tilde{r}(t_0,\tau)\exp[-2\alpha(\tau)\int_{t_0}^{t}\gamma(t)dt], \tag{2.5}$$

where

$$\begin{aligned}\tilde{r}(t_0,\tau) &= \|y(t_0,\tau)-x_\alpha(\tau)\|^2 + \|y^*(t_0,\tau)-x_\alpha^*(\tau)\|^2\\ &\leqslant 2[\|y(t_0,\tau)\|^2 + \|x_\alpha(\tau)\|^2 + \|y^*(t_0,\tau)\|^2 + \|x_\alpha^*(\tau)\|^2]\\ &\leqslant 2[\|u_0\|^2 + \|u_0^*\|^2 + \|x_0\|^2 + \|F_1(x_0)\|^2].\end{aligned}$$

Consequently, from (2.2), (2.5) and the properties of $\gamma(t), \alpha(t)$ we can obtain (see [13] for details)

$$\lim_{\tau\to+\infty}\tilde{r}(\tau,\tau) = 0$$

and the boundness of $\{y(t,\tau)\}$ and $\{y^*(t,\tau)\}$. Therefore,

$$\lim_{\tau\to+\infty} y(\tau,\tau) = x_0,$$

and there exists a positive constant d_2 such that $\|y(t,\tau)\|, \|y^*(t,\tau)\| \leqslant d_2$. Further, set

$$\begin{aligned}\tilde{R}(t,\tau) &= \tilde{R}_1(t,\tau) + \tilde{R}_2(t,\tau),\\ \tilde{R}_1(t,\tau) &= \|u(t)-y(t,\tau)\|^2,\\ \tilde{R}_2(t,\tau) &= \|u^*(t)-y^*(t,\tau)\|^2.\end{aligned}$$

On the base of (2.1) and (2.3) we can write

$$\begin{aligned}&\langle\frac{d(u(t)-y(t,\tau))}{dt}, u(t)-y(t,\tau)\rangle + \gamma(t)\Big[\langle F_1^{h(t)}(u(t)) - F_1(y(t,\tau)),\\ &u(t)-y(t,\tau)\rangle + \langle\alpha(t)u(t)-\alpha(\tau)y(t,\tau), u(t)-y(t,\tau)\rangle\\ &+\langle y^*(t,\tau)-u^*(t), u(t)-y(t,\tau)\rangle\Big] = 0,\\ &\langle\frac{d(u^*(t)-y^*(t,\tau))}{dt}, u^*(t)-y^*(t,\tau)\rangle + \gamma(t)\Big[\langle F_2^{h(t)}(u^*(t)) - F_2(y^*(t,\tau)),\\ &u^*(t)-y^*(t,\tau)\rangle + \langle\alpha(t)u^*(t)-\alpha(\tau)y^*(t,\tau), u^*(t)-y^*(t,\tau)\rangle\\ &+\langle u(t)-y(t,\tau), u^*(t)-y^*(t,\tau)\rangle\Big] = 0.\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{d\tilde{R}(t,\tau)}{dt} + 2\gamma(t)\alpha(\tau)\tilde{R}(t,\tau) \leqslant\\ &\gamma(t)[h(t)g(\|y(t,\tau)\|) + |\alpha(t)-\alpha(\tau)|\|y(t,\tau)\|]\|u(t)-y(t,\tau)\| +\\ &\gamma(t)[h(t)g(\|y^*(t,\tau)\|) + |\alpha(t)-\alpha(\tau)|\|y^*(t,\tau)\|]\|u^*(t)-y^*(t,\tau)\|.\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{d\tilde{R}(t,\tau)}{dt} \leqslant D\gamma(t)[h(t) + |\alpha(t)-\alpha(\tau)|] - 2\tilde{\alpha}(t)\tilde{R}(t,\tau),\\ &\tilde{\alpha}(t) = \gamma(t)\alpha(\tau), D = 2\max\{g(d_2)(d_1+d_2), d_2\}.\end{aligned}$$

It is not difficult to verify that

$$\begin{aligned}
&\tilde{R}(\tau,\tau) \leqslant R_1(\tau) + R_2(\tau)\\
&R_1(\tau) = D\int_{t_0}^{\tau} \gamma(t)h(t)\xi(t)dt/\xi(\tau),\\
&R_2(\tau) = D\int_{t_0}^{\tau} \gamma(t)\alpha'(t)(t-\tau)\xi(t)dt/\xi(\tau),\\
&\xi(s) = exp(\int_{t_0}^{s} \tilde{\alpha}(t)dt.
\end{aligned}$$

Therefore, $\lim_{t\to+\infty} R_1(\tau) = \lim_{t\to+\infty} R_2(\tau) = 0$. Since $\|x_0 - u(\tau)\| \leqslant \|x_0 - x_\alpha(\tau)\| + \|x_\alpha(\tau) - y(\tau,\tau)\| + \|y(\tau,\tau) - u(\tau)\|$, then $\lim_{\tau\to+\infty} u(\tau) = x_0$. Theorem is proved.

Remark. The solution existence of (2.1) or (2.3) is followed from [7], [15] and [16], when $F_i^{h(t)}$ are weakly continuous or Lipschitz continuous for each $t \geqslant t_0$.

3. FINITE-DIMENSIONAL REGULARIZATION

Consider the system of finite-dimensional problems

$$\begin{aligned}
&\frac{du_n(t)}{dt} + \gamma(t)\Big[F_{1,n}^{h(t)}(u_n(t)) + \alpha(t)u_n(t) - u_n^*(t)\Big] = \theta,\\
&\frac{du_n^*(t)}{dt} + \gamma(t)\Big[F_{2,n}^{h(t)}(u_n^*(t)) + \alpha(t)u_n^*(t) + u_n(t) - f_n(t)\Big] = \theta,\\
&u_n(t_0) = P_n u_0, u_n^*(t_0) = P_n u_0^*,
\end{aligned} \tag{3.1}$$

where $F_{1,n}^{h(t)} = P_n^* F_1^{h(t)} P_n, F_{2,n}^{h(t)} = P_n F_2^{h(t)} P_n^*$, $f_n(t) = P_n f(t)$, P_n is a linear projection from H onto its finite-dimensional subspace H_n such that $H_n \subset H_{n+1}$, $P_n x \to x$, as $n \to \infty$ for every $x \in H$, and P_n^* is the dual of P_n with $\|P_n\| \leqslant \tilde{c} =$ constant, for all n, and $u_n(t), u_n^*(t) : [t_0, +\infty) \to H_n$.

To prove

$$\lim_{n,t\to+\infty} u_n(t) = x_0,$$

as in the Section 2, we use the system of finite-dimensional equations

$$\begin{aligned}
&\frac{dy_n(t,\tau)}{dt} + \gamma(t)\Big[F_{1,n}(y_n(t,\tau)) + \alpha(\tau)y_n(t,\tau) - y_n^*(t,\tau)\Big] = \theta,\\
&\frac{dy_n^*(t,\tau)}{dt} + \gamma(t)\Big[F_{2,n}(y_n^*(t,\tau)) + \alpha(\tau)y_n^*(t,\tau) + y_n(t,\tau) - f_n\Big] = \theta,\\
&y_n(t_0,\tau) = P_n u_0, y_n^*(t_0,\tau) = P_n u_0^*, \forall t \geqslant t_0,
\end{aligned} \tag{3.2}$$

depending on the parameter $\tau \geqslant t_0$, where $F_{1,n} = P_n F_1 P_n^*, F_{2,n} = P_n^* F_2 P_n$, and $f_n = P_n f$.

We have a result.

Theorem 3.1. *Assume that the following conditions hold:*

(i) problems (3.1) *and* (3.2) *possess solutions in the class* $C^1[t_0, +\infty)$ *for any* $u_0, u_0^* \in H$ *with* $\|u_n(t)\|, \|u_n^*(t)\| \leqslant d_3, d_3 > 0, t \geqslant t_0$.

(ii) the functions $\alpha(t), h(t)$ *and* $\gamma(t)$ *satisfy the above conditions.*

(iii) $F_i, i = 1, 2$, *are Fréchet differentiable with Lipschitz continuous derivatives (common Lipschitz constant* L*), there exist* x^1 *and* x^2 *such that*

$$F_1'(x_0)^*x^1 + x^2 = x_0,$$
$$F_2'(x_0^*)^*x^2 - x^1 = x_0^*,$$

$L\max_{i=1,2}\|x^i\|/2 < 1,$ *and*

$$\lim_{n,\tau\to+\infty} \xi_n/\alpha(\tau) = 0,$$

where

$$\xi_n = \max\{\|(I-P_n)x_0\|, \|(I-P_n^*)F_1(x_0)\|, \|(I-P_n)f\|, \|(I-P_n^*)x^1\|, \|(I-P_n)x^2\|\}.$$

Then, $\lim_{n,\tau\to+\infty} u_n(\tau) = x_0$.

Proof. We recall that the finite-dimensional problems

$$x + F_{2,\alpha}^n F_{1,\alpha}^n(x) \ = \ f_n, \ x \in H_n,$$

where $F_{2,\alpha}^n = F_{2,n} + \alpha(\tau)I, F_{1,\alpha}^n = F_{1,n} + \alpha(\tau)I$, have a unique solution $x_{\alpha,n}(\tau)$. This solution and $x_{\alpha,n}^*(\tau)$ are the solution of the follwing equations

$$\begin{aligned} &F_{1,n}(x_{\alpha,n}(\tau)) + \alpha(\tau)x_{\alpha,n}(\tau) - x_{\alpha,n}^*(\tau) = \theta, \\ &F_{2,n}(x_{\alpha,n}^*(\tau)) + \alpha(\tau)x_{\alpha,n}^*(\tau) + x_{\alpha,n}(\tau) - f_n = \theta, \end{aligned} \tag{3.3}$$

and under condition (iii) plus $\lim_{\tau\to+\infty}\alpha(\tau) = 0$ we have

$$\lim_{n,\tau\to+\infty} x_{\alpha,n}(\tau) = x_0, \lim_{n,\tau\to+\infty} x_{\alpha,n}^*(\tau) = x_0^*$$

(see [5] or Appendix).

Set

$$\tilde{r}_n(t,\tau) = \tilde{r}_{1,n}(t,\tau) + \tilde{r}_{2,n}(t,\tau),$$
$$\tilde{r}_{1,n}(t,\tau) = \|y_n(t,\tau) - x_{\alpha,n}(\tau)\|^2,$$
$$\tilde{r}_{2,n}(t,\tau) = \|y_n^*(t,\tau) - x_{\alpha,n}^*(\tau)\|^2.$$

From (3.2) and (3.3) it follows

$$\begin{aligned} \langle \frac{d(y_n(t,\tau) - x_{\alpha,n}(\tau))}{dt}, &y_n(t,\tau) - x_{\alpha,n}(\tau)\rangle + \gamma(t)\Big[\langle F_{1,n}(y_n(t,\tau)) \\ &-F_{1,n}(x_{\alpha,n}(\tau)), y_n(t,\tau) - x_{\alpha,n}(\tau)\rangle + \alpha(\tau)\tilde{r}_{1,n}(t,\tau) \\ &+ \langle x_{\alpha,n}^*(\tau) - y_n^*(t,\tau), y_n(t,\tau) - x_{\alpha,n}(\tau)\rangle\Big] = 0, \\ \langle \frac{d(y_n^*(t,\tau) - x_{\alpha,n}^*(\tau))}{dt}, &y_n^*(t,\tau) - x_{\alpha,n}^*(\tau)\rangle + \gamma(t)\Big[\langle F_{2,n}(y_n^*(t,\tau)) \\ &-F_{2,n}(x_{\alpha,n}^*(\tau)), y_n^*(t,\tau) - x_{\alpha,n}^*(\tau)\rangle + \alpha(\tau)\tilde{r}_{2,n}(t,\tau) \\ &+ \langle x_{\alpha,n}(\tau) - y_n(t,\tau), y_n^*(t,\tau) - x_{\alpha,n}^*(\tau)\rangle\Big] = 0. \end{aligned}$$

Therefore,

$$\frac{d\tilde{r}_n(t,\tau)}{dt} + 2\gamma(t)\alpha(\tau)\tilde{r}_n(t,\tau) \leqslant 0.$$

Hence,

$$\tilde{r}_n(t,\tau) \leqslant \tilde{r}_n(t_0,\tau)\exp[-2\alpha(\tau)\int_{t_0}^t \gamma(t)dt]$$

with

$$\begin{aligned}\tilde{r}_n(t_0,\tau) &= \|y_n(t_0,\tau) - x_{\alpha,n}(\tau)\|^2 + \|y_n^*(t_0,\tau) - x_{\alpha,n}^*(\tau)\|^2\\ &\leqslant 2c[\|y(t_0,\tau)\|^2 + \|x_\alpha(\tau)\|^2 + \|y^*(t_0,\tau)\|^2 + \|x_\alpha^*(\tau)\|^2]\\ &\leqslant 2c[\|u_0\|^2 + \|u_0^*\|^2 + \|x_0\|^2 + \|F_1(x_0)\|^2].\end{aligned}$$

Thus,

$$\lim_{n,\tau\to+\infty} \tilde{r}_n(\tau,\tau) = 0.$$

Consequently,

$$\lim_{\tau\to+\infty} y_n(\tau,\tau) = x_0,$$

and there exists a positive constant d_4 such that $\|y_n(t,\tau)\|, \|y_n^*(t,\tau)\| \leqslant d_4$.

Further, set

$$\begin{aligned}\tilde{R}_n(t,\tau) &= \tilde{R}_{1,n}(t,\tau) + \tilde{R}_{2,n}(t,\tau),\\ \tilde{R}_{1,n}(t,\tau) &= \|u_n(t) - y_n(t,\tau)\|^2,\\ \tilde{R}_{2,n}(t,\tau) &= \|u_n^*(t) - y_n^*(t,\tau)\|^2.\end{aligned}$$

Then, from (3.1) and (3.3) we obtain the following equalities From (3.2) and (3.3) it follows

$$\begin{aligned}&\langle\frac{d(u_n(t) - y_n(t,\tau))}{dt}, u_n(t) - y_n(t,\tau)\rangle + \gamma(t)\Big[\langle F_{1,n}^{h(t)}(u_n(t)) - F_{1,n}(y_n(t,\tau)),\\ &\quad u_n(t) - y_n(t,\tau)\rangle + \langle\alpha(t)u_n(t) - \alpha(\tau)y_n(t,\tau), u_n(t) - y_n(t,\tau)\rangle\\ &\quad +\langle y_n^*(t,\tau) - u_n^*(t), u_n(t) - y_n(t,\tau)\rangle\Big] = 0,\\ &\langle\frac{d(u_n^*(t) - y_n^*(t,\tau))}{dt}, u_n^*(t) - y_n^*(t,\tau)\rangle + \gamma(t)\Big[\langle F_{2,n}^{h(t)}(u_n^*(t)) - F_{2,n}(y_n^*(t,\tau)),\\ &\quad u_n^*(t) - y_n^*(t,\tau)\rangle + \langle\alpha(t)u_n^*(t) - \alpha(\tau)y_n^*(t,\tau), u_n^*(t) - y_n^*(t,\tau)\rangle\\ &\quad +\langle u_n(t) - y_n(t,\tau), u_n^*(t) - y_n^*(t,\tau)\rangle\Big] = 0.\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{d\tilde{R}_n(t,\tau)}{dt} + 2\gamma(t)\alpha(\tau)\tilde{R}_n(t,\tau) \leqslant\\ &\quad\gamma(t)[h(t)g(\|y_n(t,\tau)\|) + |\alpha(t) - \alpha(\tau)|\|y(t,\tau)\|]\|u_n(t) - y_n(t,\tau)\| +\\ &\quad\gamma(t)[h(t)g(\|y_n^*(t,\tau)\|) + |\alpha(t) - \alpha(\tau)|\|y_n^*(t,\tau)\|]\|u_n^*(t) - y_n^*(t,\tau)\|.\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{d\tilde{R}_n(t,\tau)}{dt} \leqslant D_1\gamma(t)[h(t) + |\alpha(t) - \alpha(\tau)|] - 2\tilde{\alpha}(t)\tilde{R}_n(t,\tau),\\ &\tilde{\alpha}(t) = \gamma(t)\alpha(\tau), D_1 = 2\max\{g(d_4)(d_3 + d_4), d_4\}.\end{aligned}$$

By the argument as in the proof of the theorem 2.1, we have

$$\begin{aligned}&\tilde{R}_n(\tau,\tau) \leqslant R_1^n(\tau) + R_2^n(\tau)\\ &R_1^n(\tau) = D_1\int_{t_0}^{\tau}\gamma(t)h(t)\xi(t)dt/\xi(\tau),\\ &R_2^n(\tau) = D_1\int_{t_0}^{\tau}\gamma(t)\alpha'(t)(t-\tau)\xi(t)dt/\xi(\tau),\\ &\xi(s) = exp(\int_{t_0}^{s}\tilde{\alpha}(t)dt.\end{aligned}$$

Therefore, $\lim_{n,\tau\to+\infty} R_1^n(\tau) = \lim_{n,\tau\to+\infty} R_2^n(\tau) = 0$. Since

$$\|x_0 - u_n(\tau)\| \leqslant \|x_0 - x_{\alpha,n}(\tau)\| + \|x_{\alpha,n}(\tau) - y_n(\tau,\tau)\| + \|y_n(\tau,\tau) - u_n(\tau)\|,$$

and $\|x_0 - x_{\alpha,n}(\tau)\| \to 0$ is followed from condition (iii) of the theorem (see [5], [6]), then $\lim_{n,\tau\to+\infty} u(\tau) = x_0$. Theorem is proved.

Appendix. *If condition (iii) and* $\lim_{\tau\to+\infty}\alpha(\tau) = 0$ *are satisfied, then* $\lim_{n,\tau\to+\infty} x_{\alpha,n}(\tau) = x_0$.

Proof. Set

$$B = \|x_{0,n} - x_{\alpha,n}(\tau)\|^2 + \|x^*_{0,n} - x^*_{\alpha,n}(\tau)\|^2,$$

where $x_{0,n} = P_n x_0$, and $x^*_{0,n} = P_n^* x_0^*$. From (1.1), (3.3), the monotone property of $F_{i,n}, i = 1,2$, and $x_{0,n} + P_nF_2(x_0^*) = f_n$ it implies that

$$\begin{aligned}
B =&\big\langle x_{0,n}, x_{0,n} - x_{\alpha,n}(\tau)\big\rangle + \big\langle x^*_{0,n}, x^*_{0,n} - x^*_{\alpha,n}(\tau)\big\rangle\\
&+ \frac{1}{\alpha(\tau)}\Big[\big\langle x^*_{\alpha,n}(\tau) - F_{1,n}(x_{\alpha,n}(\tau)), x_{\alpha,n}(\tau) - x_{0,n}\big\rangle\\
&+ \big\langle f_n - x_{\alpha,n}(\tau) - F_{2,n}(x^*_{\alpha,n}(\tau)), x^*_{\alpha,n}(\tau) - x^*_{0,n}\big\rangle\Big]\\
=&\big\langle x_0, x_{0,n} - x_{\alpha,n}(\tau)\big\rangle + \big\langle x_0^*, x^*_{0,n} - x^*_{\alpha,n}(\tau)\big\rangle\\
&+ \frac{1}{\alpha(\tau)}\Big[\big\langle x^*_{\alpha,n}(\tau) - x^*_{0,n} + P_n^*x_0^* - F_{1,n}(x_{\alpha,n}(\tau)), x_{\alpha,n}(\tau) - x_{0,n}\big\rangle\\
&+ \big\langle x_{0,n} - x_{\alpha,n}(\tau) + P_nF_2(x_0^*) - F_{2,n}(x^*_{\alpha,n}(\tau)), x^*_{\alpha,n}(\tau) - x^*_{0,n}\big\rangle\Big]\\
\leqslant&\big\langle x_0, x_{0,n} - x_{\alpha,n}(\tau)\big\rangle + \big\langle x_0^*, x^*_{0,n} - x^*_{\alpha,n}(\tau)\big\rangle\\
&+ \frac{1}{\alpha(\tau)}\Big[\big\langle F_1(x_0) - F_1(x_{0,n}), x_{\alpha,n}(\tau) - x_{0,n}\big\rangle\\
&+ \big\langle F_2(x_0^*) - F_2(x^*_{0,n}), x^*_{\alpha,n}(\tau) - x^*_{0,n}\big\rangle\Big]\\
\leqslant&\big\langle x_0, x_{0,n} - x_{\alpha,n}(\tau)\big\rangle + \big\langle x_0^*, x^*_{0,n} - x^*_{\alpha,n}(\tau)\big\rangle\\
&+ \frac{\gamma_n}{\alpha(\tau)}\Big[C_1\|x_{\alpha,n}(\tau) - x_{0,n}\| + C_2\|x^*_{\alpha,n}(\tau) - x^*_{0,n}\|\Big], C_i > 0, i = 1,2.
\end{aligned}$$

Hence, $\{x_{\alpha,n}(\tau)\}$ and $\{x^*_{\alpha,n}(\tau)\}$ are bounded, as $n,\tau\to+\infty$ and $\gamma_n/\alpha(\tau)\to 0$.

On the other hand,

$$\begin{aligned}
\big\langle x_0, x_{0,n} - x_{\alpha,n}(\tau)\big\rangle \leqslant \gamma_n\|x_0\| + &\big\langle x_0, x_0 - x_{\alpha,n}(\tau)\big\rangle\\
\leqslant& O(\gamma_n) + \big\langle x^1, F_1(x_0) - F_1(x_{\alpha,n}(\tau)\big\rangle + \big\langle x^2, x_0 - x_{\alpha,n}(\tau)\big\rangle\\
&+ \frac{\tilde{L}\|x^1\|}{2}\|x_{0,n} - x_{\alpha,n}(\tau)\|^2.
\end{aligned}$$

In the similar way, we also have

$$\begin{aligned}
\big\langle x_0^*, x^*_{0,n} - x^*_{\alpha,n}(\tau)\big\rangle \leqslant& O(\gamma_n) + \big\langle x^2, F_2(x_0^*) - F_2(x^*_{\alpha,n}(\tau))\big\rangle - \big\langle x^1, x_0^* - x^*_{\alpha,n}(\tau)\big\rangle\\
&+ \frac{\tilde{L}\|x^2\|}{2}\|x^*_{0,n} - x^*_{\alpha,n}(\tau)\|^2.
\end{aligned}$$

Because of

$$\begin{aligned}&\big\langle x^1, F_1(x_0) - F_1(x_{\alpha,n}(\tau))\big\rangle = \big\langle x^1, x_0^* - x_{\alpha,n}^*(\tau)\big\rangle\\&\quad + \big\langle x^1, x_{\alpha,n}^*(\tau) - F_{1,n}(x_{\alpha,n}(\tau))\big\rangle - \big\langle (I - P_n)x^1, F_1(x_{\alpha,n}(\tau))\big\rangle\\&\quad \leqslant O\big(\gamma_n\big) + \big\langle x^1, x_0^* - x_{\alpha,n}^*(\tau)\big\rangle + \alpha(\tau)\|x^1\|\|x_{\alpha,n}(\tau)\|,\\&\big\langle x^2, F_2(x_0^*) - F_2(x_{\alpha,n}^*(\tau))\big\rangle = \big\langle x^2, -x_0 + x_{\alpha,n}(\tau)\big\rangle + \big\langle x^2, f - f_n\big\rangle\\&\quad + \big\langle x^2, f_n - x_{\alpha,n}(\tau) - F_{2,n}(x_{\alpha,n}^*(\tau))\big\rangle - \big\langle (I - P_n^*)x^2, F_2(x_{\alpha,n}^*(\tau))\big\rangle\\&\quad \leqslant O\big(\gamma_n\big) - \big\langle x^2, x_0 - x_{\alpha,n}(\tau)\big\rangle + \alpha(\tau)\|x^2\|\|x_{\alpha,n}^*(\tau)\|,\end{aligned}$$

we obtain

$$\begin{aligned}&\big(1-\frac{L\|x^1\|}{2}\big)\|x_{\alpha,n}(\tau) - x_{0,n}\|^2 \leqslant \big(1 - \frac{L\|x^1\|}{2}\big)\|x_{\alpha,n}(\tau) - x_{0,n}\|^2\\&\quad + \big(1 - \frac{L\|x^2\|}{2}\big)\|x_{\alpha,n}^*(\tau) - x_{0,n}^*\|^2 \leqslant O\big((\gamma_n + \alpha(\tau) + \gamma_n/\alpha(\tau)\big).\end{aligned}$$

Hence,

$$\lim_{n,\tau\to+\infty} x_{\alpha,n}(\tau) = x_0,$$
$$\lim_{n,\tau\to+\infty} x_{\alpha,n}^*(\tau) = x_0^*.$$

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