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How Much Information of Concurrency Can Be Got From Firing Sequences in Petri Nets

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1. Introduction.

In concerning the concurrent and distributed systems, the way in which the temporal/causal ordering of eventsis described is a problem being under discussion. In the interleaving approach, the fact that a set of events may occur in parallel is described by saying that they may occur in any order. Models based on true concurrency use instead partial orderings to explicity describe the temporal/causal relations among events [4], [6], [8], [9]. In [2], [3], a comparison between two approaches has been treated. These authors proved that in P/T nets processes (corresponding to the latter) are not recoverable from firing sequences (corresponding to the former), while in C/E systems they are. This means that in general in P/T nets true concurrency cannot be obtained from firing sequences. As firing sequences play an important role in studuing the behaviours of P/T nets, and as a part of true concurrency is carried in them, it is worth studying the way to decide what we can say about concurrency from firing sequences of P/T nets. By following the approach of Mazurkeewicz [7], Best [2] and Degano [3] to the behaviours of concurrent systems and devoloping some results in [5], the paper presents a way to study comcurrency from firing sequences. We show that in order to obtain information of true concurrency from firing sequences, only the statistical structure of nets comes into play. We also give a necessary and sufficient condition to a net for which processes are recorevable from firing sequences.

2. Quasi - dependency.

We follows Mazurkiewics's approach to the behaviours of C/E systems [1], [7] in studying firing sequences of P/T nets.

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Our starting point is the notion of so-called quasi - dependency. Intuitively speaking, when there may be causal dependencies among concurrences of two actions, we consider them to be in quasi - dependence. Formally quasi - dependency is defined below.

Let A be a finite set whose members are referred to as actions. Let A^*, (A^ω respectively) denote the set of all finite (infinite) sequences (or words) over A, $A^\infty = A^* \cup A^\omega$. The empty sequence is denoted by ϵ.

For $w \in A^\infty$ and $a \in A, \#_a w$ will denote the number of the occurrency of a in w and $\mathcal{O}(w)$ denotes the set $\{(a,i) | \#_a w > 0 \wedge 0 < i < \#_a w + 1\}$.

Definition 1. A quasi - depedency on A is a reflexive binary relation on A.

Since in general an action can depend on another action while the latter is independent from the former, a quasi - depemdency is not required to be symmetrical.

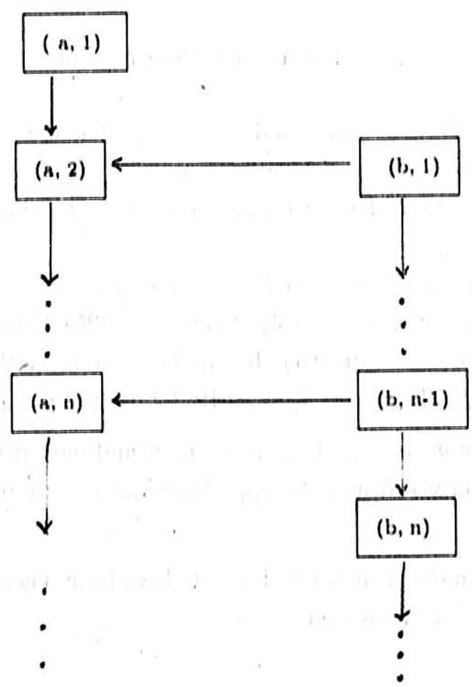
Let D be a quasi - dependency on A. Each $w \in A^*$, w may represent a computation. Then the partial ordering of causal dependency relation $\geq_w$ is defined as follows.

Definition 2. The partial ordering generated by w over D is $(\mathcal{O}(w), \leq_w)$, where $\leq_w$ is the reflexive and transitive closure F_w^* of the relation F_w defined by $(a,i)F_w(b,j)$ iff the j-th occurrence of b follows the i-th occurrence of a in w and $(a,b) \in D$.

Example 1. Let $A = \{a,b\}$, $D = \{(a,a),(b,b),(b,a)\}$.

$w = (ab)^\omega = abab... \in A^\omega$.

Then $(\mathcal{O}(w), \leq_w)$ is represented by the following graph (the transitive arcs are omitted).



Now, we introduce a partial order among members of A^∞.

Definition 3. For w, $w' \in A^\infty$, $w \sqsubseteq w'$ iff $\mathcal{O}(w) = \mathcal{O}(w')$ and $\leq_w \sqsubseteq \leq_{w'}$ ($w \sqsubseteq w'$ iff the partial ordering generated by w over D is coarser than that generated by w').

We consider the behaviour of computation system as a pair of

- A quasi - dependency, which approximately represents dependency in the system,
- A subset C of A^∞, which represents possible computations of the system (the interleaving behaviour of the system).

Then, for each $w \in C$, $(\mathcal{O}(w), \leq_w)$ represents uncertainly the causal dependencies among occurrences of actions in w, some causal dependencies of which are introduced by going extremly.

Now, we consider what the relation $\sqsubseteq$ means.

In the sequel, let *name*: $A \times \{1,2,3...\} \rightarrow A$ be defined as $name((a,i)) = a$ for all intergers $i > 0$, and let pref: $A^\infty \rightarrow 2^{A^*}$ be a mapping which returns all prefixes of its argument. The mapping *name* is extended to a homomorphism from $(A \times \{1,2,3,...\})^\infty$ to A^∞ in the obvious way. Furthermore, for w, $w' \in A^*$, we write $w \rightarrow w'$ iff there is a derivation from w to w' in the rewriting system (A, P) with $P = \{ab \rightarrow ba | (ab) \notin D\}$.

Theorem 1.

(i) Let w, $w' \in A^*$, $w \sqsubseteq w'$ if and only if $w \rightarrow w'$.

(ii) Let w, $w' \in A^\omega$, $w \sqsubseteq w'$ if and only if $(\mathcal{O}(w) = \mathcal{O}(w') \wedge (\forall v \in pref(w')\ \exists u \in pref(w)\ \exists\, x \in A^* : (u \rightarrow vx))$.

Proof.

(i) Only the 'only if 'part is not obvious and can be shown by induction on the length $|w|$ of w, and we leave it to the readers.

(ii) ($\Leftarrow$) : Let e_1, $e_2 \in \mathcal{O}(w) = \mathcal{O}(w')$ and $e_1 \leq_w e_2$. There must be v in $pref(w')$ such that e_1, $e_2 \in \mathcal{O}(v)$. Let u and x be such that $u \in pref(w)$ and $u \rightarrow vx$. From (i) it follows e_1, $e_2 \in \mathcal{O}(u)$. By the definition of $\leq_w$ we have $e_1 \leq_u e_2$, and thus $e_1 \leq_v e_2$ by (i). Hence $e_1 \leq_{w'} e_2$ by the definition of $\leq_w$.

($\Rightarrow$) : Let $v \in pref(w')$. Then $\mathcal{O}(v) \subseteq \mathcal{O}(w)$. Let $u \in pref(w)$ such that $\mathcal{O}(u) \supseteq \mathcal{O}(v)$. It follows that $\leq_u = (\leq_w) \cap (\mathcal{O}(u) \times \mathcal{O}(u)) \subseteq (\leq_{w'} \cap (\mathcal{O}(u) \times \mathcal{O}(u))$. Let α be a topology sorting of $\mathcal{O}(u) \setminus \mathcal{O}(v)$ by $\leq_{w'}$ and $x = name(\alpha)$. It can be seen from the definition of $\leq_{w'}$ that $(\leq_{w'} \cap (\mathcal{O}(u) \times \mathcal{O}(u)) \sqsubseteq \leq_{vx}$. Hence, $\leq_u \sqsubseteq \leq_{vx}$. By (i) we get $u \rightarrow vx$. ⊙

The theorem 1 says that for w, $w' \in A^\infty$, $w \sqsubseteq w'$ if and only if they have the same set of action occurrences and w' is derived from w by applying a (finite or infinite) number of rewritting rules $ab \rightarrow ba$ with $(a, b) \notin D$.

Since independent events can occur in any order, it follows from Theorem 1 that if $w \in C$, for each w' such that $w \sqsubseteq w'$, $w' \in C$ as well.

Corollary 1. *Let $\equiv$ be defined as $w \equiv w'$ iff $\leq_w \doteq \leq_{w'}$, then $w \equiv w'$ if and only if $w \rightarrow w'$, where D_S is the symmetrical closure of D.*

3. Information of True Concurrency in Firing Sequence of P/T Nets.

In this section we investigate how much information of true concurrency can be got from firing sequence of P/T nets. We shall compare the partial orderings among events introduced by process in P/T nets to the partial orderings generated by firing sequences with respect to the natural quasi - dependencies defined by the structure of nets.

A nets is a triple $< S, T; F >$, where
- $S \cap T = \emptyset$;
- $F \subseteq (S \times T) \cup (T \times S)$.

Let, as usual, $t^{\bullet} = \{s \in S | \, tFs\}$, $^{\bullet}t = \{s | \, sFt\}$ for a net $< S, T; F >$.

An occurrence net is a net $K = < S, T; F >$ such that
- The transitive closure of F, defined by F^{+}, is acyclic.
- $\forall s \in S$, $|^{\bullet}s| \leq 1 \wedge |s^{\bullet}| \leq 1$.

Furthermore,
- $S \cup T$ is considered as ordered by $<$, defined as F^{+};
- The slice of K are maximal subsets of S which do not contain elements related by $<$.

A marked place/transition net (P/T net) is a quitiple $N = < S, T; F, W, M >$, where
- $< S, T; F >$ is a net, with S and T finite;
- $W : F \rightarrow N$ assigns a positive weight to each arc;
- $M : S \rightarrow N$ is the initial marking of N.

Given a P/T net $N = < S, T; F, W, M >$, a firing sequence of N is $\{M_0 t_0 M_1 t_1 M_2 ...\}$, where for $i = 1, 2, ...$

- M_i are markings of N and $M_0 = M$; $t_i \in T$;
- $M_i[t_i > M_{i+1}$, where $M[t > M'$ implies that $\forall s \in S$, $M(s) \geq W(s, t)$ and $M'(s) = M(s) - W(s, t) + W(t, s)$.

We shall call the sequences obtained from firing sequences by dropping the markings also firing sequences without fear of confusions.

Given a P/T net $N = < S, T, F; W, M >$ and an occurrence net $K = < S', T'; F' >$, a P/T process of N is a function $p : K \rightarrow N$ such that
- $p(S') \subseteq S$, $p(T') \subseteq T$;
- $(S' \cup T', <')$ is finitly proceded. Let $^{\circ}K$ be the set of its minima;
- $\forall s \in S$, $M(s) = |p^{-1}(s) \cap {^{\circ}K}|$;
- $\forall t' \in T', \forall s \in S$
  - (i) $W(s, p(t')) = |p^{-1}(s) \cap {^{\bullet}t'}|$
  - (ii) $W(p(t'), s) = |p^{-1}(s) \cap t'^{\bullet}|$

Definition 4.

The labelled partial ordering generated by a process $p: K \to N$ of a P/T net N (denoted as above) is $(T', p|_{T'}, \leq_p)$, where $\leq_p$ is $F'^+|_{T' \times T'}$.

From the results in [2], [3] it follows:

For P/T net N, α is a firing sequence of N if and only if there exists a process $p: K \to N$ of N such that $\alpha = p(\beta)$, where β is a topology sorting of T' by $\leq_p$, p is extended to a homomorphism on sequences in the obviuous way.

It can be seen easily that if $p: K \to N$ is a process of N with $K = < S', T'; F' >$, $S' \cup T'$ is countable. Futhermore, since isomorphic processes are not distinguished, in the sequel T' is usually considered as a subset of $T \times \{1, 2, 3, ...\}$ satisfying:

(i) $t' = (a, n) \in T'$ implies $(a, i) \in T'$ for 0 ¡ i ¡ n,

(ii) $(a, n), (a, n') \in T'$ & $n < n'$ implies $(a, n') \leq_p (a, n)$,

(iii) $p(t) = name(t)$ for all $t \in T'$.

As in [4] processes are considered to be equivalent iff the partial order of event occurrences agrees in them. We give the following definition.

Definition 5.

Let K and N be denoted as above. $p: \to N$ is a process of N. $(T', \leq'_p)$ is called a concurrency characteristic of p (characteristic of p for short).

As in [2], let us denote for a firing sequence α and for a process $p: K \to N$ of N with $K = (S', T'; F')$: $Lin(p) := \{\alpha | \alpha = name(\beta)$ with β being a topology sorting of T' by $\leq'_p\}$ and $Proc(\alpha) := \{p \mid \alpha \in Lin(p)\}$.

From the results in [2], [3], we have

Theorem 2. *Let N be a P/T net, α is a firing sequence of N if and only if there exists a process $p: K \to N$ of N with $K = < S', \mathcal{O}(\alpha), F' >$ such that $\alpha = name(\beta)$ with β being a topology sorting of $\mathcal{O}(\alpha)$ by $\leq_p$. (p is said to correspond to α).*

This is the first result on the relationship between firing sequences and processes. Now we give some anothers.

Definition 6.

Let $N = < S, T; F, W, M >$ be a P/T net.

$D = \{(t, t') | t, t' \in T \wedge (t^{\bullet} \cap {}^{\bullet}t' \neq \emptyset \vee t = t')\}$ is called quasi - dependency generated by N.

Let, in the sequel, N be a P/T net, D its quasi - dependency, α a firing sequence of N with $(\mathcal{O}(\alpha), \leq_\alpha)$ being its partial ordering on D, and let $\sqsubseteq$ be defined as in Definition 3 w.r.t. D.

Theorem 3. *Let $K = (S', \mathcal{O}(\alpha), F')$ and $p: K \to N$ be a process of N corresponding to α, $\leq'_p = F'^*$. Then $(\mathcal{O}(\alpha), \leq'_p)$ is coarser than $(\mathcal{O}(\alpha), \leq_\alpha)$.*

Proof. It is sufficient to prove:

$\forall (a, i), (b, i)$ *in* $\mathcal{O}(\alpha)$: $(a, i) F'^2 (b, j) \leq_\alpha (b, j)$

We have

$(a,i)F'^{2}(b,j) \Rightarrow \exists s \in S' : (a,i)F'sF'(b,j)$

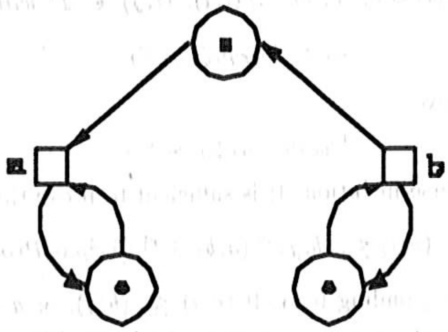
$\Rightarrow a^{\bullet} \cap b \supseteq \{p(s)\} \Rightarrow (a,b) \in D.$

Since $\alpha \in Lin(p)$, the i-th occurrence of a precedes the j - th occurrence of b in α. By Definition 2 $(a,b) \in D$ implies $(a,i) \leq_{\alpha} (b,j)$.

Theorem 3 say that if (a,i) and (b,j) are not related by $\leq_{\alpha}$, neither are they by any process corresponding to α. That means we can get some information of concurrency from firing sequences of the net by its quasi - dependency.

Example 2.

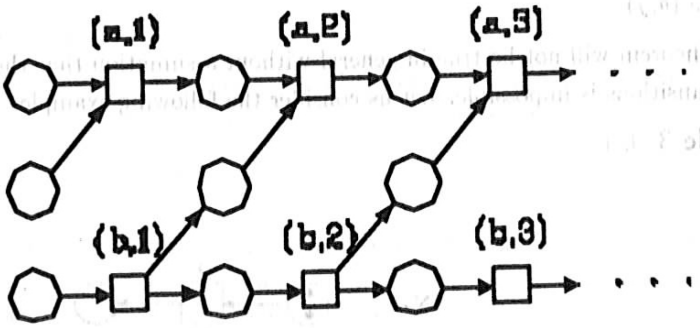
Let = N



Then,

$D = \{(a,a), (b,b), (b,a)\}$.

$\alpha = (ab)^{\omega}$ is a firing sequence of N, and $(\mathcal{O}(\alpha), \leq_{\alpha})$ is the same as in Example 1. A process corresponding to α is the following



In this case, $Proc(\alpha)$ contains one process, and $(\mathcal{O}(\alpha), \leq_{\alpha})$ is its characteristic.

Theorem 4. *If the parallel occurrence of the same transitions is impossible in N, (this means that if $p: (S', T'; F') \to N$ is a process of N, and if $(t,i), (t,j) \in T'$ with $i < j$, $(t,i) \leq_p (t,j)$), then*

$$\leq_\alpha = \cup_{p \in Proc(\alpha)} \leq_p .$$

Proof. From Theorem 3 we have

$$\cup_{p \in Proc(\alpha)} \leq_p \subseteq \leq_\alpha$$

Now we have to show the inverse inclution. It is sufficient to prove that

$$\forall (a,i), (b,j) \in \mathcal{O}(\alpha) : (a,i) \leq_\alpha (b,j) \cap (a,b) \in D \Rightarrow \exists p \in Proc(\alpha) : (a,i) \leq_p (b,j).$$

Let p be a process of N corresponding to α. If $(a,i) \leq_p (b,j)$, or $a = b$, the theorem has been proved. Suppose that $(a,i) \not\leq_p (b,j)$ and $a \neq b$. Since $(a,b) \in D \exists s \in a^\bullet \cap {}^\bullet b$. Let $s_1 \in (a,i)^\bullet$ and $s_2 \in {}^\bullet(b,j)$ such that $p(s_1) = p(s_2) = s$. Of course, $s_1 \neq s_2$. Now we construct an occurrence net $K' = (S', \mathcal{O}(\alpha), F'')$, where

$$F'' = F' \setminus \{(s_2, (b,j))\} \cup \{(s_1, (b,j))\};$$

$$p' = p.$$

It can be seen that (F'' is acyclic and $p' : K' \to N$ is a process in $Proc(\alpha)$ as well. Furthurmore, $(a,i) \leq_{p'} (b,j)$.

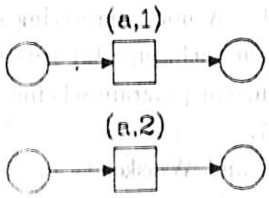
The theorem will not be true in general without assumption that the parallel occurrence of the same transitions is impossiple. Let us consider the following example.

Example 3. Let

N = 

Then aa is a firing sequence of N with $(\mathcal{O}(aa), <_{aa})$ is represented by the first figure in this page

Proc (aa) =



and its characteristic is $((a,1),(a,2)\},\emptyset)$.

Theorem 4 says, in the case when the parallel occurrence of the same transitions is impossible in N, that the information of concurrency in each firing is maximal amount derived from all processes corresponding to the firing sequence.

Theorem 5. *If α is a firing sequence of N and $\alpha \subseteq \beta$ then β is a firing sequence of N also. Moreover $Proc(\alpha) \sqsubseteq Proc(\beta)$.*

Proof. It follows from Theorem 1 that if α is a firing sequence, then $Proc(\alpha) \neq \emptyset$. Let $p \in Proc(\alpha)$. from Theorem 3 we have $\leq_p \subseteq_\alpha \subseteq \leq_\beta$, which implies $\beta \in Lin(p)$. Hence, β is a firing sequence of N, and every process in $Proc(\alpha)$ is a process in $Proc(\beta)$, *too*. ⊙

In the sequel, we assume that N be such net in which the parallel occurrence of the same transitions is impossible. We have the following corollaries.

Corollary 2. *For firing sequences α, β of N, $Proc(\alpha) = Proc(\beta)$ if and only if $\alpha \equiv \beta$, where $\equiv$ is defined as in Corollary 1.*

Proof. The ' only if ' part follows from Theorem 4, and the ' if ' part follows from Theorem 5.

Corollary 3. *For a firing sequence α of N, all processes in $Proc(\alpha)$ are equivalent (by $\equiv$ defined in [2]) if and only if $(\mathcal{O}(\alpha), \leq_\alpha)$ is their characteristic.*

Corollary 3 shows that the firing sequence approach and the process approach to the behaviour of P/T nets concide only for a restricted class of Petri nets concluding 1-safe nets.

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Abstract

How Much Information of Concurrency Can Be Got From Firing Sequences in Petri Nets

It is well know in [2], [3] that in general processes in Petri nets are not recoverable from firing sequences. However, firing sequences in Petri nets say somthing about concurrency. The paper presents a way to define comcurrency from firing sequences of nets. It turns out that the information of concurrency in a firing sequence characterise all its processes.