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Accelerated Methods for Solving Grid Equations. I

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1. Introduction

The boundary value problems for elliptic equations after discretization by the method of finite differences or by the finite elements method are reduced to the large system of linear algebraic equations

$$Au = f. \tag{1.1}$$

where A is a symmetric, positive definite matrix of order N equal to the number of nodes of the grid. There have been many effective methods for solving the system (1.1) (see e.g. [1-4]). Besides, in order to accelerate the convergence rate of the main iterative processes one proposed some interesting methods. First we should mention the group of accelerated methods by additive correction (see [3] and bibliography therein), whose most important representatives are the multigrid methods [5-7]. These methods are intensively developed now. The second group of accelerated methods are those by multiple correction [8,9]. In spite of the existing methods the elaboration of new methods either more effective or easily realized on parallel computers is of interest to researchers.

In this work we propose a method accelerating the convergence rate of certain well known iterative methods such as the alternately triangular method (ATM) and the alternating direction method (ADM) [1,4,16,17]. More precisely, we suggest new algorithms on the base of the above mentioned methods and prove that our algorithms converge faster than ATM and ADM applied directly to the system (1.1). Our idea is based on the construction of an appropriate combination of the solutions of the systems, which may be solved faster than the original system (1.1). This idea on the continuous level was used by ourselves in [10-12] to construct effective algorithms for solving biharmonic equation and the second order elliptic equation with discontinuous coefficients.

s possible to consider this idea as an expession of the new approach to numerical modelling of blems in physics and mechanics stated recently in [13,14].

Construction of approximate solution of operator equation by parametric extrapolation

n this section we construct the theoretical background of our accelerated algorithms. First we ablish some auxiliary results. Let A be a linear, symmetric, positive operator in the Hilbert ice H with the scalar product $(.,.)$. Consider the equation

$$Au = f, \; f \in R(A). \tag{2.1}$$

cording to this equation we introduce the purturbed equation

$$Au_\epsilon + \epsilon P u_\epsilon = f \tag{2.2}$$

re ϵ is a small parameter, $\epsilon > 0$; P is a linear, symmetric, positive definite operator $P = P^* \geq$ ', $\mu > 0$, E is the identity operator.

nma 2.1. *Let u_0 and u_ϵ be the solutions of the equations (2.1),(2.2) respectively, where A, P the operators described above. Then*

$$(Au_\epsilon, u_\epsilon) \leq (Au_0, u_0) = (f, u_0). \tag{2.3}$$

oof. From (2.1) and (2.2) it follows that

$$Au_\epsilon + \epsilon P u_\epsilon = Au_0.$$

ltiplying both sides of the above equality by u_ϵ and taking into account that $P > 0$ we get

$$(Au_\epsilon, u_\epsilon) \leq (Au_0, u_\epsilon). \tag{2.4}$$

om the inequality

$$(A(u_\epsilon - u_0), u_\epsilon - u_0) = (Au_\epsilon, u_\epsilon) + (Au_0, u_0) - 2(Au_0, u_\epsilon) \geq 0$$

derive

$$2(Au_0, u_\epsilon) \leq (Au_\epsilon, u_\epsilon) + (Au_0, u_0)$$

is estimate and (2.4) imply the desired inequality (2.3).

mark 2.1. In the case when H is finite-dimensional space from (2.3) we have

$$\|u_\epsilon\|^2 \leq C_1(f, u_0),$$

ere $C_1 = const.$, independent of ϵ.

Remark 2.2. If in addition to the assumptions of Lemma 2.1 we suppose that the operator A is completely continuous and the operators A and P are commute then instead of (2.3) we get the stronger estimate

$$\|u_\epsilon\| \leq \|u_0\|.$$

This result in the case $P = E$ was proved in [10,12]. Here it should take into account the commutativity of A and P.

In the future we shall need the following result.

Lemma 2.2. (see [15], Appendix). Let M be a natural number. Then the solution of the system

$$\sum_{i=1}^{M+1} \gamma_i = 1, \; \sum_{i=1}^{M+1} \frac{1}{i^l}\gamma_i = 0, \; l = \overline{1, M}$$

is given by the formula

$$\gamma_i = \frac{(-1)^{M+1-i} i^{M+1}}{i!(M+1-i)!}, \; i = \overline{1, M+1}. \tag{2.5}$$

Now we prove a theorem on asymptotic expansion of the solution of the purturbed equation (2.2).

Theorem 2.1. *Let A be a linear, symmetric, positive operator ; P be a linear, symmetric, positive definite operator in the Hilbert space. Let M be a given natural number such that the equations*

$$Av_k = -Pv_{k-1}, \; k = \overline{1, M+1}, \; v_0 = u_0 \tag{2.6}$$

have solutions. Then the solution of the equation (2.2) can be expanded in the form

$$u_\epsilon = u_0 + \sum_{k=1}^{M} \epsilon^k v_k + \epsilon^{M+1} w_\epsilon, \tag{2.7}$$

where u_0 is the solution of (2.1), v_k, $(k = \overline{1, M})$ are elements independent of ϵ, and w_ϵ satisfies the estimate

$$(Aw_\epsilon, w_\epsilon) \leq C_2 = const. \tag{2.8}$$

Proof. Substitute (2.7) into (2.2). After balancing the coefficients of ϵ we get the equations

$$Av_k = -Pv_{k-1}, \; k = \overline{1, M},$$

$$(A + \epsilon P)w_\epsilon = -Pv_M.$$

Clearly, $v_k (k = \overline{1, M})$ do not depend on ϵ and by Lemma 2.1 we have

$$(Aw_\epsilon, w_\epsilon) \leq (Aw_0, w_0),$$

where w_0 is the solution of the equation solvable by assumption

$$Aw_0 = -Pv_M.$$

Thus, the theorem is proved with $C_2 = (Aw_0, w_0)$.

In order to use Theorem 2.1 for constructing solution of grid equations we restate one in the case of finite-dimensional space with the attention that in this space positive operator is also positive definite.

Theorem 2.1a. *Let A and P be linear, symmetric operators in N-dimensional Euclidean space. Moreover, $A > 0$, $P \geq 0$. Then for any natural number M it is possible to expand the solution of the equation (2.2) in the form*

$$u_\epsilon = u_0 + \sum_{k=1}^{M} \epsilon^k v_k + \epsilon^{M+1} w_\epsilon, \tag{2.9}$$

where v_k $(k = \overline{1, M})$ are elements independent of ϵ,

$$\|w_\epsilon\| \leq C_3. \tag{2.10}$$

We now construct the solution of (2.1) by extrapolation of solutions of (2.2).

Put

$$U^E = \sum_{k=1}^{M+1} \gamma_k u_{\epsilon/k}, \tag{2.11}$$

where γ_k $(k = \overline{1, M+1})$ are coefficients given by (2.5), $u_{\epsilon/k}$ are the solutions of (2.2) with the parameters ϵ/k. These solutions are called the basic solutions and U^E is considered as an approximate solution of the equation (2.1).

Theorem 2.2. *Let A and P be operators, satisfying the assumptions of Theorem 2.1a. Then for the approximate solution of (2.1) constructed by (2.11) there holds the estimate*

$$\|U^E - u_0\| \leq C_4 \epsilon^{M+1}. \tag{2.12}$$

Proof. We use the expansion (2.9) for $u_{\epsilon/k} (k = \overline{1, M+1})$ and form U^E by (2.11). Then applying Lemma 2.1 and the estimate (2.10) we obtain (2.12) with

$$C_4 = C_3 \sum_{k=1}^{M+1} \frac{|\gamma_k|}{k^{M+1}}.$$

Thus, the theorem is proved.

We see from above that in order to construct approximate solution of the original problem (2.1) we have to solve a number of purturbed problems (2.2) with the parameters ϵ/k $(k = \overline{1, M+1})$. Usually, we can not find exact solutions of these problems but only their approximate solutions. Suppose that by iterative method we have found approximate solutions $y_{\epsilon/k}^{(n_k)}$ of the purturbed equations with accuracy θ, i.e.

$$\|y_{\epsilon/k}^{(n_k)} - u_{\epsilon/k}\| \leq \theta, \quad k = \overline{1, M+1}. \tag{2.13}$$

Then instead of the theoretical approximate solution U^E we obtain the real one

$$Y^E = \sum_{k=1}^{M+1} \gamma_k y_{\epsilon/k}^{(n_k)}. \tag{2.14}$$

It is easy to get

Lemma 2.3. *For the real approximate solution Y^E we have*

$$\|Y^E - u_0\| \le C_4\epsilon^{M+1} + C_5\theta,$$

where

$$C_5 = \sum_{k=1}^{M+1} |\gamma_k|.$$

Let M be chosen, that is we have fixed the number of basic solutions for constructing U^E. From Lemma 2.3 we obtain

Theorem 2.3. *If $\epsilon^{M+1} = \theta$ then Y^E computed by (2.14) is approximate solution of the problem (2.1) with the accuracy*

$$\|Y^E - u_0\| \le C\theta,$$

where $C = C_4 + C_5$ independent of θ.

Suppose that the equations (2.1), (2.2) are solved by such a same iterative method. Then the construction of the solution of (2.1) by the parametric extrapolation (2.11) is effective if the total computational cost needed for solving the purturbed equations is less than for solving the original equation. In Section 3 this fact will be established for some iterative methods.

Remark 2.3. The basic solutions $u_{\epsilon/k}$ are sought independently from each other. Hence, it is optimal to realize the parametric extrapolation method on parallel computers. Otherwise, on sequential computers if the purturbed equations are solved iteratively then for reducing the computation time it is proposed to take $\frac{1}{k}\sum_{i=1}^{k} y_{\epsilon/i}^{(n_i)}$ to be the starting approximation in the iterative process for finding $u_{\epsilon/(k+1)}$. Here $y_{\epsilon/i}^{(n_i)}$ is the approximation of $u_{\epsilon/i}$ with given accuracy.

3. Application to grid equations

In the N-dimensional Euclidean space consider the equation

$$Au = f, \tag{3.1}$$

where A is a linear, symmetric, positive definite operator:

$$A = A^* \ge \delta E,\ \delta > 0. \tag{3.2}$$

For solving the above equation one usually constructs two-layer iterative process of the form

$$B\frac{y^{(k+1)} - y^{(k)}}{\tau_{k+1}} + Ay^{(k)} = f,\ k = 0, 1, ..., n-1. \tag{3.3}$$

Here B is an operator, satisfying the following requirements: i) B may be fast inverted ii) the ratio of the energetic equivalence coefficients of the operators A and B reaches maximal value, i.e. $\gamma_1/\gamma_2 \to max$, where γ_1, γ_2 are the coefficients in the relation

$$\gamma_1 B \le A \le \gamma_2 B,$$

e parameters of the Chebyshev collection (see [1,16]). The convergence rate depends on γ_1/γ_2.

ually, for the construction of B one starts from an operator $R = R^* > 0$, which is energetic alent to A, i.e.

$$C_1 R \le A \le C_2 R, \quad C_2 \ge C_1 > 0. \tag{3.4}$$

the above idea and the results of Section 2 we shall construct efficient algorithm for solving quation (3.1).

ppose that the operator R can be decomposed into the sum of operators. We shall consider ollowing cases.

Case 1: R is the sum of two noncommute but conjugate each other operators i.e.

$$R = R_1 + R_2, \quad R_2^* = R_1, \quad R_1R_2 \ne R_2R_1.$$

is case we choose $P = R_1R_2$. Clearly, $P = P^* > 0$.

nsider the purturbed equation

$$A_\epsilon u_\epsilon = f, \tag{3.5}$$

e

$$A_\epsilon = A + \epsilon P, \quad \epsilon > 0. \tag{3.6}$$

shall construct an operator B, energetic equivalent to A_ϵ in the form

$$B = (E + \omega R_1)(E + \omega R_2). \tag{3.7}$$

problem is to choose ω so that the ratio γ_1/γ_2 reaches the more the better value. Here γ_1, γ_2 the coefficients in the relation

$$\gamma_1 B \le A_\epsilon \le \gamma_2 B \tag{3.8}$$

ma 3.1. *Let us choose $\omega = a\sqrt{\epsilon}$, where a is a real number, satisfying*

$$a \ge \frac{\sqrt{\epsilon}}{C_1}, \tag{3.9}$$

being the coefficient in (3.4).

Then we have the inequalities (3.8), where

$$\gamma_1 = \frac{\delta}{1 + \delta a^2}, \quad \gamma_2 = \frac{C_2}{a\sqrt{\epsilon}}. \tag{3.10}$$

$< \epsilon < C_1^2/\delta$ *then the ratio* $\xi = \gamma_1/\gamma_2$ *reaches maximum when*

$$a = \mathring{a} = 1/\sqrt{\delta} \tag{3.11}$$

In this case we have

$$\xi = \overset{\circ}{\xi} = \frac{\sqrt{\delta}}{2C_2}\sqrt{\epsilon}, \tag{3.12}$$

$$\gamma_1 = \overset{\circ}{\gamma}_1 = \delta/2, \ \gamma_2 = \overset{\circ}{\gamma}_2 = C_2\sqrt{\frac{\delta}{\epsilon}}. \tag{3.13}$$

Proof. For all $x \in R^N$ we have

$$(Bx, x) = ||x||^2 + \omega(Rx, x) + \omega^2||R_2x||^2,$$

$$(A_\epsilon x, x) = (Ax, x) + \epsilon||R_2x||^2.$$

Inserting $\omega = a\sqrt{\epsilon}$ into the above equalities and taking into account (3.4) we get

$$(Bx, x) \geq ||x||^2 + \frac{a\sqrt{\epsilon}}{2}(Ax, x) + a^2\epsilon||R_2x||^2$$

$$\geq \frac{a\sqrt{\epsilon}}{C_2}((Ax, x) + C_2a\sqrt{\epsilon}||R_2x||^2).$$

Using (3.9) we obtain

$$(Bx, x) \geq \frac{a\sqrt{\epsilon}}{C_2}((Ax, x) + \epsilon||R_2x||^2) = \frac{a\sqrt{\epsilon}}{C_2}((A_\epsilon x, x).$$

that is

$$B \geq \frac{a\sqrt{\epsilon}}{C_2}A_\epsilon. \tag{3.14}$$

Next we get an upper bound for B. We have

$$(Bx, x) \leq ||x||^2 + \frac{a\sqrt{\epsilon}}{C_1}(Ax, x) + a^2\epsilon||R_2x||^2$$

$$\leq \frac{1}{\delta}(Ax, x) + a^2(\frac{\sqrt{\epsilon}}{aC_1}(Ax, x) + \epsilon||R_2x||^2)$$

$$\leq \frac{1}{\delta}(A_\epsilon x, x) + a^2((Ax, x) + \epsilon||R_2x||^2) \leq (\frac{1}{\delta} + a^2)(A_\epsilon x, x).$$

Thus, we get

$$B \leq (\frac{1}{\delta} + a^2)A_\epsilon. \tag{3.15}$$

From (3.14),(3.15) we obtain (3.8) with γ_1, γ_2 computed by (3.10). Their ratio is a function of

$$\xi = \xi(a) = \frac{\gamma_1}{\gamma_2} = \frac{\delta\sqrt{\epsilon}a}{C_2(1 + \delta a^2)}.$$

If $0 < \epsilon < C_1^2/\delta$ then $\max_{a \geq \sqrt{\epsilon}/C_1}\xi(a)$ is reached when $a = \overset{0}{a} = 1/\sqrt{\delta}$. In this case we have $\xi = \overset{0}{\xi}$, $\gamma_1 = \overset{0}{\gamma}_1$, $\gamma_2 = \overset{0}{\gamma}_2$ computed by (3.12), (3.13).

The lemma is proved.

For solving the purturbed equation we use the iterative process

$$B\frac{y_\epsilon^{(k+1)}-y_\epsilon^{(k)}}{\tau_{k+1}}+A_\epsilon y_\epsilon^{(k)}=f,\ k=\overline{0,n-1}, \tag{3.16}$$

$^{0)}$ is given.

From the theory of two-layer iterative processes in [1,16] we obtain the following

heorem 3.1. If

$$B=(E+\sqrt{\frac{\epsilon}{\delta}}R_1)(E+\sqrt{\frac{\epsilon}{\delta}}R_2)$$

d $\{\tau_k\}_{k=1}^n$ is the Chebyshev parameters collection, constructed according to the bounds γ_1, γ_2 fined in Lemma 3.1 then for the iterative process (3.16) there holds the estimate

$$\|y_\epsilon^{(n)}-u_\epsilon\|\le q_n\|y_\epsilon^{(0)}-u_\epsilon\|, \tag{3.17}$$

here

$$q_n=\frac{2\rho_1^n}{1+\rho_1^{2n}},\ \rho_1=\frac{1-\sqrt{\overset{0}{\xi}}}{1+\sqrt{\overset{0}{\xi}}},\ \overset{0}{\xi}=\frac{\sqrt{\delta}\sqrt{\epsilon}}{2C_2} \tag{3.18}$$

order to achieve the relative accuracy θ, that is to have

$$\|y_\epsilon^{(n)}-u_\epsilon\|\le\theta\|y_\epsilon^{(0)}-u_\epsilon\| \tag{3.19}$$

is needed to implement n iterations, $n\ge n_c(\theta)$,

$$n_c(\theta)=\ ln\tfrac{2}{\theta}/ln\tfrac{1}{\rho_1}\sim ln\tfrac{2}{\theta}/(\sqrt{\tfrac{2}{C_2}}\delta^{1/4}\epsilon^{1/4}).$$

r the stationary iterative process with

$$\tau_k\equiv\tau=\frac{2}{\overset{0}{\gamma}_1+\overset{0}{\gamma}_2}$$

tead of (3.17) we have

$$\|y_\epsilon^{(k)}-u_\epsilon\|\le\rho^k\|y_\epsilon^{(0)}-u_\epsilon\|,\ k=1,2,\ldots$$

ere

$$\rho=\frac{1-\overset{0}{\xi}}{1+\overset{0}{\xi}}$$

his case the number of iterations needed to achieve the relative accuracy θ is n, $n\ge n_d(\theta)$,

$$n_d(\theta)=\ ln\tfrac{2}{\theta}/(\frac{\sqrt{\delta}}{C_2}\sqrt{\epsilon}).$$

Example 3.1 We illustrate the proposed method on the Dirichlet problem for the second order elliptic equation

$$\sum_{\alpha=1}^{p} \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x)\frac{\partial u}{\partial x_\alpha}\right) = -f(x), \quad x \in \Omega, \tag{3.22}$$

$$u = 0, \quad x \in \Gamma.$$

Here $\overline{\Omega}$ is a parallelepiped in p-dimensional Euclidean space ($p = 2, 3$)

$$\overline{\Omega} = \{0 \le x_\alpha \le l_\alpha, \ \alpha = \overline{1,p}\},$$

Γ is the boundary of $\overline{\Omega}$.

We introduce over $\overline{\Omega}$ the grid

$$\overline{\omega} = \{x_\alpha = i_\alpha h_\alpha, \ 0 \le i_\alpha \le N_\alpha, \ \alpha = \overline{1,p}\}$$

with the boundary γ. Here $N_\alpha h_\alpha = l_\alpha$, $\alpha = \overline{1,p}$.

We approximate the problem (3.22) by the difference scheme

$$\Lambda y \equiv \sum_{\alpha=1}^{p} (a_\alpha y_{\bar{x}_\alpha})_{x_\alpha} = -f(x), \quad x \in \omega, \tag{3.23}$$

$$y = 0 \quad x \in \gamma.$$

Here and henceforward we use the difference notations taken from [1,16].

Let us denote by $\overset{0}{H}_h$ the space of grid functions defined on $\overline{\omega}$ and vanishing on γ. Then the difference scheme may be written in the form of operator equation

$$Ay = f. \tag{3.24}$$

where $Ay = -\Lambda y, \ y \in \overset{0}{H}_h$.

In $\overset{0}{H}_h$ we also define the operators $R, \ R_1, \ R_2$ as follows

$$R_1 y = \sum_{\alpha=1}^{p} \frac{y_{\bar{x}_\alpha}}{h_\alpha}, \quad R_2 y = -\sum_{\alpha=1}^{p} \frac{y_{x_\alpha}}{h_\alpha}, \quad Ry = \sum_{\alpha=1}^{p} y_{\bar{x}_\alpha x_\alpha}.$$

If in (3.22) it is assumed that

$$C_1 \le k_\alpha(x) \le C_2, \ C_2 \ge C_1 > 0, \ \alpha = \overline{1,p}$$

then we have

$$C_1 R \le A \le C_2 R.$$

For δ in (3.2) we can take

$$\delta = 8 \sum_{\alpha=1}^{p} \frac{1}{l_\alpha^2}.$$

Since the difference scheme (3.23) of the problem (3.22) has accuracy of order h^2, $(h^2 = \sum_{\alpha=1}^{p} h_\alpha^2)$, in order to have consistence we also implement iterative processes with accuracy $\theta = h^2$. With this accuracy when fixing $M = 1$, i.e. when solving two purturbed problems it should be taken $\epsilon = h$. Then the purturbed problem (2.2) has the form

$$Ay + hR_1R_2y = f.$$

It is easy to verify that the number of iterations needed for solving this problem with accuracy $\theta = h^2$ is

$$n_c(\theta) = \mathcal{O}(\frac{1}{h^{1/4}} \ln \frac{1}{h}).$$

Hence, the total number of iterations for obtaining the approximate solution of the problem (3.22) with the same accuracy is a quantity of order $\mathcal{O}(\frac{1}{h^{1/4}} \ln \frac{1}{h})$. This number is less than $\mathcal{O}(\frac{1}{h^{1/2}} \ln \frac{1}{h})$, which the alternately triangular method with the Chebyshev parameters collection requires (see [1,16]) if applying directly this method to the original difference problem (3.24).

When $M > 1$ we must solve more than two purturbed problems. In this case, choosing $\epsilon = h^{2/(M+1)}$ we shall reduce the number of iterations needed for obtaining the approximate solution of (3.22) with accuracy h^2 to $\mathcal{O}(\frac{1}{h^{1/2(M+1)}} \ln \frac{1}{h})$.

3.2 Case 2 : R is decomposed into the sum of two symmetric, commute operators

$$R = R_1 + R_2,\ R_\alpha^* = R_\alpha\ (\alpha = 1, 2),\ R_1R_2 = R_2R_1.$$

Suppose $\delta_\alpha E \le R_\alpha \le \Delta_\alpha E$, $\delta_\alpha > 0, \alpha = 1, 2$.

Taking $P = R_1R_2$ and making the same arguments as in Case 1 we obtain similar results.

Example 3.2. Consider the Dirichlet problem (3.22) with $p = 2$. We also approximate it by the difference scheme (3.23). The operators R, R_1, R_2 are defined as follows

$$Ry = -y_{\bar{x}_1x_1} - y_{\bar{x}_2x_2},\ R_1y = -y_{\bar{x}_1x_1},\ R_2y = -y_{\bar{x}_2x_2}$$

The estimate for the number of iterations is similar to that in Example 3.1, where R is the sum of two operators conjugate each other. Comparing with ADM applied directly to the original difference problem we see the effectiveness of our parametric extrapolation technique.

The case, where R is split into the sum of three commute operators, which we meet when solving elliptic problems in three-dimensional domains and a comparative analysis of the proposed algorithms on numerical examples will be presented in an another paper.

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Abstract. *In this paper we propose a technique accelerating the convergence rate of the known iterative schemes for solving grid equations such as the alternately triangular method and the alternating direction method. Our idea is by the parametric extrapolation of the solutions of equations, which can be solved faster than the original ones. The efficiency of the accelerated methods is shown on examples.*