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ON MONOTONE ILL-POSED PROBLEMS IN HILBERT SPACES

NGUYEN BUONG

Summary. The main aim of this paper is to study convergence rates for an operator method of regularization to solve nonlinear ill-posed problems involving monotone operators in infinite-dimensional Hilbert space without needing closeness conditions. Then these results are presented in form of combination with finite-dimensional approximations of the space. An iterative method for solving regularized equation is given and an example in the theory of singular integral equations is considered for illustration.

I. INTRODUCTION

Let H be a real Hilbert space with norm and scalar product denoted by $\|.\|$ and $\langle .,. \rangle$, respectively. Let A be a nonlinear operator in H with domain of definition $D(A) = H$ and range $R(A) \subseteq H$, and f_0 be an element of $R(A)$.

Consider the nonlinear ill-posed problem

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$$A(x) = f_0. \tag{1.1}$$

By this we mean that solutions of (1.1) do not depend continuously on the data f_0. Various aspects about regularization of (1.1) were studied in detail when A is compact (see, for instance, [6], [13]-[15], [18], [19] and their bibliographies). Here to study convergence rates of variational method of Tikhonov regularization minimizing the functional

$$F_\alpha^\delta(x) = \|A(x) - f_\delta\|^2 + \alpha\|x\|^2, \tag{1.2}$$

where $\alpha > 0$ is the parameter of regularization and f_δ are the approximations of f_0 with the wellknown informations

$$\|f_\delta - f_0\| < \delta, \ \delta \to 0,$$

one needs to have the following conditions (see [6]): (i) A is Fréchet differentiable, (ii) there exists a constant $L > 0$ such that $\|A'(x) - A'(y)\| \leq L\|x - y\|$, $x, \ y \in D(A)$ (iii) there exists an element $\omega \in H$ such that $A'^*(x_0)\omega = x_0$, where $A'^*(x_0)$ denotes the adjoint of derivative of A at x_0 and x_0 is a norm-minimal solution of (1.1), and (iv) $L\|\omega\| < e$ $(= 1)$ which is called the closeness condition. In [15] A. Neubauer estimated e < 1 for a modification of (1.2).

In [5], when A is a monotone operator, the author obtained e=2 for the operator version of Tikhonov regularization

$$A(x) + \alpha x = f_\delta, \tag{1.3}$$

but convergence rates are a bit weaker. It is clear that the equation in condition (iii) is not defined explicitly because the operator $A'^*(x_0)$ and the right-hand side x_0 are not known. Therefore, the verification of (iv) is almost too difficult to realize. So, it is natural to propose the question if there exists a some way excepting condition (iv). In [13] A. Neubauer developed an approach of [11] in the linear case for problems involving compact operators. A big advantage of this approach is that rates are obtained by merely requiring smoothness conditions for the exact solution as in the linear case. In this paper, we shall show that by using a modification of (1.3) that is the regularized equation (see [18])

$$A(x) + \alpha \partial\varphi(x) = f_\delta, \tag{1.4}$$

where $\partial\varphi$ is the subdifferential of the uniformly convex functional φ on H, and replacing the smoothness condition (iii) by another one we can exclude condition (iv). Main results about convergence rates are presented in Section 2. An iterative process for (1.4) is given in Section 3 and an numerical example is considered in Section 4 for illustration.

II. MAIN RESULTS

Consider the uniformly convex functional $\varphi(x) = \|x\|^\mu$, $2 < \mu < 3$. Then (see [7], Lemma 2)

$$\langle \partial\varphi(x) - \partial\varphi(y), x - y \rangle \geq 2^{2-\mu}\|x - y\|^\mu, \ \forall x, \ y \in H.$$

Since $\|x - y\| \geq \|x\| - \|y\|$ it is wellknown in [18] that Eq. (1.4) has a unique solution $x_{\alpha\delta}$ for every fixed $\alpha > 0$ and $f_\delta \in H$ and the sequence $\{x_{\alpha\delta}\}$ converges in norm of H to x_0 if δ/α and α tend to zero, where

$$\langle \partial\varphi(x_0), x - x_0 \rangle \geq 0, \ \forall x \in S_0,$$

S_0 is the set of sulutions of (1.1).

We shall prove the following result.

Theorem 2.1. *Let the following conditions hold:*

(i) A is twice-Fréchet differentiable in some neighbourhood of x_0,

(ii) there exists a constant $L > 0$ such that

$$\|A''(x) - A''(z)\| \leq L\|x - z\|$$

for $x,\ z \in S(x_0, r)$, where $S(x_0, r)$ is a ball with center x_0 and radius r, and

(iii) the equation

$$(A'^*(x_0) - \frac{1}{2}A''^*(x_0)y)\omega = \partial\varphi(x_0)$$

has a bounded solution $\omega(y),\ y \in S(x_0, r)$.

Then for the choice $\alpha = 0(\delta^p),\ 0 < p < 1$ we obtain

$$\|x_{\alpha\delta} - x_0\| = 0(\delta^q),\ q = \min\ ((1-p)/(\mu - 1), p/\mu).$$

Proof. By virtue of Eqs (1.1) and (1.4) and the monotone property of A we have

$$\alpha 2^{2-\mu}\|x_{\alpha\delta} - x_0\|^\mu \leq \delta\|x_{\alpha\delta} - x_0\| - \alpha\langle\partial\varphi(x_0), x_{\alpha\delta} - x_0\rangle.$$

From this inequality and condition (iii) of the theorem it follows

$$\alpha 2^{2-\mu}\|x_{\alpha\delta} - x_0\|^\mu \leq \delta\|x_{\alpha\delta} - x_0\| + \alpha\langle\omega(y), (A'(x_0) - \frac{1}{2}A''(x_0)y)(x_0 - x_{\alpha\delta})\rangle,\ y \in S(x_0, r).$$

Using Taylor expression (see [20]) and taking $y = x_{\alpha\delta}$ we can write

$$A'(x_0)(x_0 - x_{\alpha\delta}) - \frac{1}{2}A''(x_0)(x_0 - x_{\alpha\delta})^2 = A(x_0) - A(x_{\alpha\delta}) + r_{\alpha\delta},\ \|r_{\alpha\delta}\| \leq L\|x_{\alpha\delta} - x_0\|^3/6.$$

Therefore,

$$\alpha 2^{2-\mu}\|x_{\alpha\delta} - x_0\|^\mu \leq \delta\|x_{\alpha\delta} - x_0\| + \alpha\|\omega(x_{\alpha\delta} - x_0)\|\left(\delta + \alpha\|x_{\alpha\delta}\| + L\|x_{\alpha\delta} - x_0\|^3/6\right). \quad (2.1)$$

Hence

$$\left(2^{2-\mu} - \frac{L}{6}\|\omega(x_{\alpha\delta} - x_0)\|\|x_{\alpha\delta} - x_0\|^{3-\mu}\right)\|x_{\alpha\delta} - x_0\|^\mu \leq [(\delta + \alpha^2\|\omega(x_{\alpha\delta} - x_0)\|)/\alpha]\|x_{\alpha\delta} - x_0\|$$
$$+ \|\omega(x_{\alpha\delta} - x_0)\|\left(\delta + \alpha\|x_0\|\right).$$

Since $x_{\alpha\delta} \to x_0$ and $\mu < 3$, for sufficiently small α and δ, we have

$$\|x_{\alpha\delta} - x_0\|^\mu \leq (\delta/\alpha + \alpha\|\omega(x_{\alpha\delta} - x_0)\|) + \|\omega(x_{\alpha\delta} - x_0)\|(\delta + \alpha\|x_0\|)).$$

Using the relation in [12]:

$$a,\ b,\ c\ \geq\ 0,\ p\ >\ q,\ a^p \leq ba^q + c \Longrightarrow a^p = O(b^{p/(p-q)} + c)$$

we obtain

$$\|x_{\alpha\delta} - x_0\| = O(\delta^q),\ \ q\ =\ \min\ ((1-p)/(\mu - 1), p/\mu) \qquad \otimes$$

Remark 1. If $A''(x_0) = 0$ condition (iii) of Theorem will be written in the common form (iii) in Introduction with the righ-hand side $\partial\varphi(x_0)$. We shall see this in an example in Section 4.

For numerical approximations one has to approximate the infinite dimensional Hilbert space H by a sequence of finite-dimensional subspaces H_n:

$$H_1 \subset H_2 \subset \ldots \subset H_n \ldots \subset H, \; P_n x \to x, \; n \to +\infty, \; \forall x \in H,$$

where P_n denotes the orthogonal projections from H onto H_n. Now, in place of (1.4), consider the finite-dimensional problems

$$A_n(x) + \alpha \partial \varphi_n(x) \; = \; f_\delta^n, \; x \in H_n \tag{2.2}$$

where $A_n = P_n^* A P_n$, $\partial\varphi_n = P_n^* \partial\varphi P_n$ and $f_\delta^n = P_n^* f_\delta$. It is easy to verify that A_n and $\partial\varphi_n$ are monotone and continuous in H_n. Hence Eq. (2.2) has a unique solution $x_{\alpha\delta}^n$ for $\alpha > 0$ and, for arbitrary $\alpha > 0$ and $f_\delta \in H$, the sequence $\{x_{\alpha\delta}^n\}$ converges to $x_{\alpha\delta}$, as $n \to \infty$ (see [17]).

Theorem 2.2. *Let the following conditions hold:*

(i) A is twice-Fréchet differentiable at some neighbourhood $\mathcal{U}_0$ of S_0,

(ii) there exists a constant $L > 0$ such that

$$\|A''(x) - A''(y) \leq L\|x - y\|, \; x, \; y \; \in \; \mathcal{U}_0,$$

and

(iii) $\alpha = \alpha(n, \delta)$ is such that $\alpha, \delta/\alpha \to 0$ and

$$\Big(\gamma_n(x)\|(I - P_n)x\| + L\|(I - P_n)x\|^3/6\Big)/\alpha \; \to 0, \; \forall x \in S_0$$

as $n \to \infty$, where $\gamma_n(x)$ is denoted by

$$\gamma_n(x) \; = \; \max\{\|A'(x)(I - P_n)\|, \|A''(x)(I - P_n)\|\|x\|\},$$

I is the identity operator in H. Then the sequence $\{x_{\alpha\delta}^n\}$ converges to x_0.

Proof. From (2.2) we have

$$\begin{aligned} \big\langle A_n(x_{\alpha\delta}^n) - A_n(x^n) + \alpha(\partial\varphi_n(x_{\alpha\delta}^n) - \partial\varphi_n(x^n)), x_{\alpha\delta}^n - x^n\big\rangle \; &= \; \big\langle f_\delta^n - A_n(x^n), x_{\alpha\delta}^n - x^n\big\rangle \\ &+ \alpha\big\langle \partial\varphi_n(x^n), x^n - x_{\alpha\delta}^n\big\rangle, \; x^n = P_n x, \; x \in S_0. \end{aligned} \tag{2.3}$$

As

$$A(P_n x) = A(x) + A'(x)(P_n x - x) + \frac{1}{2}A''(x)(P_n x - x)^2 + r^n, \;\; \|r^n\| \leq L\|(I - P_n)x)\|^3/6, \; x \in S_0,$$

from (2.3) it implies

$$2^{2-\mu}\|x^n_{\alpha\delta} - x^n\|^\mu \le (\delta + 1.5\gamma_n(x)\|(I-P_n)x\| + L\|(I-P_n)x\|^3/6)\|x^n_{\alpha\delta} - x^n\|/\alpha \qquad (2.4)$$

$$+\langle \partial\varphi(x^n), x^n - x^n_{\alpha\delta}\rangle.$$

Consequently, the sequence $\{x^n_{\alpha\delta}\}$ is bounded. Let $x^n_{\alpha\delta} \rightharpoonup x_1$ as $\alpha,\ \delta/\alpha \to 0$ and $n \to \infty$. Then $A_n(x^n_{\alpha\delta}) \to f_0$ follows from (2.2). We write the monotone property for A_n:

$$\langle A_n(x^n_{\alpha\delta}) - A_n(x^n), x^n_{\alpha\delta} - x^n\rangle \ge 0,\ \forall\ x\ \in\ H,\ x^n = P_n x.$$

Therefore,

$$\langle A_n(x^n_{\alpha\delta}) - A(x^n), x^n_{\alpha\delta} - x^n\rangle \ge 0.$$

From the last inequality and the continuity of A it follows

$$\langle f_0 - A(x), x_1 - x\rangle\ \ge\ 0,\ \forall\ x \in\ H,$$

i.e. $x_1 \in\ S_0$. Replacing x^n by x^n_1 $(= P_n x_1)$ in (2.4) we can conclude that the sequence $\{x^n_{\alpha\delta}\}$ converges strongly to x_1 and

$$2^{2-\mu}\|x^n_{\alpha\delta} - x^n\|^\mu \le \Big(\delta + 1.5\gamma_n(x)\|(I-P_n)x\| + L\|(I-P_n)x\|^3/6\Big)\|x^n_{\alpha\delta} - x^n\|/\alpha$$

$$+\langle \partial\varphi(x^n), x^n - x^n_{\alpha\delta}\rangle.$$

After passing $\alpha,\ \delta \to 0$ and $n \to \infty$ in the last inequality we obtain

$$\langle \partial\varphi(x), x - x_1\rangle \ge 0,\ \forall x \in S_0.$$

The last variational inequality is equivalent to $\langle \partial\varphi(x_1), x - x_1\rangle \ge 0,\ \forall x \in S_0$. Then $x_1 = x_0$ and the entire sequence $\{x^n_{\alpha\delta}\}$ converges strongly to x_0. $\otimes$.

Remark 2. From the above froof we can see that Theorem is still true if condition (iii) is replaced by

$$\gamma^1_n(x)/\alpha \to 0,\ \gamma^1_n(x) = \|(I-P_n)x)\|,\ x \in S_0. \qquad (iii^*)$$

We prove the following theorem in this case.

Theorem 2.3. *Assume that the following conditions hold:*

(i) conditions (i) - (iii) of Theorem 2.1 with $\omega(y) \equiv \omega_1,\ \forall y \in S(x_0, r)$ *and* (iii^*)

(ii) there exist two constants $L' > 0$, $\gamma' > 0$ such that

$$\langle \partial\varphi(y) - \partial\varphi(x_0), z\rangle \le L'\|y - x_0\|^{\gamma'}\|z\|, \ \forall y, \ z \in S(x_0, r).$$

If we choose $\alpha = 0\Big((\delta + \gamma_n^1)^p\Big)$, $0 < p < 1$, and denote

$$\gamma_n^1 = \max\{\|(I - P_n)x_0\|, \|(I - P_n)\omega_1\|, \|(I - P_n)f_0\|\}.$$

Then

$$\|x_{\alpha\delta}^n - x_0\| = O\Big(\delta^{q_1} + (\gamma_n^1)^{q_2}\Big), \quad q_1 = \min\{(1-p)/(\mu - 1), p/\mu\} \text{ and } q_2 = \min\{\gamma'/(\mu - 1), q_1\}.$$

Proof. From (2.3) (with $x = x_0$),

$$\|A(P_n x_0) - f_\delta\| \le \delta + 1.5\tilde{\gamma}_0\|(I - P_n)x_0\| + L\|(I - P_n)x_0\|^3/6$$

where

$$\tilde{\gamma}_0 = \max\ \{\|A'(x_0)\|, \|A''(x_0)\|\},$$

and the monotonicity of A_n it follows

$$\alpha 2^{2-\mu}\|x_{\alpha\delta}^n - x_0^n\|^\mu \le \alpha\langle\partial\varphi(x_0^n), x_0^n - x_{\alpha\delta}^n\rangle + \|f_\delta - A(x_0^n)\|\|x_{\alpha\delta}^n - x_0^n\|$$

$$\le (\delta + \tilde{\gamma}_0\gamma_n^1 + (\gamma_n^1)^3)\|x_{\alpha\delta}^n - x_0^n\| + \alpha\langle\partial\varphi(x_0), x_0^n - x_{\alpha\delta}^n\rangle + \alpha\langle\partial\varphi(x_0^n) - \partial\varphi(x_0), x_0^n - x_{\alpha\delta}^n\rangle$$

$$\le (\delta + \tilde{\gamma}_0\gamma_n^1 + (\gamma_n^1)^3 + \alpha L'(\gamma_n^1)^{\gamma'})\|x_{\alpha\delta}^n - x_0^n\| + \alpha\langle\omega_1, A(x_0) - A(x_{\alpha\delta}^n)\rangle + \alpha\|\omega_1\|\|r_{\alpha\delta}^n\|,$$

where

$$r_{\alpha\delta}^n = A(x_{\alpha\delta}^n) - A(x_0) + A'(x_0)(x_0 - x_{\alpha\delta}^n) - \frac{1}{2}A''(x_0)(x_{\alpha\delta}^n - x_0)^2,$$

$$\|r_{\alpha\delta}^n\| \le L\|x_{\alpha\delta}^n - x_0\|^3/6 \le L\|x_{\alpha\delta}^n - x_0^n\|^3/6 + O(\gamma_n^1)$$

and

$$\langle\omega_1, A(x_0) - A(x_{\alpha\delta}^n)\rangle \le \|\omega_1\|(\delta + \gamma_n^1 + \|f_\delta^n - A_n(x_{\alpha\delta}^n)\|) + \langle(P_n - I)\omega_1, A(x_{\alpha\delta}^n)\rangle.$$

Because of locally bounded property of every hemicontinuous and monotone operator (see [20])

$$\langle\omega_1, A(x_0) - A(x_{\alpha\delta}^n)\rangle \le \|\omega_1\|(\delta + \gamma_n^1 + C_1\alpha + C_2\|(I - P_n)\omega_1\|, \ C_1, C_2 > 0.$$

Consequently

$$\alpha\Big(2^{2-\mu}-\frac{L\|\omega_1\|}{6}\|x^n_{\alpha\delta}-x^n_0\|^{3-\mu}\Big)\|x^n_{\alpha\delta}-x^n_0\|^{\mu}\le\Big(\delta+\tilde{\gamma}_0\gamma^1_n+(\gamma^1_n)^3+\alpha L'(\gamma^1_n)^{\gamma'}\Big)$$
$$\|x^n_{\alpha\delta}-x^n_0\|+\alpha\Big(\|\omega_1\|((1+C_2)\gamma^1_n+\delta+C_1\alpha\Big).$$

Using, again, the relation in [12] we have

$$\|x^n_{\alpha\delta}-x^n_0\|=O(\delta^{q_1}+(\gamma^1_n)^{q_2})$$

and

$$\|x^n_{\alpha\delta}-x_0\|=O(\delta^{q_1}+(\gamma^1_n)^{q_2}) \qquad \otimes$$

III. ITERATIVE METHOD

Now consider an iterative method to solve the equation

$$F(x)\equiv A(x)+\partial\varphi(x)=f,\ f\in R(A+\partial\varphi), \tag{3.1}$$

where A and $\partial\varphi$ are defined as above. In the case $\partial\varphi=I$, the unique solution $\tilde{x}$ of (3.1) can be found by iterative methods in [2], [3] and [4] since in this cases $\mu=2$.

Let x^1 be an arbitrary element of H. The sequence of iterations x^k is constructed by the formula

$$x^{k+1}=x^k-\beta_k(F(x^k)-f)/\|F(x^k)-f\|. \tag{3.2}$$

Theorem 3.1. *If the real numbers β_k satisfy the conditions*

$$1>\beta_n>0,\ \beta_n\searrow 0,\ \sum_{n=1}^{\infty}\beta_n=+\infty,\ \sum_{n=1}^{\infty}\beta_n^2<+\infty,$$

then the sequence $\{x^k\}$ converges to $\tilde{x}$, as $k\to+\infty$.

Proof. Put

$$\lambda_k:=\|x^k-\tilde{x}\|^2.$$

It is easy to see that

$$\lambda_{k+1}\le\lambda_k+2\langle x^{k+1}-x^k,\ x^k-\tilde{x}\rangle+\|x^{k+1}-x^k\|^2.$$

From this inequality, the uniformly monotone property of F (that is caused by that one of $\partial\varphi$) and (3.2) we get

$$\lambda_{k+1} \leq \lambda_k - 2^{3-\mu}\beta_k\lambda_k^{\mu/2}/\|F(x^k) - f\| + \beta_k^2. \tag{3.3}$$

Therefore, the sequence $\{\lambda_k\}$ is bounded. Consequently, the sequences $\{x^k\}$ and $\{F(x^k)\}$ are bounded, too. Hence there exist constants $G_1,\ G_2 > 0$ such that

$$\lambda_k \leq G_1 \text{ and } \|F(x^k) - f\| \leq G_2.$$

We can write (3.3) in the form

$$\Delta_{k+1} \leq \Delta_k - \frac{2^{3-\mu}G_1^{\mu/2-1}}{G_2}\beta_k\Delta_k^{\mu/2} + \beta_k^2/G_1, \quad \Delta_k = \lambda_k/G_1.$$

Repeating the proof of Lemma 3 in [17] to the last inequality with $s_k = \alpha_1\Delta_1^{\mu/2-1} + ... + \alpha_k\Delta_k^{\mu/2-1}$ we can conclude that the sequence $\{\Delta_k\}$ tends to zero, as $k \to +\infty$. Theorem is proved.

IV. APPLICATION

We now apply the obtained results of the previous sections to study the singular integral equation in form (see [8])

$$\int_0^t |s-t|^{-\lambda}x(s)ds + F(x(t)) = f_0(t), \ 0 < \lambda < 1, \tag{4.1}$$

where $f_0(t) \in L_2([0,1])$ and $F(t)$ satisfies the following conditions:

(i) $F(t)$ is a differentiable function,

(ii)$|F(t)| \leq c_0 + c_1|t|; c_i > 0;\ F(t_1) \leq F(t_2),\ t_1 \leq t_2$.

Let the operators K and F define by

$$Ky(t) = \int_0^1 k(t,s)x(s)ds, \ (Fx)(t) = F(x(t)),$$

where $k(t,s) = 0$ if $t \geq s$ and $k(t,s) = (t-s)^{-\lambda}$ if $s < t$. Then K and F are the monotone operators in $H = L_2([0,1])$. In addition, suppose F is compact, therefore (4.1) is an ill-posed problem, because K also is compact (see [10]). It is easy to see that $A = K + F$ is monotone and Fréchet differentiable. In this case the condition (iii) of Theorem 2.1 is described by

$$(K^* + F'^*(x_0) - \frac{1}{2}F''^*(x_0)y)\omega = \partial\varphi(x_0) \tag{4.2}$$

and if F'' is Lipschitz continuous A'' also is Lipschitz continuous. If $F''(x_0) = 0$, for istance, F is linear on the set of solutions of (4.1) , then (4.2) has a simple form

$$(K^* + F'^*(x_0))\omega = \partial\varphi(x_0).$$

And, in particular, if $F'(x_0) \equiv 0$ that condition has a very simple form $K^*\omega = \partial\varphi(x_0)$.

Consider a concrete example, when

$$F(t) = \begin{cases} t, & t \leq 0, \\ 0, & 0 < t < 1, \\ 2(t-1), & t \geq 1, \end{cases} \tag{4.3}$$

with $x_0(t) = \tilde{c}_0 t^\beta$, $\beta > 0$, where $\tilde{c}_0$ is a constant satisfying the condition

$$\mu\|x_0\|^{\mu-1}_{L_2([0,1])}\tilde{c}_0(\beta + 3/2) = 1.$$

Then

$$f_0(t) = c_0\int_0^t |t-s|^{-\lambda}s^{\beta+1/2}ds, \ \partial\varphi(x_0)(t) = t^{\beta+1/2}.$$

In this case, $\omega(t) = \Gamma(\beta + 3/2)t^\beta/\Gamma(\beta+1)$ (see [7]).

Without loss of generality, consider $\tilde{x}_0(t) = x_0(t)/\tilde{c}_0$ and $\tilde{f}_0(t) = f_0(t)/\tilde{c}_0$ with $\lambda = \beta = 0.5$ and $\mu = 2.5$.

The values of $f_\delta(t)$ are chosen as perturbations of the values $\tilde{f}_0(t)$ according to

$$f_\delta(t) = \tilde{f}_0(t) + \delta.$$

We compute the regularized solution $x^n_{\alpha\delta}$ for this problem using the iterative method (3.2) with error estimate 0.001 and approximating the Hilbert space $L_2[0,1]$ by the sequence of linear subspaces H_n, where

$$H_n = L\{\psi_1, \psi_2, \ldots, \psi_n\},$$

$$\psi_j = \begin{cases} 1, & t \in [t_{j-1}, t_j], \\ 0, & t \notin [t_{j-1}, t_j], \ j = 1, \ldots n. \end{cases}$$

It is wellknow that

$$\|(I - P_n)y_0\| = O(n^{-2}), \quad \text{where } P_n y = \sum_{j=1}^n y(t_j)\psi_j(t).$$

Now we apply Theorem 2.3 for $\alpha(n) = O(n^{-1})$ and $\delta = O(n^{-2})$. We should obtain the convergence rates $e^n = \|\tilde{x}_0 - x^n_{\alpha\delta}\|$.

All numerical results are obtained with FORTRAN programs on an IBM 3031.

n	Calc.Results	
points	$\alpha(n)$	e^n
15	0.007143	0.115264
21	0.005000	0.097654
33	0.003125	0.083582
65	0.001563	0.067732

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