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ACCELERATED METHOD FOR SOLVING GRID EQUATIONS II (3-D CASE)

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Summary. In this paper following the parametric extrapolation technique [1] we consider the case when the starting operator is split into the sum of three operators. A result on estimating iteration numbers needed for solving the perturbed problem is obtained. Finally, the advantage of our method over the direct use of the alternating directions method is shown on examples.

This paper is a continuation of [1], in which the theoretical bacground of the accelerated method by parametric extrapolation was elaborated and the case of splitting the operator R into the sum of two operators was investigated. Here we consider the case when R is split into the sum of three symmetric and commute operators. This case encounters in solving boundary value problems for elliptic equations in three-dimensional domains.

For convenience all the formulas in the paper are numbered anew.

Thus, we assume that

$$\begin{aligned} &R = R_1 + R_2 + R_3,\ \ R_iR_j = R_jR_i,\ \ i,j = 1,2,3, \\ &\delta_i E \leq R_i = R_i^* \leq \Delta_i E,\ \ \delta_i > 0,\ \ i = 1,2,3. \end{aligned} \tag{1}$$

Assume also that after discretization of a differential problem on a grid with stepsize h we obtain the operator equation

$$Au = f \tag{2}$$

with $A = A^* \geq \delta E$

We put

$$\begin{aligned} &P = R_1R_2 + R_1R_3 + R_2R_3 + \sqrt{h}R_1R_2R_3, \\ &A_\varepsilon = A + \varepsilon P. \end{aligned} \tag{3}$$

We construct the operator B, energetic equivalent to A_ε in the form

$$B = (E + \omega R_1)(E + \omega R_2)(E + \omega R_3). \tag{4}$$

The parameter ω will be found so that the ratio of the energetic equivalent coefficients of B and A_ω is maximal. As in the cases 1 & 2 [1] we set $\omega = a\sqrt{\varepsilon}$ and seek a.

Lemma. *Assume that*

$$C_1 R \leq A \leq C_2 R \tag{5}$$

and

$$\varepsilon \leq C_1^2 \sqrt{h}/C_2. \tag{6}$$

Then

i) if $a \in \mathcal{A}$, *where*

$$\mathcal{A} = \{a, \frac{\sqrt{\varepsilon}}{C_1} \leq a \leq \frac{\sqrt{h}}{\sqrt{\varepsilon}}, a \geq \frac{h^{1/4}}{\sqrt{C_2}}\} \tag{7}$$

we have

$$\gamma_1 A_\varepsilon \leq B \leq \gamma_2 A_\varepsilon, \tag{8}$$

where

$$\gamma_1 = \frac{\delta}{1 + \delta a^2}, \quad \gamma_2 = \frac{C_2}{a\sqrt{\varepsilon}}.$$

Hence the ratio of the energetic equivalent coefficients of B *and* A_ε *is*

$$\xi(a) = \frac{\gamma_1}{\gamma_2} = \frac{\delta a \sqrt{\varepsilon}}{C_2(1 + \delta a^2)} \tag{9}$$

ii) if except (6) ε *additionally satisfies the condition*

$$\varepsilon \leq \min \left(\frac{C_1^2}{\delta}, \delta h\right), \tag{10}$$

then in the case

$$h \leq C_2^2/\delta^2 \tag{11}$$

the ratio $\xi(a)$ *reaches maximum at* $a = \overset{0}{a} = 1/\sqrt{\delta}$ *and we have*

$$\gamma_1 = \overset{0}{\gamma_1} = \frac{\delta}{2}, \quad \gamma_2 = \overset{0}{\gamma_2} = \frac{C_2\sqrt{\delta}}{\sqrt{\varepsilon}},$$

$$\xi = \gamma_1 = \overset{0}{\xi} = \frac{\sqrt{\delta\varepsilon}}{2C_2},$$

iii) if (11) is not satisfied then $\xi(a)$ *reaches maximum when* $a = \bar{a} = h^{1/4}/\sqrt{C_2}$ *and we have*

$$\bar{\xi} = \xi(\bar{a}) = \frac{\delta h^{1/4}\sqrt{\varepsilon}}{\sqrt{C_2}(C_2 + \delta\sqrt{h})}. \tag{12}$$

Proof. The estimate (8) is obtained in the same way as in Lemma 3.1 of [1]. Thus, if (6) and (7) are satisfied then we have (9).

Under (10) and (11) it is easy to verify that $\overset{0}{a} = 1/\sqrt{\delta} \in \mathcal{A}$. Set

$$\varphi(a) = \frac{a}{1 + \delta a^2}.$$

From the fact that

$$\max_{a \geq 0} \varphi(a) = \varphi(\overset{0}{a})$$

follows the assertion ii) of the lemma.

Finally, notice that $\overline{a} = h^{1/4}/\sqrt{C_2} \leq \overset{0}{a}$ and $\overline{a} \in \mathcal{A}$. From the property of monotone increase of the function $\varphi(a)$ in $(0, \overset{0}{a})$ we obtain the assertion iii) of the lemma.

Now for solving the perturbed equation

$$A_\varepsilon u_\varepsilon = f, \tag{13}$$

where A_ε has the form (3) we use the iterative process

$$B \frac{y_\varepsilon^{(k+1)} - y_\varepsilon^{(k)}}{\tau_{k+1}} + A_\varepsilon y_\varepsilon^{(k)} = f, \tag{14}$$

$y_\varepsilon^{(0)}$ is given. Here B is defined by (4) in which $\omega = a\sqrt{\varepsilon}$ and a is given by the above lemma.

From the theory of two-layer iterative processes in [4] we get the following result.

Theorem. *Assume that ε is chosen so that (6) and (10) are valid. Then*

i) if h satisfies (11) and $a = \overset{0}{a}$ then the process (14) with the Chebysev collection of parameters $\{\tau_{k+1}\}$ constructed by $\overset{0}{\gamma}_1$ and $\overset{0}{\gamma}_2$ reaches the relative accuracy θ, that is,

$$\|y_\varepsilon^{(k+1)} - u_\varepsilon\| \leq \theta \, \|y_\varepsilon^{(0)} - u_\varepsilon\|$$

after $n_c(\theta)$ iterations

$$n_c(\theta) = \ ln\frac{2}{\theta}/(\sqrt{\frac{2}{C_2}}(\delta\varepsilon)^{1/4}).$$

For the stationary iterative process with

$$\tau_k \equiv \tau = 2/(\overset{0}{\gamma}_1 + \overset{0}{\gamma}_2)$$

to reach the relative accuracy θ it is needed $n_d(\theta)$ iterations

$$n_d(\theta) = \ ln\frac{2}{\theta}/\frac{\sqrt{\delta\varepsilon}}{C_2}.$$

ii) in the case if (11) is not satisfied, with the selection

$$a = \bar{a} = h^{1/4}/\sqrt{C_2}$$

the numbers of iterations are estimated as follows:

- for the iterative process with the Chebyshev parameters

$$n_c(\theta) = \ ln\frac{2}{\theta}/(2\sqrt{\bar{\xi}}),$$

- for the stationary iterative process

$$n_d(\theta) = \ ln\frac{2}{\theta}/(2\bar{\xi}),$$

where $\bar{\xi}$ is given by (12).

Example 1. Consider the Dirichlet problem for the oisson equation in the unitary cûb $\overline{\Omega} = \{x,\ 0 \le x_\alpha \le 1,\ \alpha = 1,2,3.\}$ with boundary Γ

$$-\Delta u = -(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}) = f(x),\ x \in \Omega,$$
$$u|_\Gamma = 0.$$

We approximate this problem by the difference scheme on the grid $\bar{\omega}$, defined as in Example 3.1 [1]

$$Ay = -y_{\bar{x}_1 x_1} - y_{\bar{x}_2 x_2} - y_{\bar{x}_3 x_3} = f(x),\ x \in \omega,$$
$$y|_\gamma = 0.$$

The operators $R,\ R_\alpha$ are defined as follows

$$R = A,\ R_\alpha y = -y_{\bar{x}_\alpha x_\alpha},\ \alpha = 1,2,3.$$

The values of quantities $C_1,\ C_2$ in the Lemma and Theorem above are

$$C_1 = C_2 = 1,\ \delta = 24.$$

The parameter in the Theorem should be $\varepsilon \le\ \min(1/24, 24h, \sqrt{h})$.
If $h \le 1/576$ then (11) is satisfied. We choose $\varepsilon = 24h,\ a = 1/\sqrt{24}$. Then we have $\overset{0}{\xi} = 12\sqrt{h}$. Hence

$$n_c(\theta) = \ \ln\frac{2}{\theta}/(\sqrt{48}h^{1/4})$$
$$n_d(\theta) = \ \ln\frac{2}{\theta}/(24h^{1/2}). \tag{15}$$

If $1/576 < h \leq 1/24$ we choose $\varepsilon = h,\ a = h^{1/4}$ Then we have

$$\bar{\xi} = \frac{24h^{3/4}}{1+24h^{1/2}}.$$

Hence

$$\begin{aligned} n_c(\theta) &= (1+24h^{1/2})^{1/2}\ln\frac{2}{\theta}/(4\sqrt{6}h^{3/8}), \\ n_d(\theta) &= (1+24h^{1/2})^{1/2}\ln\frac{2}{\theta}/(48h^{3/4}). \end{aligned} \tag{16}$$

Notice that in [2] it was shown that for solving Dirichlet problem for the Poisson equation in unitary cub with the relative accuracy θ they need perform n iterations

$$n = n_c(\theta) = O(\ln\frac{1}{\theta}/h^{3/4}). \tag{17}$$

Of course, here we have in mind the alternating directions method (ADM) with the Chebyshev collection of parameters.

From (15)-(17) we see the apparent advantage of our parametric extrapolation technique over the direct use of ADM.

Below we give some more example for the application of our technique.

Example 2. Consider the first boundary value problem for the Lame equation in the elasticity theory

$$\begin{aligned} &\mu\Delta u + (\lambda+\mu)\text{grad div}(u) + f = 0,\ x \in \Omega, \\ &u = g(x),\ x \in \Gamma. \end{aligned} \tag{18}$$

Here Ω and Γ are the same as in Example 1, $u = (u^1, u^2, u^3)^T$ is the displacement vector, $f = (f^1, f^2, f^3)^T$ the body strength vector, λ and μ are the Lame constants.

Usually, the problem (18) is approximated by the difference scheme (see [2-4])

$$\begin{aligned} &(\Lambda y)^s = -\mu\sum_{\alpha=1}^{3} y^s_{\bar{x}_\alpha x_\alpha} - 0.5(\lambda+\mu)\sum_{\beta=1}^{3}(y^\beta_{\bar{x}_\beta x_s} + y^\beta_{x_\beta \bar{x}_s}) = f^s(x),\ x \in \omega_h, \\ &y^s(x) = g^s(x),\ x \in \gamma_h,\ s = 1,2,3. \end{aligned} \tag{19}$$

where $y = (y^1, y^2, y^3)^T$ is a grid vector-function to be sought.

After eliminating the boundary condition on γ_h we can rewrite (19) in the form

$$Ay = F,\ y \in \Omega_h, \tag{20}$$

where Ω_h is the space of grid vector-functions defined in ω_h, A is the operator, which coincides with Λ on the set of grid vector-functions equal to zero on γ_h.

Introduce the operator R, R_α by the formulas

$$Ry = -\sum_{\alpha=1}^{3} y_{\bar{x}_\alpha x_\alpha}, \ R_\alpha y = -y_{\bar{x}_\alpha x_\alpha}, \ \alpha = 1, 2, 3.$$

Then we have $C_1 R \leq A \leq C_2 R$ with $C_1 = \mu$, $C_2 = \lambda + 2\mu$. We have also $\delta = 24.$. Following the Theorem we get the estimates for iteration numbers for the perturbed equation corresponding to (20), which are like to those in Example 1.

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