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THE INVERSE SOURCE PROBLEMS FOR THE HELMHOLTZ OPERATORS (*)

LE TRONG LUC

Abstract. The paper is devoted to studying the inverse source problem for the Helmholtz equations as investigating solution sets (or information content) because of its ill-posedness, consisting of source distributions located on a closed domain considered, which create the potentials known outside that closed domain and unknown on it.

I - INTRODUCTION AND NOTATIONS

In this paper we study the inverse source problems for the positive Helmholtz operator

$$P = P^{+} := \Delta_3 + k^2,\ k > 0\,,$$

and the negative Helmholtz operator

$$Q = P^{-} := \Delta_3 - k^2,\ k > 0\,,$$

in the sense of distributions, where Δ_3 is the Laplace Operator in the Euclidean space R^3. The inverse source problem consists in determining the distributions creating the potentials which are known exterior to a given closed domain. This problem has been studied in the sense of measures in [3], [9]. Here we generalize it into the sense of distributions as in [5], [6], [7].

Let R^n be the n-dimensional Euclidean space. Denote by $C_0(R^n)$ the space of all continuous functions with compact support, by $C_0^1(R^n), C_0^m(R^n), C_0^\infty(R^n)$ the space of all functions of $C_0(R^n)$ with continuous derivatives of order $1, m$, and derivatives of arbitrary order. Further let us denote by $C_0^{'}(R^n), C_0^{1'}(R^n), C_0^{m'}(R^n)$, $C_0^{\infty'}(R^n)$ the dual space of $C_0(R^n), C_0^1(R^n), C_0^m(R^n), C_0^\infty(R^n)$, respectively. As usual, we say that the elements of $C_0^{\infty'}(R^n), C_0^{m'}(R^n), C_0^{1'}(R^n), C_0^{'}(R^n)$ are distributions, distributions of order $\leq m$, distributions of order ≤ 1, distributions of order zero or (Radon) measures, respectively, (see [4],Theorem 2.1.6 and [9]).

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Let $\alpha = (\alpha_1, ..., \alpha_n)$ be a multi-index, $\alpha_i \geq 0$. We denote by $|\alpha| = \alpha_1 + ... + \alpha_n$ and by $(\mu_\alpha)_{|\alpha| \leq m}$ a vector of measures, where $\mu_\alpha \in C_0'(R^n)$. Then the distribution T defined by

$$T : \varphi \to \sum_{|\alpha| \leq m} \int D^\alpha \varphi(x) d\mu_\alpha(x), \; \varphi \in C_0^m(R^n) \tag{1}$$

is a special distribution of $C_0^{m\prime}(R^n)$.

Let $C_0^{m\prime}(R^n)_j$ stand for the set of all distributions of the special form (1) with $|\alpha| = j, j = 0, 1, 2, ..., m$. Then

$$C_0^{m\prime}(R^n) \supseteq \bigcup_{j=0}^{m} C_0^{m\prime}(R^n)_j \, .$$

For $m = 1$ we have

$$C_0'(R^n) \subset C_0'(R^n) \cup C_0^{1\prime}(R^n)_1 \subseteq C_0^{1\prime}(R^n).$$

A measure μ is called positive if $\mu(f) \geq 0$ for every $f \geq 0$. Each measure μ can be represented as the difference of its positive part μ^+ and negative part μ^-, i.e. $\mu = \mu^+ - \mu^-$, where

$$\mu^+(f) = \sup\{\mu(g) : g \in C_0(R^n), \; 0 \leq g \leq f\},$$
$$\mu^-(f) = \sup\{-\mu(g) : g \in C_0(R^n), \; 0 \leq g \leq f\}.$$

Further, we denote by $C_0'(R^n)^+$ the set of all positive measures in R^n and by $C_0^{\infty\prime}(R^n, A)$, or $\mathcal{D}'(R^n, A)$, for an arbitrary set $A \subset R^n$, the set of all distributions ν with supp$\nu \subseteq A$. Analogously we denote by $C_0^{m\prime}(R^n, A)_j$ the set of all elements ν of $C_0^{m\prime}(R^n)_j$ with supp$\nu \subseteq A$.

Throughout this paper we consider a simply connected bounded domain G containing the origin in R^3 with sufficiently smooth boundary ∂G. Denote by $\overline{G}$ the closure of G and by $G_1 = R^3 \setminus \overline{G}$ the complement of $\overline{G}$. We shall discuss the inverse problems with respect to $\overline{G}$ and $\overline{G}_1$ which we will call the inner and outer inverse source problems. The coupling of the inner with the outer inverse problems will be discussed in another occasion. The paper is divided into two sections concerning the positive and negative Helmholtz operators.

II - THE INVERSE SOURCE PROBLEM FOR THE POSITIVE HELMHOLTZ OPERATOR

It is well known that for the positive Helmholtz operator there are two following fundamental solutions

$$E(x) = -\frac{e^{ik|x|}}{4\pi|x|}, \text{ and } \overline{E}(x) = -\frac{e^{-ik|x|}}{4\pi|x|} \, .$$

One says that a function $u \in C^1(G_1)$ satisfies the Sommerfeld emission condition at the infinity $S(\infty)$, or $\overline{S}(\infty)$, if the following condition is fulfilled, respectively,

$$\frac{\partial u(x)}{\partial |x|} - iku(x) = 0(|x|^{-1}),$$

or

$$\frac{\partial u(x)}{\partial |x|} + iku(x) = 0(|x|^{-1}), \; |x| \to \infty .$$

For simplicity of representation we introduce the following sets

$$S(\infty) := \{u : u \in C^1(G_1), \frac{\partial u(x)}{\partial |x|} - iku(x) = 0(|x|^{-1})\},$$
$$\overline{S}(\infty) := \{u : u \in C^1(G_1), \frac{\partial u(x)}{\partial |x|} + iku(x) = 0(|x|^{-1})\}.$$

It is easy to verify that $E \in S(\infty)$ and $\overline{E} \in \overline{S}(\infty)$.

1. The inner inverse source problems (inverse scattering problems)

Let us first recall the set of possible solutions $\mathcal{B}(\mu)$ introduced in [9] (p. 102):

$$\mathcal{B}(\mu) := \{\eta \in \mathcal{M}^+(\bar{\Omega}) : \pi\eta = \pi\mu\},$$

where μ is a positive given measure located on a bounded given domain $\bar{\Omega}$ of the Euclidean space R^3 (or R^n); and the Balayage-operator $\pi : \mathcal{M}(\bar{\Omega}) \to \mathcal{M}(\partial\Omega)$, acting from the set $\mathcal{M}(\bar{\Omega})$ of (Radon) measures on $\bar{\Omega}$ into the set $\mathcal{M}(\partial\Omega)$ of measures located on the boundary $\partial\Omega$, means

$$\pi\mu = \pi\eta \Longleftrightarrow \phi\mu(z) = \phi\eta(z) \quad \forall z \in \mathbf{C}\Omega \quad (\text{or } \forall z \notin \Omega),$$

with ϕ being the fundamental solution of the Laplace operator Δ. In [3] (p. 136) the same set of such possible solutions has been defined (with some slight changement of notations), namely

$$\mathcal{B}(\mu) := \{\nu \geq 0 : \ \mathrm{supp}\nu \subset \bar{\Omega}, G_L\nu(x) = G_L\mu(x), x \notin \Omega\},$$

where $G_L = \phi$, which is equivalent to ([3] p. 144)

$$\mathcal{B}(\mu) := \{\nu \geq 0 : \ \mathrm{supp}\nu \subset \bar{\Omega}, \int f d\nu = \int f d\mu ,$$
$$f \in C(\bar{\Omega}) \cap C^2(\Omega), \Delta f = 0 \text{ in } \Omega\} .$$

Here we understand the same inverse source problem above in the converse way as the following. We introduce the following set

$$H := \overline{H} := \{u : u \in \mathcal{D}'(G_1),\ Pu(x) = 0 \quad in \quad G_1\}.$$

Definition. The inner inverse source problem relative to an element $v \in H$ and to the fundamental solution E, or $\overline{E}$, respectively, consists in studying the following possible solutions set,

$$L(v) := \{\nu : \nu \in \mathcal{D}'(R^3, \overline{G}),\ E * \nu(x) = v(x),\ x \in G_1\},$$

or

$$\overline{L}(v) := \{\nu : \nu \in \mathcal{D}'(R^3, \overline{G}),\ \overline{E} * \nu(x) = v(x),\ x \in G_1\}.$$

The description of the distribution equation such as $E * f(x) = g(x),\ x \in G$ is referred to [12], [13].

Theorem 1.

(i) *For each* $v \in H \cap \mathcal{D}'(R^3, \overline{G}_1)$ *with the existence of the convolution* $E * v$ *there exists a solution* $\nu \in L(v) \cap \mathcal{D}'(R^3, \partial G)$, *which is represented by* $\nu = Pv$.

(ii) *For each* $v \in \overline{H} \cap \mathcal{D}'(R^3, \overline{G}_1)$ *with the existence of the convolution* $\overline{E} * v$ *there exists a solution* $\nu \in \overline{L}(v) \cap \mathcal{D}'(R^3, \partial G)$, *which is also represented by* Pv.

Proof. (i) Since $v \in H$, that is $Pv(x) = 0, \forall x \in G_1$, we have $\text{supp}Pv \subseteq R^3 \setminus G_1 = \overline{G}$. By assumption $v \in \mathcal{D}'(R^3, \overline{G}_1)$ we obtain $\text{supp}Pv \subseteq \overline{G}_1$. Consequently, we get $\text{supp}Pv \subseteq \overline{G} \cap \overline{G}_1 = \partial G$ or $Pv \in \mathcal{D}'(R^3, \partial G)$. It is known (cf.[12]) that from the existence of the convolution $f * g,\ f,\ g \in \mathcal{D}'(R^3)$ follows the existence of the convolution $D^\alpha f * g$ and $f * D^g\ \forall \alpha = (\alpha_1, \alpha_2, \alpha_3)$ with integers $\alpha_1,\ \alpha_2,\ \alpha_3$ satisfying $D^\alpha(f * g) = D^\alpha f * g = f * D^\alpha g$. Moreover, our operator P is a linear one with constant coefficients. Hence, from the existence of the convolution $E * v$ follows the existence of the convolutions $E * Pv$ and $(PE) * v$ which satisfy the condition

$$\begin{aligned} P[E * v](x) = E * Pv(x) = PE * v(x) = \\ = \delta * v(x) = v(x),\ x \in R^3. \end{aligned}$$

In particular, we obtain $E * Pv(x) = v(x),\ x \in G_1$, or $\nu = Pv \in L(v) \cap \mathcal{D}'(R^3, \partial G)$. q.e.d.

(ii) The proof of this part is quite similar to that of (i).

By additional information on $v \in H \cap C(\overline{G}_1) \cap S(\infty)$ and $v \in H \cap C^1(\overline{G}_1) \cap S(\infty)$ one can get classi al solutions in form of double and simple layers in the classes

$C_0^{1'}(R^3, \partial G)_1$ and $C'(\partial G)$, respectively. This is also the same for the functions of the classes $\overline{H} \cap C(\overline{G}_1) \cap \overline{S}(\infty)$, and $\overline{H} \cap C^1(\overline{G}_1) \cap \overline{S}(\infty)$, respectively.

Let A be a closed set of R^3, and $\varepsilon > 0$ a positive number. Then there exists a function $\eta_{A,\varepsilon}(x) \in C^\infty(R^n)$ with (cf. [8], p. 16)

$$\eta_{A,\varepsilon}(x) = 1 \qquad \forall x \in A,$$
$$\eta_{A,\varepsilon}(x) = 0 \qquad \forall x \notin A_\varepsilon$$

where

$$A_\varepsilon = \{x + y : x \in A, \quad |y| < \varepsilon\}.$$

Let f be an arbitrary distribution of $\mathcal{D}'(R^3)$. We call a distribution g of $\mathcal{D}'(R^3)$ the restriction of f on the closed set A, if for each sequence $\{\eta_{A,\varepsilon}\}$, $\varepsilon > 0$ the following condition holds :

$$g = \lim_{\varepsilon \to 0} f\eta_{A,\varepsilon} \text{ in } \mathcal{D}'(R^3),$$

or

$$(g, \varphi) = \lim_{\varepsilon \to 0}(f\eta_{A,\varepsilon}, \varphi) \text{ in } C^1 \qquad \forall \varphi \in \mathcal{D}(R^3).$$

We denote by $f_{|A}$ the restriction of f on the closed set A .

For each continuous function $f \in C(R^n)$ and each closed set A there exists a distribution $R_A f \in \mathcal{D}'(R^n)$ defined by

$$(R_A f, \varphi) = \int_A f(x)\varphi(x)dx \qquad \varphi \in \mathcal{D}(R^n).$$

Denoting by $f|_A$ the usual restriction of f on A, and by $\tilde{f}|_A$ the corresponding extension of $f|_A$ by zero on the whole space R^n, i.e.

$$\tilde{f}|_A(x) = \begin{cases} f|_A(x) = f(x), & x \in A \\ 0 & x \notin A, \end{cases}$$

there exists a distribution $T_A f$ of $\mathcal{D}'(R^n)$ carried by A:

$$(T_A f, \varphi) := \int_A \tilde{f}|_A(x)\varphi(x) = \int_A f(x)\varphi(x)dx \qquad \varphi \in \mathcal{D}(R^n).$$

By definition we have $R_A = T_A$ in $\mathcal{D}'(R^n)$. So we can identify the usual restriction $f|_A$ as a distribution of $\mathcal{D}'(R^n)$ with $R_A f$ or $T_A f$; and we can write

$$(f|_A, \varphi) = (T_A f, \varphi) = (R_A f, \varphi) \qquad \varphi \in \mathcal{D}(R^n)$$

for the convenience of presentation. In the following we shall use this notation for continuous functions in the whole space R^n.

It is not hard to show that if the restriction $f_{|A}$ exists, then it belongs to $\mathcal{D}'(R^n, A)$. The problem of posing conditions on the closed set A so that the restriction $f_{|A}$ exists for each distribution $f \in \mathcal{D}'(R^n)$ is still open.

Theorem 2 (Sweeping-Out Principle). *Let ν be a distribution of $\mathcal{D}'(R^n, \overline{G})$ so that the convolution $E * \nu$ exists. Suppose the existence of the restriction $(E * \nu)_{|\overline{G}_1}$. Moreover, assume the existence of the convolution $E * (E * \nu)_{|\overline{G}_1}$. Then there exists a so-called swept-out distribution ν' carried by ∂G so that*

$$E * \nu(x) = E * \nu'(x), \; x \in G_1.$$

Proof. The Theorem follows from Theorem 1 since $(E * \nu)_{|\overline{G}_1} \in H \cap \mathcal{D}'(R^3, \overline{G}_1)$ and the assumption that the convolution $E * (E * \nu)_{|\overline{G}_1}$ exists.

The Theorem can be formulated analogously for $\overline{H}$,and for $\overline{E}$.

2. Regarding to the outer inverse source problems we can deal with them in the same manner as to the inner ones with the validity of the Theorems 1, 2.

III - THE INVERSE SOURCE PROBLEM FOR THE NEGATIVE HELMHOLTZ OPERATOR

1. Fundamental solutions by negative Helmholtz operator

Setting $k = \lambda i$ in the fundamental solutions of the positive Helmholtz operator we obtain the two following fundamental solutions for the negative Helmholtz operator:

$$E(x) = -\frac{e^{-\lambda|x|}}{4\pi|x|}, \text{ and } \widetilde{E}(x) = -\frac{e^{\lambda|x|}}{4\pi|x|}.$$

Hereafter we only take the fundamental solution $E(x)$ with the real distributions ν of $\mathcal{D}'(R^n)$, i.e. (ν, φ) being real for each real valued function $\varphi \in \mathcal{D}(R^n)$, into consideration. The another one is more difficult to study since it is a continuous function as a regular distribution tending to infinity as x does so. This fundamental solution has the following property

$$E(x) = 0(\frac{1}{|x|^2}), \text{ and } \frac{\partial E(x)}{\partial |x|} = 0(\frac{1}{|x|^2}) \quad \text{as} \quad |x| \to \infty.$$

2. On classical boundary value problems

Lemma. *The outer classical Dirichlet and Neumann problems for the homogeneous negative Helmholtz equation*

$$Qu = (\Delta - \lambda^2)u(x) = 0, \quad x \in G_1$$

with real differentiable function u such that $u(x) = 0(\frac{1}{|x|^2})$ and $\frac{\partial u(x)}{\partial |x|} = 0(\frac{1}{|x|^2})$ as $|x| \to \infty$ are uniquely solvable in the class of real regular distributions.

Proof.

a. The Uniqueness. At first we show the uniqueness of the Dirichlet and Neumann problems considered.

Take a sufficiently great number R such that the ball

$$K(0, R) := \{x : |x| < R\},$$

contains the closure $\overline{G}$. Denote by S the boundary of G. Suppose there are two solutions v_1, v_2 of the Dirichlet or Neumann problem, we will show that $v_1 = v_2$. Indeed, setting $w = v_1 - v_2$ and applying Gauss-Ostrogradskij formula we get

$$\begin{aligned}\int_{S\cup\partial K(0,R)} w\frac{\partial w}{\partial n} dS &= \int_{K(0,R)\setminus\overline{G}} [w\Delta w + \sum_{i=1}^{3}(\frac{\partial w}{\partial x_i})^2]\, dx \\ &= \int_{K(0,R)\setminus\overline{G}} [\lambda^2 w^2 + \sum_{i=1}^{3}(\frac{\partial w}{\partial x_i})^2]\, dx\,,\end{aligned}$$

where dS and dx is the surface element and the volume element, respectively. (The similar uniqueness statement has been proved for the Laplace operator in [12]). By the assumption that $w = 0$ or $(\partial w/\partial n) = 0$ on S with respect to the Dirichlet or Neumann problem, we obtain

$$\begin{aligned}\int [\lambda^2 w^2 + \sum_{i=1}^{3}(\frac{\partial w}{\partial x_i})^2]\, dx &= -\int_{\partial K(0,R)} w\frac{\partial w}{\partial n_R} ds \\ &\le \frac{C}{R^2}\frac{1}{R^2}\frac{4}{3}\pi R^2 = \frac{4C}{3}\frac{\pi}{R^2}\,,\end{aligned}$$

where C is some constant, and $\frac{\partial w}{\partial n_R}$ is the outer normal derivative of w on the sphere centered at the origin with some sufficiently great radius R. Let $R \to \infty$, we have $w \equiv 0$, which proves the uniqueness of the solution.

b. The solvability. The solvability of the outer classical Dirichlet and Neumann problems can be found in [8], [10], [11]. Furthermore, the solution u of the outer Dirichlet problem is expressed in the form of potential of double layer

$$u(x) = E * \frac{\partial}{\partial n}(\rho\delta_S)(x),$$

and the solution of the outer Neumann problem is given by the potential of simple layer

$$u(x) = E * \rho\delta_S(x).$$

where

$$(\rho\delta_S, \varphi) = \int_S \rho(x)\varphi(x)dS(x) \quad \forall\varphi \in \mathcal{D}(R^3)$$

with $\rho \in C(S)$ (cf.[12],[13]).

3. The inner inverse problem for the negative Helmholtz operator

We introduce the potential set H as follows

$$H := \{u : u \in \mathcal{D}'(G_1), Pu(x) = 0 \text{ in } G_1\}$$

Definition. The inverse source problem relative to an element v of H consists in studying the following solution set

$$L(v) := \{\nu : \nu \in \mathcal{D}'(R^3, \overline{G}),\ E * \nu(x) = v(x) \quad in \quad G_1\}.$$

Theorem 3. *For each distribution v of H with $v \in C^1(G^1)$ satisfying condition $v(x) = 0(1/|x|^2), (\partial v(x)/\partial|x|) = 0(1/|x|^2)$ as $|x| \to \infty$, and $v(x) \leq 0$ there exists a uniquely determined positive measure solution carried by ∂G.*

Proof. We construct a sequence of domains $G^1, G^2, ..., G^j, ...$ with sufficiently smooth boundaries $S_j := \partial G^j$ and with the property: $G^1 \supset G^2 \supset \cdots \supset G^j \cdots \supset G$, and *lim* $G^j = G$ as $j \to \infty$. Since ∂G is supposed to be sufficiently smooth, such a sequence of domains G^j exists. Here we use the method of sweeping potentials from the whole $R^3 \setminus \overline{G}$ on the boundary ∂G (cf.[1][2]). Then there is for any j a measure $\nu_j \in C'(S_j)$ with

$$v(x) = E * \nu_j(x),\ \forall x \in G_1^j := R^3 \setminus \overline{G}^j. \tag{2}$$

We now show that $\nu_j \geq 0$ and the set $\{\nu_j\}$ is weakly bounded.

At first, let us consider the following sets

$$F(S_j) := \{\lambda : \lambda \in C'(S_j), \quad E * \lambda \in C(R^3)\},$$
$$D(S_j) := \{g : g = E * \lambda|_{S_j}, \quad \lambda \in F(S_j)\},$$

where $E*\lambda|_{S_j}$ is the restriction of $E*\lambda$ on S_j. Note that the set $F(S_j)$ is nonempty, for instance, for any $\rho \in C(S_j)$ the potential $E * \rho\delta_{S_j}$ belong to $C(R^3)$. We will show that $D(S_j)$ is dense in $C(S_j)$.

Indeed, since S_j is sufficiently smooth, the outer and inner classical Dirichlet problems for the homogeneous Helmholtz equation

$$Qu(x) = (\Delta - \lambda^2)u(x) = 0\,, \tag{3}$$

are uniquely determined for every boundary continuous function on S_j. On the other hand, the function

$$E(x-y) = \frac{e^{-\lambda|x-y|}}{4\pi|x-y|}\,,$$

satisfies the negative Helmholtz equation (3), too, with respect to y for the following separated cases

$$Q_y E(x-y) = 0 \quad for \quad y \in G_1^j, \quad x \in G^j;$$
$$Q_y E(x-y) = 0 \quad for \quad y \in G^j, \quad x \in G_1^j,$$

where $G_1^j = R^3 \setminus \overline{G^j}$. Note that for $x \in G^j$ we have $E(x-y) \in H \cap C^1(\overline{G_1^j})$. Thus there is a (harmonic) measure $\gamma_x \in C'(S_j)$ for any $x \in G^j$ with

$$E(x-y) = E * \gamma_x(y) \quad \forall y \in \overline{G_1^j}\,. \tag{4}$$

In the same way, there is a (harmonic) measure $\beta_x \in C'(S_j)$ for any $x \in G_1^j$ with

$$E(x-y) = E * \beta_x(y) \quad \forall y \in \overline{G^j}\,. \tag{5}$$

From this for each $x \notin S_j$ there is a measure $\mu_x \in C'(S_j)$ satisfying condition

$$E(x-y) = E * \mu_x(y) \quad \text{for} \quad y \in S_j \quad (x \notin S_j). \tag{6}$$

Suppose now contrarily that $D(S_j)$ is not dense in $C(S_j)$. Then, by Hahn-Banach theorem there is a measure $\mu \in C'(S_j)$, $\mu \neq 0$, such that

$$\mu(f) = 0 \quad \forall f \in D(S_j).$$

Since $E * \mu_x|_{S_j} \in D(S_j)$ we have

$$\mu(E * \mu_x|_{S_j}) = 0 \quad \forall x \notin S_j. \tag{7}$$

Combining (6) with (7) we obtain

$$\mu(E * \mu_x|_{S_j}) = \int_{S_j} E(x-y)d\mu(y) = E * \mu(x) = 0 \quad \forall x \notin S_j. \tag{8}$$

Since S_j is sufficiently smooth and the function $e^{-\lambda|x|}$ is continuous, (8) implies

$$E * \mu(x) = 0 \quad \forall x \in R^3.$$

From this it follows that

$$(E * \mu, Q^*\varphi) = 0 \quad \forall\varphi \in C_0^\infty(R^3),$$

consequently,

$$(E * \mu, Q^*\varphi) = (QE * \mu, \varphi) = \quad [illegible], \varphi) = (\mu, \varphi) = 0 \quad \forall\varphi \in C_0^\infty(R^3),$$

where Q^* is the adjoin operator of Q.

Since $C_0^\infty(R^3)$ is dense in $C_0(R^3)$, we obtain $\mu = 0$. This contradicts the assumption that $\mu \neq 0$. Thus, $D(S_j)$ is dense in $C(S_j)$.

We can easily show, as by Laplace operator [5], that the set

$$D^-(S_j) := \{E * \lambda|_{S_j}, \quad \lambda \in F(S_j), \quad \lambda \geq 0\},$$

is dense in

$$C^-(S_j) := \{g : g \in C(S_j), \quad g \leq 0\}.$$

By virtue of (2), we obtain

$$E' * \nu_j(x) = -v(x) \geq 0 \quad \forall x \in \overline{G}_1^j, \tag{9}$$

where

$$E'(x) = -E(x) = \frac{e^{-\lambda|x|}}{4\pi|x|} > 0.$$

Now we show that $\nu_j \geq 0$. Integrating both sides of (9) with respect to the measures $\lambda \in F(S_j)$, $\lambda \geq 0$, we obtain

$$\int E' * \nu_j d\lambda = -\int v d\lambda \geq 0 \quad \forall\lambda \in F^+(S_j),$$

where $F^+(S_j)$ denotes the set of all positive measures λ of $F(S_j)$. Using Fubini's theorem we get

$$\int E' * \lambda d\nu_j \geq 0 \quad \forall \lambda \in F^+(S_j). \tag{10}$$

Since $D(S_j)$ is dense in $C(S_j)$, then the set

$$D^+(S_j) := \{E' * \lambda|_{S_j}, \quad \lambda \in F^+(S_j)\},$$

is dense in

$$C^+(S_j) := \{g : g \in C(S_j), \quad g \geq 0\}.$$

Taking (9), (10) into account we obtain

$$\int g d\nu_j \geq 0 \quad \forall g \in C^+(S_j),$$

or $\nu_j \geq O$.

Further, we will show that the set $\{\nu_j\}$, $j = 1, 2, ...$ is weakly compact. Let R be a sufficiently great real number such that the open ball

$$K(0, R) := \{x : x \in R^3, \quad |x| < R\},$$

contains all domains G^j. Integrating both sides of (9) with respect to the measure γ_R defined by

$$(\gamma_R, f) := \frac{1}{4\pi R^2} \int_{\partial K(0,R)} f(x) dS(x), \quad f \in C_0(R^3),$$

and applying Fubini's theorem we arrive at

$$\int E' * \gamma_R d\nu_j = \int -v(x) d\gamma_R(x) \leq \|v\|_{K(0,R)} \int d\gamma_R = 4\pi R^2 \|v\|_{K(0,R)}, \tag{11}$$

where $\|v\|_{K(0,R)} := max\{|v(x)| : x \in \partial K(0, R)\}$.

Calculating the integral of the left-hand side of (11) we obtain the following estimate

$$\int E' * \gamma_R d\nu_j = \int \left(\int_{\partial K(0,R)} \frac{e^{-\lambda|x-y|}}{4\pi|x-y|} d\gamma_R(x)\right) d\nu_j(y)$$
$$\geq h_R \left(\int_{\partial K(0,R)} d\gamma_R\right) \int d\nu_j = h_R \,, \tag{12}$$

where

$$h_R := \min_{x\in\overline{S}, y\in\partial K(0,R)} \frac{e^{-\lambda|x-y|}}{4\pi|x-y|} = \text{ const } > 0\,.$$

Since ν_j is positive, we have

$$\int d\nu_j = \|\nu_j\|\,.$$

From (11), (12) we obtain

$$\|\nu_j\|h_R \leq 4\pi R^2\|v\|_{K(0,R)}\,,$$

and

$$\|\nu_j\| \leq \frac{1}{h_R}4\pi R^2\|v\|_{K(0,R)} = C = \text{ const}\,.$$

Thus we have shown that the set $\{\nu_j\}$, $j = 1, 2, 3, \ldots$ is weakly compact. In view of (2) there is a subsequence $\{\nu_{j_i}\}$, $j_i = 1, 2, \ldots$, converging to a positive measure $v_0 \in C'(\partial G)$ with

$$v(x) = E * \nu_0(x) \quad \forall x \in G_1\,.$$

Hence, the existence of a solution of Theorem 3 is proved.

Now we prove the uniqueness. Suppose there are two positive measures $\nu_1, \nu_2 \in C'(\partial G)^+$ with

$$v(x) = E * \nu_1(x) = E * \nu_2(x) \quad \forall x \in G_1\,. \tag{13}$$

We shall show that $\nu_1 = \nu_2$.

Indeed, as we have shown above, there are two positive measures $\nu_1^j, \nu_2^j \in C'(S_j)^+$ with

$$v(x) = E * \nu_1^j(x) = E * \nu_2^j(x) \quad \forall x \in G_1^j, \quad j = 1, 2\ldots. \tag{14}$$

On the other hand, for each $\lambda \in F^+(S)$ there is $\lambda_j \in C'(S_j)^+$ with

$$E * \lambda_j(x) \quad \forall x \in G_1^j\,. \tag{15}$$

We first show that

$$\int E * \lambda d\nu_i = \lim_{j\to\infty} \int E * \lambda d\nu_i^j, \quad i = 1, 2\,. \tag{16}$$

Using Fubini's th‹ ›rem and taking (14) (15) into account we obtain

$$\begin{aligned}|\int E*\lambda d\nu_i - \int E*\lambda d\nu_i^j| &= |\int E*\lambda d\nu_i - \int E*\lambda_j d\nu_i^j| \\ &= |\int E*\lambda d\nu_i - \int E*\nu_i^j d\lambda_j| \\ &= |\int E*\lambda d\nu_i - \int E*\nu_i d\lambda_j| \\ &= |\int E*\lambda d\nu_i - \int E*\lambda_j d\nu_i| \\ &\leq \int\int |E*\lambda - E*\lambda_j| d\nu_i, \ i = 1, 2\,.\end{aligned}$$

Since $E*\lambda$ and $E*\lambda_j$ are continuous, we obtain $E*\lambda_j \to E*\lambda$ as $j \to \infty$. Thus (16) is proved. From (14) we have

$$\int E*\nu_1^j d\lambda_j = \int E*\nu_2^j d\lambda_j\,.$$

From (15) we obtain

$$\int E*\lambda d\nu_1^j = \int E*\lambda d\nu_2^j \quad \forall\lambda \in F^+(S).$$

Taking (13), (14), and (16) into account we arrive at

$$\int E*\lambda d\nu_1 = \int E*\lambda d\nu_2 \quad \forall\lambda \in F^+(S). \tag{17}$$

As we have shown that $D^+(S)$ is dense in $C^+(S)$ and $D(S)$ is dense in $C(S)$, thus (17) implies $\nu_1 = \nu_2$, which completes the proof of Theorem 3.

Corollary (sweeping-out principle). *For each positive measure ν carried by the closure $\overline{G}$ there exists a uniquely determined positive measure ν' carried by ∂G such that*

$$E*\nu(x) = E*\nu'(x), \ x \in G_1.$$

The proof of this corollary follows directly from the fact that the potential $(E*\nu)|_{G_1}$ satisfies the hypothese of Theorem 3.

4. The outer inverse source problem

We introduce the following potential set

$$H := \{u : u \in \mathcal{D}'(G), \ Pu(x) = 0, \quad x \in G\}.$$

Definition. The inverse source problem relative to an element $v \in H$ consists in studying the following solution set

$$L(v) := \{\nu : \nu \in \mathcal{D}'(R^3, \overline{G}_1),\ E * \nu(x) = v(x), \quad \forall x \in G\}.$$

Similarly to Theorem 1, and taking the classical results into account, we have the following theorem with its corollary.

Theorem 4.

(i) *For each $v \in H \cap \mathcal{D}'(R^3, \overline{G})$ with the existence of the convolution $E * v$ there exists a solution $\nu \in L(v) \cap \mathcal{D}'(R^3, \partial G)$.*

(ii) $\forall v \in H \cap C(\overline{G}),\ \exists \nu \in L(v) \cap C_0^{1\prime}(R^3, \partial G)_1$,

(iii) $\forall v \in H \cap C^1(\overline{G}),\ \exists \nu \in L(v) \cap C'(\partial G)$.

Corollary (Sweeping-out principle). *For each $\mu \in \mathcal{D}'(G_1)$ with compact support there are two distributions $\mu_1 \in C_0^{1\prime}(R^3, \partial G)_1$ and $\mu_2 \in C'(\partial G)$ with*

$$E * \mu(x) = E * \mu_1(x) = E * \mu_2(x), \quad \forall x \in G.$$

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Institute of Mathematics
P.O. Box 631 Bo Ho
Hanoi, Vietnam.

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