Tạp chí Tin học và Điều khiển học T 12, S 1 (1996) (7 - 20)

LATTICE CHARACTER OF THE REFINEMENT STRUCTURE OF HEDGE ALGEBRAS*

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Hedge algebras were introduced in order to model the natural structure of domains of linguistic variables. In N. Cat Ho and H. Van Nam [7], we introduced a refinement structure of hedge algebras (RSHA, for short), and some fundamental properties of this structure were examined. In this paper, the main property of the RSHA is given, and it is also shown that many linguistis values, which contain the disjunction and conjunction, can be expressed in the RSHA.

Keywords: PN-consistent hedge algebras, Refinement structure of hedge algebras, Linguistic - valued logic, Linguistic reasoning.

I. INTRODUCTION

In the traditional approach to human reasoning, vague concepts are expressed by fuzzy sets, and linguistic hedges could be viewed as operators on fuzzy sets. In [9], Zadeh pointed out that the set of linguistic values of linguistic variables can be regarded as a formal language generated by a context-free grammar. These ideas suggested the authors in [5] to consider the sets of such linguistic values as formal algebras with operations to be linguistic hedges, and an axiomatization for the so-called hedge algebras was given. It is shown that the meaning of vague concepts can be expressed by elements in a suitable algebraic structure. Furthermore, the meaning of vague concepts can be also expressed by their relative position in this structure. Notice that the meaning of 'true' and 'false' in the classical logic may be expressed by the relationship between the elements in a two-elements Boolean algebra.

In this approach, every linguistic domain can be interpreted as an algebra $AX = (X, G, H \leq)$, where $(X, \leq)$ is a poset and G is a set of primary generators and H is a set of unary operations representing linguistic hedges under consideration.

The assumptions adopted on H are simple and natural: H can be decomposed into two disjoint subsets H^+ and H^- so that each hedge in H^+ is converse w.r.t. every one in H^- and that $H^+ + I$ and $H^- + I$, where I is the identity, i. e. $Ix = x$ for every x in X, are lattices of finite length. It can be seen that many finite sets of linguistic hedges satisfy these conditions.

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* This research is supported in part by Vietnam National Program for Fundamental Research in Natural Sciences.

In [7] we investigated and gave an axiomatization for the so-called refinement structure of hedge algebras. In order to obtain this structure, we restricted our consideration to the specific hedge algebras called the PN-consistent hedge algebras, we constructed the distributive lattices of hedge operations by the action of operations $\cup$ and $\cap$ on $H^+ + I$ and $H^- + I$, which are modular lattices of finite length and satisfy the condition (C_0). It is shown that any PN-consistent hedge algebra can be refined to a RSHA, and some fundamental properties of the RSHA were examined. In this paper, we continue our investigation on the RSHA. It will be shown that every RSHA with a finite chain of the primary generators is a lattice.

The paper is organized as follows. In Section 2, we shall recall the way of constructing the distributive lattices of hedge operations, and some necessary notions will be introduced. In Section 3, we shall review the RSHA and some properties of this structure will be also reformulated. The main property of the RSHA, which says that every RSHA with a finite chain of the primary generators is a lattice, will be given in Section 4. Finally, some concluding remarks are presented in Section 5.

II. PRELIMINARIES

For the sake of convenience, we will recall some notions introduced in [1]. Let P be a partial ordered set (poset, for short).

Definition 1. *By "a covers b" in a poset P, it is meant that $a > b$, but that $a > x > b$ for no $x \in P$.*

In a poset P of finite length with the least element denoted by $\mathbf{O}$, the height $h(x)$ of an element $x \in P$ is, by definition, the l. u. b. of the lengths of the chains $\mathbf{O} = x_0 < x_1 < \ldots < x_n = x$ between $\mathbf{O}$ and x. If P has a universal upper bound 1 then clearly $h(1) = l(P)$, where $l(P)$ denotes the length of the poset P. Clearly also, $h(x) = 1$ iff x covers $\mathbf{O}$.

Definition 2. *A poset P is said to be graded poset if there exists a function g: $P \to Z$ from P to the chain of all integers (in their natural order) such that:*

G1. $x > y$ implies $g(x) > g(y)$

G2. If x covers y then $g(x) = g(y) + 1$.

It is known that any modular lattice of finite length is graded by its height function $h(x)$. Let L be a modular lattice of finite length, we can define a relation R on L as follows :

$\forall x, y \in L,\ (x,y) \in R$ iff $h(x) = h(y)$.

One easily shows that R is an equivalence relation and then we have the following:

$L = \cup_{i=0}^{l(L)} L_i$, where $L_i = \{x \in L |\ h(x) = i\}$ for $i = 0, ..., l(L)$ are the equivalence classes by the relation R.

We shall need the following condition, which is satisfied by any set of hedges used in application:

(C_0)*Either* $x > y$ *or* $x < y$ *for any* $x \in L_i$ *and* $y \in L_j$ *and* $i \neq j$.

Proposition 1. *Let L be a modular lattice of finite length and it satisfies (C_0). Then the following condition satisfies:*

If there exists an index $i \in \{1, ..., l(L) - 1\}$ such that $|L_i| > 1$ then $|L_{i-1} = |L_{i+1}| = 1$, where $|L_i|, |L_{i-1}|, |L_{i+1}|$ denote the cardinalities of L_i, L_{i-1}, L_{i+1}, respectively. Moreover, $L_{i+1} = \{\vee_{x\in L_i} x\}$ and $L_{i-1} = \{\wedge_{x\in L_i}\}$, where $\wedge$ and $\vee$ are infimum and supremum in L, respectively.

Proof. The proof is straightforward.

For more details on lattice theory we refer the reader to [1]. We proceed now to consider a hedge algebra $AX \doteq (X, G, H, \leq)$, where $(X, \leq)$ is a poset, G is a set of the primary generators and H is a set of unary operations representing linguistic hedges under consideration. It is assumed that H can be decomposed into two disjoint subsetsH^+ and H^- such that $H^+ + I$ and $H^- + I$ are modular lattices of finite length, where I is the identity, i.e. $Ix = x$ for every x in X. By our convention, the identity I will only stand in a prefix of an expression, for instance I...Ih...h'x, and it means that if I occurs explicitly in an expression, then every hedge operation applying to I has no effect, i.e. $hIu = Iu$.

We will denote by N^+ and N^- the lengths of H^+I and $H^- + I$, respectively, and denote by Nat the set of all non-negative integers. Suppose that f^+ and f^- are graded functions of H^+I and H^-+I, respectively, then $f^{+(-)}:\ H^{+(-)}+I \to$ Nat satisfies $f^{+(-)} = 0$ and if h covers k in $H^{+(-)} + I$ then $f^{+(-)}(h) = f^{+(-)}(k) + 1$.

Unless otherwise stated, in the sequel we shall always adopt the assumption that H^+I and $H^- + I$ are modular lattices of finite length and they satisfy (C_0). From now on, V and L stand for the unit-operations in H^+I and $H^- + I$, respectively. With this notation we have $f^+(V) = N^+$ and $f^-(L) = N^-$. Moreover, we have the following representations:

$H^+ + I = \cup_{i=0}^{N^+} H_i^+$, where $H_i^+ = \{h \in H^+ |\ f^+(h) = i\}$ for $i = 0, 1, ..., N^+$,

$H^- + I = \cup_{i=0}^{N^-} H_i^-$, where $H_i^- = \{h \in H^+ |\ f^-(h) = i\}$ for $i = 0, 1, ..., N^-$, We will now construct lattices of hedge operations, which are generated from H^+I and $H^- + I$ by the action of binary operations $\cap$ and $\cup$ as follows:

The first, we see that the following restrictions are reasonably imposed on the two binary operations $\cap$ and $\cup$ on the elements of H^+I:

For any $h_1, h_2, h_3 \in H^+I$ and $H^- + I$

1. $h_1 \cup h_2 = h_2 \cup h_1,\ \ h_1 \cap h_2 = h_2 \cap h_1$.

2. $h_1 \cup (h_2 \cup h_3) = (h_1 \cup h_2) \cup h_3$.

$h_1 \cap (h_2 \cap h_3) = (h_1 \cap h_2) \cap h_3$.

3. $h_1 \cup (h_2 \cap h_3) = (h_1 \cup h_2) \cap (h_1 \cup h_3)$.

$h_1 \cap (h_2 \cup h_3) = (h_1 \cap h_2) \cup (h_1 \cap h_3)$.

4. $h_1 \cup (h_1 \cap h_3) = h_1,\ \ h_1 \cap (h_1 \cup h_2) = h_1$.

If $h \le k$ in $H^+ + I$ for $h, k \in H^+ + I$ then we set $h \cup k = k$ and $h \cap k = h$. If there exists an index $i \in \{1, ..., N^+ - 1\}$ such that $|H_i^+| > 1$, suppose that $H_i^+ = \{h_1^i, ..., h_n^i\}$, by Pro.1 we have $H_{i-1}^+ = \{h^{i-1}\}$ and $H_{i+1}^+ = \{h^{i+1}\}$, where $h^{i-1} = \text{Inf } H_i^+$ and $h^{i+1} = \sup H_i^+$. And the following graphical representation is a sublattice of $H^+ + I$.

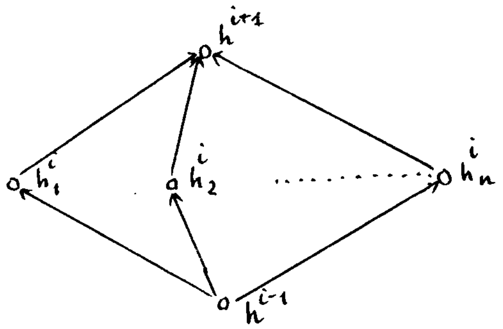

Fig. 1.

Let us denote by LH_i^+ and L_u^+ the sets of elements generated from H_i^+ and $H^+ + I$ by the action of operations $\cap$ and $\cup$, respectively.

Let $LH^+ + I = (L_u^+, H_i^+ + 1, \cap, \cup)$, it is easily seen that $LH^+ + I$ is an algebra with two the binary operations $\cap$ and $\cup$. We can define a relation on $LH^+ + I$ as follows: for any $h, k \in LH^+ + I$, $h \le k$ iff $h \cup k = k$. It is easy to check that $h \cup k = k$ iff $h \cap k = h$ and so $\le$ is ordering relation on $LH^+ + I$. It is known that $LH^+ + I$ is a lattice with $\cap$ and $\cup$ to be supremum and infimum, respectively. Moreover, we have:

$h^{i-1} \cup h = h$ and $h^{i-1} \cap h = h^{i-1}$ for any $h \in LH^+$, and $h^{i+1} \cup h = h^{i+1}$ and $h^{i+1} \cap h = h$ for any $h \in LH^+$.

It is shown that H_i^+ is the free distributive lattice with n generators of H_i^+ and the sublattice of $H^+ + I$ in Fig.1 will be replaced by the following representation in $LH^+ + I$.

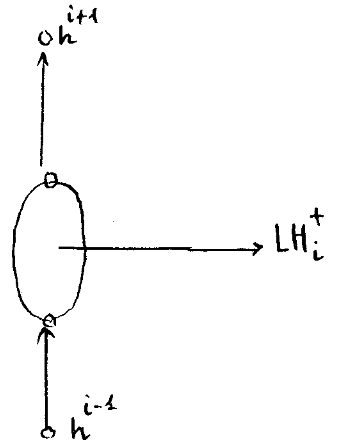

Fig. 2.

By an analogous way, we can construct the lattice, which is generated from $H^- + I$ and we will also write $LH^- + I = (L_u^-, H^- + I, \cap, \cup)$. This causes no confusion because H^+ and H^- are assumed to be disjoint and so are LH^+ and LH^-, where $LH^+ = LH^+ + I \setminus \{I\}$ and $LH^- = LH^- + I \setminus \{I\}$. We obtain the following result.

Theorem 1. *$(LH^+ + I, \cap, \cup, I, V \leq)$ and $(LH^- + I, \cap, \cup, I, V \leq)$ are distributive lattices of finite length.*

Let $I^+ = \{0, 1, ..., N^+\}$ and $I^- = \{0, 1, ..., N^-\}$, and $SI^+ = \{i \in I^+ \setminus |H_i^+| > 1\}$ and $SI = \{i \in I \setminus |H_i^+| > 1\}$.

For any i in $SI^+(SI^-), LH_i^+(LH_i^-)$ is the free distributive lattice with the generators of $H_i^+(H_i^-)$ and is a sublattice of $LH^+ + I(LH_i^-)$.

In the rest of this section, we make some necessary preparations. We first recall the following notion as defined in [5].

Definition 3. *For any $h, k \in LH$, we shall write $hx <\leq kx$ $(hx <\leq Ix)$ if for any h', $k' \in UOS$ and any $m, n \in Nat$, $V^n h'hx \leq V^m k'kx$ $(V^n h'hx \leq Ix)$. If the latter inequalities are always strict then we shall write $hx << kx$ $(hx << Ix)$.*

For any two hedges h, k in LH, if $x \leq hx$ implies $hx \leq khx$ and $hx \leq x$ implies $khx \leq hx$ then k is said to be *positive* w.r.t. h. In the opposite case, k is said to be *negative* w.r.t. h. Recall that we only consider hedge algebras in which $H^+ + I$and $H^- + I$ are modular lattices of finite length and satisfy (C_0). A more important definition is the following.

Definition 4. *A hedge algebra $AX = (X, G, H, \leq)$ is said to be PN-consistent if the unit operation V in $H^+ + I$ is positive(negative) w.r.t. h in H^+ or in H^- for i in SI^+ or in SI^-, respectively, then V is also positive(negative) w.r.t any operation in H_i^+ or in H_i^-, respectively.*

Let $LH = LH^+ \cup LH^-$ and $k(hx) = khx$ for any $h, k \in LH$ and $x \in LH(G)$, where $LH(G)$ denotes the set of all elements generated from G by means of hedges in LH.

Definition 5. *For any $h, k \in LH$ and x in $LH(G)$. If $h \geq k$ in $LH^+ + I$ or in $LH^- + I$ then $hx \geq kx$ if $hx \geq x$ and $hx \leq kx$ if $hx \leq x$.*

If $i \in I^+ \setminus SI^+ (I^- \setminus SI^-)$ then $HI^+(HI^-)$ has only one element and we also write $LH_i^+ = H_i^+ (LH_i^- = H_i^-)$. With this notation we have:

$LH^+ + I = \cup_{i=0}^{N^+} LH_i^+$ and $LH^- + I = \cup_{i=0}^{N^-} LH_i^-$. Let us denote by UOS the set of two elements V and L, which are unit-operations in $LH^+ + I$ and in $LH^- + I$, respectively.

As usual, the notation $x \leq y$ means that $x > y$ or x and y are incomparable.

Remark. From the construction of the lattices $LH^+ + I$ and $LH^- + I$ and these notations, it is easly seen that the lattice $LH^+ + I$ (or $LH^- + I$) also satisfy condition (C_0) with replacing L_i and L_j in (C_0) by LH_i^+ and LH_j^+ (or LH_i^- and LH_j^-), respectively.

III. REFINEMENT STRUCTURE OF HEDGE ALGEBRAS

This section reviews refinement structure of hedge algebras. We start with introducing an axiomatization for refinement structure of hedge algebras. Let us make necessary notations.

For every subset Y of $LH(G)$, $LH(Y)$ denotes the set of all elements generated from Y by means of hedges in LH. If Y consists of a single element, say x, then we shall simply write $LH(x)$. For any two hedges h, k in LH, if $x \leq hx$ iff $kx \leq x$ for every x in $LH(G)$ then h and k are said to be converse, or h is converse to k and vice versa. If $x \leq hx$ iff $x \leq kx$ for every x in $LH(G)$ then h and k are said to be *compatible*.

Let us denote by LH^* the set of all strings of hedges in LH. For any $\delta, \delta' \in LH^*$, we shall write $\delta \leq \delta'$ if for every x in $LH(G)$, $x \leq \delta x$ or $\leq \delta' x$ implies $x \leq \delta x \leq \delta' x$ and $\delta x \leq x$ or $\delta' x \leq x$ implies $\delta' x \leq \delta x \leq x$.

We now recall the axiomatization for refinement structure of hedge algebras.

Definition 6. *An algebraic structure $AX = (X, G, LH, \leq)$ is said to be a refinement structure of hedge algebras (or, briefly, RSHA) if $(H(G), G, H, \leq)$ is a PN-consistent hedge algebra and the following conditions hold:*

(C1) Every operation in LH^+ is a converse operation of the operations in LH^-.

(C2) The unit operation V in LH^+ is either positive or negative w.r.t any operation.

(C3) If u and v are independent , i.e. $u \notin LH(v)$ and $v \notin LH(u)$, then $x \notin LH(v)$ for any $x \notin LH(u)$. Especially, if $a, b \in G$ and $a < b$ then $LH(a) < LH(b)$.

(C4) For $x \neq hx$, $x \notin LH(hx)$. For any $h \neq k$ and $hx \leq kx$:

(i) If $h, k \in LH_i^+(LH_i^-)$ for $i \in SI^+(SI^-)$ and $hx \neq kx$ then $\delta hx < \delta kx$ for any $\delta \in LH^*$. Furthermore, for any $y \in LH(kx)$ such that $y \geq \delta kx$, δhx and y are incomparable, and for any $z \in LH(hx)$ such that $z \leq \delta hx$, δkx and z are incomparable.

(ii) $h'hx \leq k'kx$ for any $h', k' \in UOS$, otherwise.

Furthermore, if $hx \neq kx$ and $h \in LH_i^+(LH_i^-)$ for $i \in I^+(I^-)$ then h_1x and kx are independent for any $h_1 \in LH_i^+(LH_i^-$ and $h_1 \neq k$.

(C5) If $u \in LH(x)$ and for each $i \in I^+(I^-)$ such that $u \notin LH(hx)$ for any $h \in LH_i^+(LH_i^-)$ and $u \geq v(u \leq v)$ for $v \in LH(hx)$ then $u \geq h'v(u \leq h'v)$ for each $h' \in UOS$.

Definition 7. *Let x and u be two elements in a RSHA $AX = (X, G, LH, \leq)$. The expression $h_n...h_1u$ is said to be a canonical representation of x w.r.t. u in AX if:*

(i) $x = h_n...h_1u$;

(ii) $h_i...h_1u \neq h_{i-1}...h_1u$ for every $i \leq n$.

All results in the rest of this section (with more details and proofs) can be found in N. C. Ho and H.V. Nam [7].

Theorem 2. *Let $AX = (X, G, LH, \leq)$ be a RSHA. Then the following statements hold:*

(0) If $hx <\leq kx$ then $hx \leq kx$.

(i) The operations in LH^+ or the ones in LH^- are compatible, i.e. for any $h, k \in LH^+(LH^-)$, for any $x \in X, x \leq hx$ iff $x \leq kx$.

(ii) If $x \in X$ is a fixed point of an operation h in LH, i.e $hx = x$, then it is also a fixed point of the orthers.

(iii) If $x = h_n...h_1u$ then there exists an index i such that the suffix $h_i...h_1u$ of x is a canonical representation of x w.r.t. u and $h_jx = x$ for all $j > i$.

(iv) For any $h, k \in LH$, if $x \leq hx(x \geq hx)$ then $Ix <\leq hx$ $(Ix \geq> hx)$ and if $hx \leq kx$ and $h \neq k$ and there is no i in SI^+ or in SI^- such that both h and k together belong to LH_i^+ or LH_i^-, respectively, then $hx <\leq kx$.

(v) If $h \neq k$ and $hx = kx$ then hx is a fixed point.

Theorem 3. *For every operation $h \in LH$, there exist two unit-operations h^- and h^+ such that h^- is negative and h^+ is positive w.r.t. h and for any $h_1, ..., h_n \in LH$, $V^n h^- h \leq h_n...h_1 h \leq V^n h^+ h$.*

The following proposition, which has been proved in [7], shows that we can deduce om the fact that hx is a fixed point, that kx is also a fixed point and vice versa, if both h and k together belong to $LH_i^+(LH_i^-)$ for $i \in SI^+(SI^-)$.

Proposition 2. *For any $x \in X$ and $i \in SI^+(SI^-)$. If there exists a hedge $h \in LH_i^+(LH_i^-)$ such that hx is a fixed point then so is kx for any $k \in LH_i^+(LH_i^-)$.*

Recall that the RSHA is constructed from a given PN-consistent hedge algebra. Naturally, one may ask whether the PN-consistent property of the unit-operation V in $LH^+ + I$ still holds if we replace H_i^+ and H_i^- by LH_i^+ and LH_i^-, respectively. The following proposition answers this question.

Proposition 3. *If the unit operation V in $LH^+ + I$ is positive(negative) w.r.t. h in H_i^+ or in H_i^- for i in SI^+ or in SI^-, respectively. Then V is also positive(negative) w.r.t. any operation in LH_i^+ or in LH_i^-, respectively.*

Proof. The proof for this proposition we refer the reader to [7].

Proposition 4. *For any $h, k \in LH_i^+(LH_i^-)$, here $i \in SI^+(SI^-)$. We have the following assertions:*

(i) $\delta hx \geq x$ $(\delta hx \leq x)$ iff $\delta kx \geq x$ $(\delta kx \leq x$ $)$ for any $\delta \in LH^*$ and $x \in X$.

(ii) If $hx \neq kx$ then δhx and $\delta' hx$ are incomparable iff δkx and $\delta' kx$ are incomparable for any $\delta, \delta' \in LH^*$ and $x \in X$.

(iii) $\delta h \geq \delta' h$ iff $\delta k \geq \delta' k$ for any δ, $\delta' \in LH^*$.

IV. MAIN THEOREM

In this section, we shall formulate a main property of RSHA, which say that a RSHA is a lattice if the set of primary operators is a finite chain. Before stating the main theorem, the following theorem, which will give us a characterization to determine the relative position of elements in a RSHA, is necessary.

Theorem 4 *Let $x = h_n...h_1u$ and $y = k_m...k_1u$ be two arbitrary canonical representations of x and y w.r.t. u, respectively. Then there exists an index $j \leq \min\{m,n\} + 1$ such that $h_i = k_i$ for all $i < j$ and*

(1) $x < y$ iff either $h_jx_j < k_jx_j$ and $\delta k_jx_j \leq \delta' k_ix_j$, if there exists i_0 in SI^+ or in SI^- such that both h_j and k_j together belong to $LH_{i_0}^+$ or $LH_{i_0}^-$, respectively, or $h_jx_j < k_jx_j$ for otherwise, where $x_j = h_{j-1}...h_1u$, $\delta = h_n...h_{j+1}$, $\delta' = k_m...k_{j+1}$.

(2) $x = y$ iff $m = n = j$ and $h_j x_j = k_j x_j$.

(3) x and y are incomparable iff there exists i_0 in SI^+ or in SI^- such that both h_j and k_j together belong to $LH^+_{i_0}$ or $LH^-_{i_0}$, respectively, and one of the fowllowing conditions holds:

(3') $h_j x_j$ and $k_j x_j$ are incomparable

(3") $h_j x_j < k_j x_j$ and $\delta k_j x_j \leq \delta' k_j x_j$,

(3"') $h_j x_j > k_j x_j$ and $\delta k_j x_j \geq \delta' k_j x_j$,

Proof. We refer the reader to [7] for the proof of this theorem.

Corollary 1. If x is not a fixed point and u is an arbitrary element in X, then the canonical representation of x w.r.t. u, if it exists, is unique, i.e. if $h_n...h_1 u$ and $k_m...k_1 u$ are two canonical representations of x w.r.t. u then $m = n$ and $h_i = k_i$ for all $i \leq n$.

We can now formulate our main result.

Theorem 5 (Main Theorem). *Let $AX = (X, G, LH, \leq)$ be a RSHA. If G is a finite chain then AX is a lattice. Moreover, for any two incomparable elements x and y in X, then there exist two hedge operations h and k in LH and an element w in $LH(a)$, where $a \in G$, such that both h and k together belong to $LH^+_{i_0}$ or $LH^-_{i_0}$ for an index i_0 in SI^+ or in SI^-, respectively, and $x = \delta h w, y = \delta' k w$, where $\delta, \delta' \in LH*$, and*

$$x \vee y = \begin{cases} \delta(h \cup k)w \vee \delta'(h \cup k)w, & \text{if } hw > w \text{ and } hw \text{ and } kw \text{ are incomparable} \\ \delta(h \cap k)w \vee \delta'(h \cap k)w, & \text{if } hw < w \text{ and } hw \text{ and } kw \text{ are incomparable} \\ \delta hw \vee \delta' hw, & \text{if } hw > kw \\ \delta hw \vee \delta' hw, & \text{if } hw < kw. \end{cases}$$

$$x \vee y = \begin{cases} \delta(h \cap k)w \wedge \delta'(h \cap k)w, & \text{if } hw > w \text{ and } hw \text{ and } kw \text{ are incomparable} \\ \delta(h \cup k)w \wedge \delta'(h \cup k)w, & \text{if } hw < w \text{ and } hw \text{ and } kw \text{ are incomparable} \\ \delta hw \wedge \delta' hw, & \text{if } hw > kw \\ \delta hw \wedge \delta' hw, & \text{if } hw < kw. \end{cases}$$

Where $\vee$ and $\wedge$ stand for supremum and infimum, respectively.

Proof. Since G is a finite chain and by (C3), it follows that if x and y are incomparable in X then there exists an element a in G such that both x and y together belong to $LH(a)$. Thus, there exist two canonical representations of x and y w.r.t. a as follows:

$x = h^x_n...h^x_1 a$, $h^x_i \in LH$, here $i = 1, ..., n$

and $y = h^y_m...h^y_1 a$, $h^y_i \in LH$, here $i = 1, ..., m$.

By Theorem 4, there exists an index $j \leq min\ \{m, n\} + 1$ such that $h^x_i = h^y_j$ for any $i < j$ and set $h^x_i = h^y_j = h_i$. Moreover, by also Theorem 4, there exists an index i_0 in SI^+ or in SI^- such that both h^x_i and h^y_j together belong to $LH^+_{i_0}$ or $LH^-_{i_0}$, respectively. Let $\delta = h^x_n...h^x_{j+1}, \delta' = h^y_m...h^y_{j+1}$, with this notations we have: $x = \delta hw$ and $y = \delta' kw$ here $w = h_{j-1}...h_1 a$. By Theorem 4, we have the following cases:

(1) hw and kw are incomparable.

(2) $hw > kw$ and $\delta' hw \leq \delta hw$.

(3) $hw < kw$ and $\delta' hw \geq \delta hw$.

We shall prove the theorem for the supremum. The proof for the infimum is by duality.

Case (1). Assume that hw and kw are incomparable. If $hw > w$, by Pro. 2 it follows that $(h \cup k)w > \{hw, kw\}$. By (C4) we have $\delta(h \cup k)w > \delta hw$ and $\delta'(h \cup k)w > \delta' hw$. Thus, $\delta(h \cup k)w \vee \delta'(h \cup k)w > \{x, y\}$. For any $t \in LH(a)$ such that $t > \{x, y\}$, we have to prove the following assertion: $t \geq \delta(h \cup k)w \vee \delta'(h \cup k)w$.

We have possible cases of t as follows:

(1') $t \notin LH(w)$

(1") $t \in LH(w)$ and $t \notin LH(h'w)$ for any $h' \in LH^+_{i_0}(LH^-_{i_0})$.

(1"') $t \in LH(h_0 w)$ for some $h_0 \in LH^+_{i_0}(LH^-_{i_0})$.

Suppose that $t = k_p ... k_1 a$ is the canonical representation of t w.r.t. a. Clearly, $w = h_{j-1} ... h_1 a$ is the canonical representation of w w.r.t. a.

First, we shall prove the assertion for the case $t \notin LH(w)$, by Theorem 4, it follows that there exists an index $j' < j$ such that $h_i = k_i$ for any $i < j'$ and $k_{j'} h > h_{j'} u$, where $u = h_{j'-1} ... h_1 a$. If there is no index i_1 in SI^+ or in SI^- such that both $h_{j'}$ and $k_{j'}$ together belong to $LH^+_{i_j}$ or $LH^-_{i_j}$, respectively, then by Theorem 2 (iv) and (C3) and (C4), it implies that $k_{j'} u >> h_{j'} u$, i.e. $V^q k'_{j'} k_{j'} u > V^{q'} h'_{j'} h_{j'} u$ for any $k'_{j'}, h'_{j'} \in$ UOS and q, $q' \in$ Nat. By Theorem 3, there exist $h', h'', h''' \in$ UOS such that:

$\delta(h \cup k) h_{j-1} ... h_{j'} u \leq V^{n-j'-1} h' h_{j'} u$, and

$\delta'(h \cup k) h_{j-1} ... h_{j'} u \leq V^{m-j'-1} h'' h_{j'} u$, and

$k_p ... k_{j'} u \geq V^{p-j'-1} h''' k_{j'} u$.

Thus, we have $t > \{\delta(h \cup k)w, \delta'(h \cup k)w\}$ which shows that $t \geq \delta(h \cup k)w \vee \delta'(h \cup k)w$. If there exists an index $i_1 \in SI^+(SI^-)$ such that $h_{j'}, k_{j'} \in LH^+_{i_1}(LH^-_{i_1})$, by (C4) we have

$\delta h h_{j-1} ... h_{j'+1} k_{j'} u > \delta h h_{j-1} ... h_{j'+1} h_{j'} u$, and

$\delta' k h_{j-1} ... h_{j'+1} k_{j'} u > \delta' k h h_{j-1} ... h_{j'+1} h_{j'} u$, and

$\delta(h \cup k) h_{j-1} ... h_{j'+1} k_{j'} u > \delta(h \cup k) h_{j-1} ... h_{j'+1} h_{j'} u$, and

$\delta'(h \cup k) h_{j-1} ... h_{j'+1} k_{j'} u > \delta'(h \cup k) h_{j-1} ... h_{j'+1} h_{j'} u$.

Since $t \in LH(k_{j'} u)$ and $t > \{x, y\}$, it implies that $t \geq \delta h h_{j-1} ... h_{j'+1} k_{j'} u, \delta' k h_{j-1} ... h_{j'+1} k_{j'} u\}$. By Pro.3(ii) it follows that $\delta h h_{j-1} ... h_{j'+1} k_{j'} u$ and $\delta' k h_{j-1} ... h_{j'+1} k_{j'} u$ are incomparable, and hence $t > \{\delta h h_{j-1} ... h_{j'+1} k_{j'} u, \delta' k h_{j-1} ... h_{j'+1} k_{j'} u\}$ (*).

We now consider $h_{j'+1}$ and $k_{j'+1}$. If $h_{j'+1} \neq k_{j'+1}$, we have $h_{j'+1}k_{j'}u \neq k_{j'+1}k_{j'}u$. Moreover, since (*) and by Theorem 4, we have $k_{j'+1}k_{j'}u > h_{j'+1}k_{j'}u$. If there is no index $i_2 \in SI^+(SI^-)$ such that $h_{j'+1}k_{j'+1} \in LH^+_{i_2}(LH^-_{i_2})$, then by Theorem 2(iv) and (C3) and (C4), it implies that $k_{j'+1}k_{j'+1}u >> h_{j'+1}k_{j'}u$. By also an analogous argument as in the case of $k_{j'}$ and $h_{j'}$ we obtain: $t > \{\delta(h \cup k)h_{j-1}...h_{j'+1}k_{j'}u, \delta'(h \cup k)h_{j-1}...h_{j'+1}k_{j'}u\}$. Moreover, by (C4) we have:

$$\delta(h \cup k)h_{j-1}...h_{j'+1}k_{j'}u > \delta(h \cup k)h_{j-1}...h_{j'+1}k_{j'}u\} \text{ and}$$
$$\delta'(h \cup k)h_{j-1}...h_{j'+1}h_{j'}u > \delta'(h \cup k)h_{j-1}...h_{j'+1}h_{j'}u\}.$$

Thus, we have: $t > \{\delta(h \cup k)w, \delta'(h \cup k)w\}$, it follows that $t > \{\delta(h \cup k)w \vee \delta'(h \cup k)w\}$. If there exists an index $i_2 \subset SI^+(SI^-)$ such that $h_{j'+1}k_{j'+1} \in LH^+_{i_2}(LH^-_{i_2})$, since (*) and by Theorem 4, it follows that $t \geq \delta hh_{j-1}...h_{j'+2}k_{j'+1}k_{j'}u, \delta' kh_{j-1}...h_{j'+2}k_{j'+1}k_{j'}u\}$, since $\delta hh_{j-1}...h_{j'+1}k_{j'}u$ and $\delta' kh_{j-1}...h_{j'+1}k_{j'}u\}$ are incomparable and by Pro.3 (ii) it implies that $\delta hh_{j-1}...h_{j'+2}k_{j'+1}k_{j'}u$ and $\delta' kh_{j-1}...h_{j'+2}k_{j'+1}k_{j'}u\}$ are also incomparable. It follows that $t \geq \{\delta hh_{j-1}...h_{j'+2}k_{j'+1}k_{j'}u, \delta' k$
$h_{j-1}...h_{j'+2}k_{j'+1}k_{j'}u\}$ (**). If $h_{j'+1} = k_{j'+1}$, then we consider $h_{j'+2}$ and $k_{j'+2}$. By also an analogous argument, for the case $h_{j'+2} \neq k_{j'+2}$, we can show that if there is no index $i_3 \in SI^+(SI^-)$ such that $h_{j'+2}, k_{j'+2} \in LH^+_{i_3}(LH^-_{i_3})$, then we obtain $t \geq \delta(h \cup k)w \vee \delta'(h \cup k)w$. Conversely, if there exists an index $i_3 \in SI^+(SI^-)$ such that $h_{j'+2}k_{j'2} \in LH^+_{i_3}(LH^-_{i_3})$ then we also obtain (**). For the case $h_{j'+2} = k_{j'+2}$, we consider $h_{j'+3}$, $k_{j'+3}$ and repeating this argument. From (**), we shall consider cases of $h_{j'+2}$ and $k_{j'+2}$ if $h_{j'+1} \neq k_{j'+1}$ and consider cases of $h_{j'+3}$ and $k_{j'+3}$ if $h_{j'+1} = k_{j'+1}$ and repeating the argument above.

From the cases proved above, we can see that if there exists an index $i \in \{j', ..., j\}$ and there is no an index $i' \in SI^=(SI^-)$ such that $h_i, k_i \in LH^+_{i'}(LH^-_{i'})$, then we obtain $t \geq \delta(h \cup k)w \vee \delta'(h \cup k)w$. Thus, if $p < j - 1$, then the assertion is proved, since we have $k_{p+1} = I \neq h_{p+1}$ which is the case proved above. It remains to prove the assertion for the case $p \geq j - 1$ and for any $i \in \{j', ..., j\}$ then either h_i and k_i are identical or there exists an index $i' \in SI^+(SI^-)$ such that $h_i, k_i \in LH^+_{i'}(LH^-_{i'})$.

Let $w' = k_{j-1}...k_{j'}u$, we have $t \in LH(w')$ and $|w'| = |w|$, where $|w'|$ and $|w|$ denote the lengths of the canonical representations of w' and w w.r.t. u, respectively. From the condition on h_i and k_i for $i \in \{j', ..., j\}$ and the proof above, we have $k_{j'+n}...k_{j'}u > h_{j'+n}k_{j'+n-1}...k_{j'}u$ for any $n = 0, ..., j - j' - 1$. In addition, we have:

$$t = k_p...k_j w' \geq \{\delta hw', \delta' kw'\}, \text{ if } p > j - 1,$$
$$t = w' \geq \{\delta hw', \delta' kw'\}, \text{ if } p = j - 1,$$

and $\delta hw' > \delta hw$, $\delta' kw' > \delta' kw$, $\delta(h \cup k)w' > \delta(h \cup k)w$, $\delta'(h \cup k)w' > \delta'(h \cup k)w$.

If $p = j - 1$, we have $t = w'$, by Corollary 1(ii) in [5] it implies that $w' \geq \{hw', kw'\}$, since $h \cup k \in LH^+_{i_0}(LH^-_{i_0})$ it follows that h and $h \cup k$ are compatible, hence $w' \geq (h \cup k)w'$. By also Corollary 1(ii) in [5], we have $w' \geq \{\delta(h \cup k)w', \delta'(h \cup k)w'\}$, which implies that $t > \delta(h \cup k)w', \delta'(h \cup k)w'\}$. In other words, we obtain the assertion: $t \geq \delta(h \cup k)w' \vee \delta'(h \cup k)w'$.

If $p > j - 1$, if w' is a fixed point then we have $t = w'$ and it follows by above that $w' > \{\delta(h \cup k)w', \delta'(h \cup k)w'\}$, which shows that $t = \{\delta(h \cup k)w' \vee \delta'(h \cup k)w'\}$. If w' is not a fixed point, the possible cases of t is as follows:

(i) $t \notin LH(h'w')$ for any $h' \in LH^+_{i_0}(LH^-_{i_0})$.

(ii) $t \in LH(h'w')$ for some $h' \in LH^+_{i_0}(LH^-_{i_0})$.

If (i) holds, using the argument at the begining of this proof, one easily verifies that $t > \{\delta(h \cup k)w', \delta'(h \cup k)w'\}$. Hence, it implies that $t > \{\delta(h \cup k)w' \vee \delta'(h \cup k)w'\}$, which shows that $t \geq \{\delta(h \cup k)w' \vee \delta'(h \cup k)w'\}$. If (ii) holds, it follows that $k_j = h'$. We can easily seen that $\delta hw'$ and $\delta' kw'$ are incomparable and we have $t > \{\delta hw', \delta' kw'\}$. It implies by Theorem 4 that $h'w' > \{hw', kw'\}$, since $h, k, h', h \cup k \in LH^+_{i_0}(LH^-_{i_0})$, so $h'w' \geq (h \cup k)w'$.

If $h'w' = (h \cup k)w'$, $h' \neq (h \cup k)$ then it follows by (C4) and Pro.2 that $h'w' = hw' = kw' = (h \cup k)w'$. Hence $h' = (h \cup k)$, it follows by Theorem 4 that

$$t = k_p ... k_{j+1}(h \cup k)w'\{\delta'(h \cup k)w', \delta(h \cup k)w'\}.$$

Thus, it implies that $t > \{\delta(h \cup k)w', \delta'(h \cup k)w'\}$, since if one of the opposite cases holds then it leads to a contradiction by Pro.3 and Theorem 4. It implies that $t > \{\delta(h \cup k)w', \delta'(h \cup k)w'\}$, which shows that $t \geq \{\delta(h \cup k)w' \vee \delta'(h \cup k)w'\}$. As the cases considered above, we have proved the assertion for the case (1'). The remaining cases (1") and (1"') will be proved by an analogous argument as in the cases (i) and (ii).

The proof for the case $hw < w$ is obtained by duality. Consequently, the proof for Case (1) is complete.

Case (2). Assume that $hw > kw$ and $\delta' hw \leq \delta hw$. By (C4) we have $\delta' hw > \delta' kw$, hence it follows that $\delta hw \vee \delta' hw \geq x \vee y$. For any $t \in LH(a)$ such that $t > \{x, y\}$, we have to prove that $t \geq \delta hw \vee \delta' hw$. We have also the analogous cases of t as in Case (1), and then the proof for this case is obtained by the same way.

Case (3). The proof for this case is similar to the proof for Case (2). Consequently, the proof of the theorem is complete.

As a consequence of Theeorem 5, we have the foolowing

Theorem 6. *Let $AX = (X, G, LH, \leq)$ be a RSHA and G is finite chain. For any $x \in X$, for any two operations h and k such that h and k are compatible, the following assertions are hold:*

(i) If $Vx \geq x$ then

$$(h \cup k) = \begin{cases} hx \vee kx, & \text{if } h, k \in LH^{+} + I. \\ hx \wedge kx, & \text{if } h, k \in LH^{-} + I. \end{cases}$$
$$(h \cap k) = \begin{cases} hx \wedge kx, & \text{if } h, k \in LH^{+} + I. \\ hx \vee kx, & \text{if } h, k \in LH^{-} + I. \end{cases}$$

(ii) If $Vx \leq x$ then

$$(h \cup k) = \begin{cases} hx \wedge kx, & \text{if } h, k \in LH^{+} + I. \\ hx \vee kx, & \text{if } h, k \in LH^{-} + I. \end{cases}$$
$$(h \cap k) = \begin{cases} hx \vee kx, & \text{if } h, k \in LH^{+} + I. \\ hx \wedge kx, & \text{if } h, k \in LH^{-} + I. \end{cases}$$

V. CONCLUSION

In this paper we have investigated the refinement structure of hedge algebras and examined the main property of these structure, which says that every RSHA with a finite chain of the primary generators is a lattice. Notice that the assumption, which says that the set of the primary generators is a finite chain, is not stringent, since the primary generators of many languistic variables constitute linearly ordered sets. Consequently, the RSHA have a good algebrical structure, and then they can also be used as logical basis for some kind of Linguistic-valued logic and Linguistic reasoning.

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Nhận ngày 3, 12 1995