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ON THE STOPAGE RULE IN SOLUTION FOR MONOTONE ILL-POSED PROBLEMS

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Abstract. The purpose of this note is to present an iterative method for solving a regularized equation for nonlinear monotone ill-posed problems in Banach space and to study its stopage rule so that iteration sequence converges to a solution of initial problem, as the noisy data of the right-hand side converges to its exact.

1. INTRODUCTION

Let X be a real reflexive Banach space and X^* be its dual space. For the sake of simplicity norms of X and X^* will be denoted by one symbol $\|\cdot\|$. Let A be a continuous and monotone operator with domain of definition $D(A) = X$ and range $R(A) \subseteq X^*$. Let f_0 be an element of $R(A)$.

Consider the nonlinear ill-posed problem.

$$A(x) = f_0 \,. \tag{1.1}$$

By ill-posedness we mean that solution of (1.1) do not depend continuously on the data f_0. To solve (1.1) we can use variational method of Tikhonov regularization that consists of minimizing the functional

$$\|A(x) - f_\delta\|^2 + \alpha\Omega(x), \text{ over } D(A)\,, \tag{1.2}$$

where $\Omega(x)$ is a some functional that plays the role of regularization, $\alpha > 0$ is a small parameter and the noisy data f_δ satisfies the condition

$$\|f_0 - f_\delta\| \leq \delta \to 0\,.$$

In [1] and [2] they showed another version of Tikhonov regularization, that is the use of regularized equation

$$A(x) + \alpha U(x) = f_\delta \tag{1.3}$$

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instead of (1.2), where U is a uniformly monotone operator or dual mapping of X. The regularized equation (1.3) is difficultly solved by iteration methods, because iterative parameters must satisfy very complex condition (see [3]). To overcome this difficulty [4] we showed another approach of regularization for (1.1). That is the use of linear and strongly monotone operator B instead of U in (1.3), i.e. the use of equation

$$A(x) + \alpha Bx = f_\delta, \tag{1.4}$$

with $\overline{D}(B) = X$, $S_0 \subset D(B)$, S_0 denotes the set of solutions of (1.1), B is linear and

$$|x|^2 := \langle Bx, x\rangle \geq m_B\|x\|^2, \ x \in D(B), \ m_B > 0. \tag{1.5}$$

Without loss of generality, assume that $m_B = 1$.

In [4], we presend an iteration method for solving (1.4) with variable parameters of iteration. In this note, we consider another one for (1.4) and the stopage rule for it, when $\delta \to 0$ and α is shosen such that $\delta/\alpha \to 0$. Our methods is a generation of [5] and [6] in Hilbert spaces for a Banach spaces.

Below the symbols $\to$ and $\rightharpoonup$ denote the strong and the weak convergence for any sequence, respectively.

2. MAIN RESULTS

As in [5], we require an addtional condition on A:
For each $N > 0$ $\exists C_B(N) > 0$ such that

$$\langle A(x) - A(y), w\rangle \leq C_B(N)|x-y|\,|w|, \ x, y \in D(B), \quad |x|, |y| \leq N, \ \forall w \in X. \tag{21}$$

Note that this condition was considered in [5] under $D(B) = X = H$, is a Hilbert space, and B is bounded.

For finding x_α^δ, the regularized solution of (1.4), we study the iteration method

$$x^{n+1} = x^n - \rho B^{-1}(A_\alpha(x^n) - f_\delta), \ A_\alpha = A + \alpha B, \tag{2.2}$$

$\alpha > 0$ and f_δ are fixed $(x^0 \in D(B))$.

Theorem 2.1. *Let* $\langle Bx, y\rangle = \langle x, By\rangle$, $x, y \in D(B)$ *and* $0 < \rho < \frac{2\alpha}{(\alpha + C_B(N_0))^2}$, $N_0 : \|x_\alpha^\delta\| \leq N_0/2$. *Then*

$$x^n \to x_\alpha^\delta, \quad \text{as } n \to +\infty.$$

Proof. From (2.2) it implies that

$$\begin{aligned}\lambda_{n+1}^2 &:= \langle B(x^{n+1}-x_\alpha^\delta),\, x^{n+1}-x_\alpha^\delta\rangle\\ &= \langle B(x^{n+1}-x^n),\, x^{n+1}-x^n\rangle + \lambda_n^2 + 2\langle B(x^{n+1}-x^n),\, x^n - x_\alpha^\delta\rangle\\ &= \lambda_n^2 - 2\rho\langle A_\alpha(x^n) - A_\alpha(x_\alpha^\delta),\, x^n - x_\alpha^\delta\rangle\\ &\quad + \rho^2\langle A_\alpha(x^n) - A_\alpha(x_\alpha^\delta),\, B^{-1}(A_\alpha(x^n) - A_\alpha(x_\alpha^\delta))\rangle\,.\end{aligned}$$

Since

$$\begin{aligned}&\langle A_\alpha(x^n) - A_\alpha(x_\alpha^\delta),\, x^n - x_\alpha^\delta\rangle \ge \alpha\langle B(x^n - x_\alpha^\delta),\, x^n - x_\alpha^\delta\rangle = \alpha\lambda_n^2\,,\\ &\langle A_\alpha(x^n) - A_\alpha(x_\alpha^\delta),\, B^{-1}(A_\alpha(x^n) - A_\alpha(x_\alpha^\delta))\rangle =\\ &= \langle A_\alpha(x^n) - A(x_\alpha^\delta) + \alpha B(x^n - x_\alpha^\delta),\, B^{-1}(A(x^n) - A(x_\alpha^\delta)) + \alpha(x^n - x_\alpha^\delta)\rangle =\\ &= \alpha^2\lambda_n^2 + 2\alpha\langle A(x^n) - A(x_\alpha^\delta),\, x^n - x_\alpha^\delta\rangle\\ &\quad + \langle A(x^n) - A(x_\alpha^\delta),\, B^{-1}(A(x^n) - A(x_\alpha^\delta))\rangle\,,\end{aligned}$$

we have

$$\begin{aligned}&\lambda_{n+1}^2 \le \lambda_n^2 - 2\rho\alpha\lambda_n^2 + \rho^2\times\\ &\left[\alpha^2\lambda_n^2 + 2\alpha\langle A(x^n) - A(x_\alpha^\delta),\, x^n - x_\alpha^\delta\rangle + \langle A(x^n) - A(x_\alpha^\delta),\, B^{-1}(A(x^n) - A(x_\alpha^\delta))\rangle\right].\end{aligned}$$

Let N_0 be a number so that $|x_\alpha^\delta| \le N_0/2$ and $C_0 = C_B(N_0)$. We suppose that the recurrent hypothese $|x^n - x_\alpha^\delta| \le N_0/2$, then $|x^n| \le N_0$ and

$$\langle A(x^n) - A(x_\alpha^\delta),\, B^{-1}(A(x^n) - A(x_\alpha^\delta))\rangle \le C_0^2\lambda_n^2\,.$$

We have

$$\begin{aligned}\lambda_{n+1}^2 &\le \lambda_n^2 - 2\rho\alpha\lambda_n^2 + \rho^2[\alpha^2\lambda_n^2 + 2C_0\alpha\lambda_n^2 + C_0\lambda_n^2]\\ &= [1 - 2\rho\alpha + \rho^2(\alpha^2 + 2\alpha C_0 + C_0^2)]\lambda_n^2\,.\end{aligned}$$

If we choose $0 < \rho < 2\alpha/(\alpha + C_0)^2$, then $\theta_S = 1 - 2\rho\alpha + \rho^2(\alpha + C_0)^2 < 1$. Therefore, the hypothese is verifies. Consequently, $\lambda_n \to 0$. Hence, the convergence of x^n to x_α^δ follows from (1.5) and the definition of λ_n.

Remark. In much cases, we can choose the unbounded operator B such that $\|x\|, \|y\| \le N \to |x|, |y| \le N$ in condotion (2.1). In fact, for example $Bx(t) = -d^2x(t)/dt^2 + c_0x(t)$, $c_0 > 0$, where $D(B)$ is the closure in the norm W_q^2, $1 < q < 2$ of all functions from $C^2[0,\, 1]$ satisfying the condition $u(0) = u(1) = 0$. Then $B^{-1}v(t) = \int_0^1 g(t,\, s)\, v(s)\, ds$ with

$$g(t,\, s) = \begin{cases} u_1(t)\, u_2(s), & t \le s\,,\\ u_2(t)\, u_1(s), & t \ge s\,,\end{cases}$$

where u_1, u_2 are the nontrivial solutions of $Bu = 0$ such that $u(0) = u(1) = 0$. The derivatives are understood in generalized. Then, $|x|^2 = \langle Bx, x\rangle = \int_0^1 c_0\, x(t)^2\, dt = c_0\|x\|_{L_2[0,1]} \geq c_0\, \tilde{c}_p\|x\|_{L_p[0,1]}$, $2 < p < +\infty$ with some $\tilde{c}_p > 0$, since in these cases $L_p[0,1]$ is conctinuously embeded in $L_2[0,1]$.

For each $\delta > 0$, the value $\alpha = \overline{\alpha} = \overline{\alpha}(\delta)$ is chosen that $\delta/\overline{\alpha}(\delta) \to 0$, then $x^{\delta}_{\overline{\alpha}(\delta)} \to x_0$, the solution of (1.1), as $\delta \to 0$. In order to approximate the solution $x^{\delta}_{\overline{\alpha}(\delta)}$ of (1.3) with $\alpha = \overline{\alpha}$, we can use the iterative process (2.2). It is important to indicate how many iterations (depending on δ) are performed. Choices of $n = n(\delta)$ are also called "stopping rules" in the literature.

We have the result

Theorem 2.2. *If the first integer $n = n(\delta)$ satisfying the condition $|x^{n+1} - x^n| \leq a\delta$, where $a > 0$, then $x^{n(\delta)} \to x_0$, as $\delta \to 0$.*

Proof. Indeed,

$$\begin{aligned}
|x^{n+1} - x^n| &= \langle B(x^{n+1} - x^n),\, x^{n+1} - x^n\rangle \\
&= \rho^2\langle A_{\overline{\alpha}}(x^n) - f_\delta,\, B^{-1}(A_{\overline{\delta}}(x^n) - f_\delta)\rangle \\
&= \rho^2\langle A_{\overline{\alpha}}(x^n) - f_0 + f_0 - f_\delta,\, B^{-1}(A_{\overline{\delta}}(x^n) - f_0 + f_0 - f_\delta)\rangle \\
&\geq \rho^2\{\delta^2 - 2\delta\rho|x^{n+1} - x^n| + \langle A_{\overline{\alpha}}(x^n) - A_{\overline{\alpha}}(x_{\overline{\alpha}}),\, B^{-1}(A_{\overline{\alpha}}(x^n) - A_{\overline{\alpha}}(x_{\overline{\alpha}})\rangle,
\end{aligned}$$

where $x_{\overline{\alpha}} : A(x_{\overline{\alpha}}) + \overline{\alpha}Bx_{\overline{\alpha}} = f_0$.

Since

$$\begin{aligned}
&\langle A_{\overline{\alpha}}(x^n) - A_{\overline{\alpha}}(x_{\overline{\alpha}}),\, B^{-1}(A_{\overline{\alpha}}(x^n) - A_{\overline{\alpha}}(x_{\overline{\alpha}})\rangle\overline{\alpha}|x^n - x_{\overline{\alpha}}| \\
&+ \overline{\alpha}\langle A(x^n) - A(x_{\overline{\alpha}}),\, x^n - x_{\overline{\alpha}}\rangle \geq \overline{\alpha}^2|x^n - x_{\overline{\alpha}}|
\end{aligned}$$

we have

$$|x^{n+1} - x^n|^2 \geq \overline{\alpha}^2\rho^2|x^n - x_{\overline{\alpha}}|^2 - 2\rho\delta|x^{n+1} - x^n| - \rho^2\delta^2 .$$

Therefore,

$$|x^n - x_{\overline{\alpha}}| \leq \frac{2\delta}{\overline{\alpha}} .$$

This means that $x^n \to x_0$, as $\delta \to 0$.

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