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ITERATIVE METHODS FOR SOLVING DEGENERATE SYSTEM OF GRID EQUATIONS*

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Abstract. In this paper we construct an iterative method for solving degenerate system of gird equations and after that we use parametric extrapolation technique for accelerating its convergence rate. The efficiency of the method is shown on examples.

I. INTRODUCTION

In mathematical physics besides boundary value problem with unique solutions we meet also problems having infinite set of solutions, for example, the Neumann problem for elliptic equation and the problem for Lame equation in elasticity when stress is given on whole boundary. After discretization of these problems by difference or variational methods we get a system of linear algebraic equations with symmetric, nonnegative matrix

$$Au = f \tag{1}$$

The system, usually, is nonconsistent because due to the errors of computation the consistence condition of differential problem is not preserved yet. For degenerate system of linear algebraic equations in particular and for degenerate operator equation in general, in the nonconsistent case, one introduced the concept of generalized solution [5, 7] (or pseudosolution ([6, 11]). There always exists an infinite number of generalized solutions of operator equation. Among them the solution with minimal norm is called the normal solution. The normal solution exists and is unique and is orthogonal to the kernel of the operator.

The construction of stable method for finding normal solution of the operator equation of the first kind when the operator and the right-hand side are given not exactly is the subject of the theory of ill-posed problems (see c. f. [11, 12]). The latter problems often arise in processing experimental data. For them the matrix usually is complete and has not any special structure. Therefore, the problem of estimating the computational work for obtaining the normal solution with a given

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accuracy is not set yet.

Nevertheless, for the degenarate system of grid equations obtained after discretization of boundary value problem in mathematical physics one always is interested in the computational work for achieving the normal solution with a given accuracy as the size of this system is very large (for example 10^5). One wishes to construct methods with minimal cost. In this direction in [7, 9] there have been proposed and investigated some iterative method for finding normal solution and solving the two-dimensional Neumann problem as an application.

In this paper we shall construct an iterative method for degenerate equation with help of a regulizator and apply the method to the Neumann problem in two and three dimensions. After that we shall accelerate the convergence rate of the method by a parametric extrapolation technique. It is the technique that we have developed to construct accelerated methods for solving equation with positive definite operator or only positive operator in Hilbert space of infinite dimension (see [1-4]).

Throughout the paper all the operators are assumed to be linear.

II. SOME AUXILIARY RESULTS

We shall consider the system of grid equations (1) as an operator equation in Hilbert space of finite dimension $H = H_N$, where N is the number of unknown values of grid functions. Denote by $\mathrm{Ker}A$ and $\mathrm{Im}A$ the kernel and the image of A, respectively. There are the following expansions of H into orthogonal sums:

$$H = \mathrm{Ker}A \oplus \mathrm{Im}A^*,$$
$$H = \mathrm{Ker}A^* \oplus \mathrm{Im}A,$$

where A^* is the conjugate operator of A.

If $A = A^*$ we have

$$H = \mathrm{Ker}A \oplus \mathrm{Im}A. \tag{2}$$

Lemma 1. *Let R be a symmetric, nonnegative operator and B be symmetric, positive definite operator in H and $RB = BR$. Then:*

$$B^{-1}u \in \mathrm{Im}R \ \ if \ u \in \mathrm{Im}R,$$
$$B^{-1}u \in \mathrm{Ker}R \ if \ u \in \mathrm{Ker}R.$$

This lemma is easily followed from Lemmas 2, 3 in §3, Chapter XII of [9].

Now assume that

$$R = \sum_{\alpha=1}^{p} R_\alpha \tag{3}$$

with

$$R_\alpha^* = R_\alpha \geq 0\,, \;\; R_\alpha R_\beta = R_\beta R_\alpha, \;\; \alpha, \beta = 1, ..., p \geq 2 \tag{4}$$

and

$$\mathrm{Ker} R_1 = \cdots = \mathrm{Ker} R_p\,. \tag{5}$$

We consider the operator B of the form

$$B = (E + \omega_1 R_1) \cdots (E + \omega_p R_p)\,, \tag{6}$$

where $\omega_\alpha > 0$, $\alpha = 1, \ldots, p$.

It is clear that $B = B^* > 0$. From Lemma 1 it follows

Lemma 2. *Assume that R and B are defined above. Then:*

$$B^{-1}u \in \mathrm{Im}R \;\; if \;\; u \in \mathrm{Im}R\,,$$
$$B^{-1}u \in \mathrm{Ker}R \;\; if \;\; u \in \mathrm{Ker}R\,.$$

Lemma 3. *Assume that A and P are symmetric, nonnegative in H*, $\mathrm{Ker}A = \mathrm{Ker}P$. *Let $A_\varepsilon = A + \varepsilon P$, $\varepsilon > 0$. Then:*

i) $\mathrm{Ker}A_\varepsilon = \mathrm{Ker}A$,

ii) $\mathrm{Im}A_\varepsilon = \mathrm{Im}A$.

Proof: Since the assertion ii) may be followed from i) and (2) it is sufficient to prove the assertion i).

Obviously, if $u \in \mathrm{Ker}A$ then $u \in \mathrm{Ker}P$. Hence, $A_\varepsilon u = 0$, i.e. $u \in \mathrm{Ker}A_\varepsilon$.

Conversely, let $A_\varepsilon u = 0$, $u \neq 0$. It means $Au + \varepsilon Pu = 0$. Therefore, $(Au, u) + \varepsilon(Pu, u) = 0$. Since by the assumptions of the lemma $A \geq 0$, $P \geq 0$ it follows that

$$(Au, u) = 0\,. \tag{7}$$

We shall show that $Au = 0$.

Assume the contradiction, that is $Au \neq 0$. Then we are able to decompose $u = \overline{u} + \widetilde{u}$, where $\overline{u} \in \mathrm{Im}A$, $\widetilde{u} \in \mathrm{Ker}A$, $\overline{u} \neq 0$. In result, we have $(Au, u) = (A\overline{u}, \overline{u})$.

Let $e_1, e_2, ..., e_N$ be the orthogonal basis of H, consisting of eigenvectors of A and $0 = \lambda_1 = \cdots = \lambda_m < \lambda_{m+1} \leq \cdots \leq \lambda_N$ are the corresponding eigenvalues, i.e. $Ae_i = \lambda_i e_i$, $i = 1, \cdots, N$. Then we have $\overline{u} = \sum\limits_{i=m+1}^{N} c_i e_i$ with $\sum\limits_{i=m+1}^{N} c_i^2 > 0$. Consequently,

$$(A\overline{u}, \overline{u}) = \sum_{i=m+1}^{N} \lambda_i c_i^2 > \lambda_{m+1} \sum_{i=m+1}^{N} c_i^2 > 0 .$$

Hence, $(Au, u) > 0$. It contradicts (7). Thus, $Au = 0$, i.e. $u \in \mathrm{Ker}A$. The lemma is proved.

III. ALTERNATING DIRECTIONS ITERATIVE METHOD FOR SOLVING DEGENERATE EQUATION

In order to construct iterative method for solving the operator equation (1) we start from an operator $R = R^* \geq 0$, which we term the regulizator. On R we assume that:

i) $\mathrm{Ker}R = \mathrm{Ker}A$,

ii) there exist constants $0 < c_1 \leq c_2$ such that

$$c_1 (Rx, x) \leq (Ax, x) \leq c_2 (Rx, x) , \ \forall x, \ Ax \neq 0 , \tag{9}$$

that is

$$c_1 R \leq A \leq c_2 R \quad \text{in } \mathrm{Im}A \tag{9'}$$

We denote by $\overline{u}$ the normal solution of (1).

First, let us consider the case when (1) is consistent, i.e.

$$f \perp \mathrm{Ker}A . \tag{10}$$

It is equivalent to $f \in \mathrm{Im}A$. Therefore, we have

$$A\overline{u} = f , \ \ \overline{u} \in \mathrm{Im}A . \tag{11}$$

We shall consider the iterative process

$$B \frac{y_{k+1} - y_k}{\tau_{k+1}} + Ay_k = f , \ \ k = 0, 1, ... \tag{12}$$

$$y_0 \in \mathrm{Im}A ,$$

where B is a symmetric, positive definite operator constructed in dependence on R. If R has the from (3) - (5) we choose B in the form (6). The coefficient ω will be determined afterwards.

From (12) we derive

$$y_{k+1} = y_k - \tau_{k+1} B^{-1}(Ay_k - f)\,, \quad k = 0,\, 1... \tag{13}$$

Since $f \in \mathrm{Im}A$, we have $Ay_k - f \in \mathrm{Im}A$ for arbitrary y_k. Hence $Ay_k - f \in \mathrm{Im}R$. By Lemma 2 $B^{-1}(Ay_k - f) \in \mathrm{Im}R$. Consequently, if $y_k \in \mathrm{Im}A$ we have $y_{k+1} \in \mathrm{Im}A$.

Thus, starting from $y_0 \in \mathrm{Im}A$ by the iterative scheme (12) we obtain the sequence $\{y_k\} \subset \mathrm{Im}A$, or in other words, the iterative process is implemented entirely in subspace $\mathrm{Im}A$ of the space H. Therefore, it is possible to use the results on convergence of the iterative scheme for the case of symmetric, positive definite operator in the whole space to the case considered in $\mathrm{Im}A$.

Below we state the results on convergence of (12).

Case 1. Under Case 1 or 2D-case we shall understand the case, when the following conditions are satisfied:

$$R = R_1 + R_2\,, \quad R_\alpha = R_\alpha^* \geq 0\,, \ \alpha = 1,\, m2, \quad R_1R_2 = R_2R_1\,, \tag{14}$$

$$\delta_\alpha E \leq R_\alpha \leq \Delta_\alpha E\,, \ \delta_\alpha > 0, \ \alpha = 1,\, 2 \quad \text{in } \mathrm{Im}A\,. \tag{15}$$

For δ_α and Δ_α we can choose the minimal nonzero and maximal eigenvalues of R_α respectively.

Lemma 4. *In Case 1 taking*

$$B = (E + \omega_1 R_1)(E + \omega_2 R_2)\,, \tag{16}$$

where

$$\begin{gathered}\omega_1 = \frac{\omega_0}{1 + \omega_0 c_0}\,, \ \omega_2 = \frac{\omega_0}{1 - \omega_0 c_0}\,, \ \omega_0 = \frac{1}{\sqrt{\delta_1'\Delta_1'}} = \frac{1}{\sqrt{\delta_2'\Delta_2'}}\,, \\ \delta_1' = \delta_1 + c_0\,, \ \Delta_1' = \Delta_1 + c_0\,, \ \delta_2' = \delta_2 - c_0\,, \ \Delta_2' = \Delta_2 - c_0\,, \\ c_0 = \frac{\delta_2\Delta_2 - \delta_1\Delta_1}{\delta_1 + \delta_2 + \Delta_1 + \Delta_2}\,,\end{gathered} \tag{17}$$

we have

$$\overset{\circ}{\gamma}_1 B \leq R \leq \overset{\circ}{\gamma}_2 B\,, \tag{18}$$

with

$$\begin{gathered}\overset{\circ}{\gamma}_1 = \frac{1-\bar\rho}{\tau}\,, \ \overset{\circ}{\gamma}_2 = \frac{1+\bar\rho}{\tau}\,, \ \tau = \omega_1 + \omega_2\,, \\ \bar\rho = \frac{1-\sqrt{\eta_1'}}{1+\sqrt{\eta_1'}}\frac{1-\sqrt{\eta_2'}}{1+\sqrt{\eta_2'}}\,, \ \eta_\alpha' = \frac{\delta_\alpha'}{\Delta_\alpha'} \ (\alpha = 1,\, 2)\,.\end{gathered}$$

The coefficients of energetic equivalence in (18) are optimal in the sence that the ratio $\mathring{\gamma}_1/\mathring{\gamma}_2$ is maximal.

The content of Lemma 4 was proved in [10, §2, Chapter VIII] although it was not formulated as a lemma.

In the sequel under $\|\cdot\|$ we mean $\|\cdot\|_D$ where $D = A$, B in ImA.

Theorem 1. *If ω_1, ω_2 are chosen as in Lemma 4 then the iterative method (12) is convergent and there holds the estimate*

$$\|y_n - \overline{u}\| \leq \rho_0^n \|y_0 - \overline{u}\|, \tag{19}$$

where

$$\rho_0 = \frac{1-\xi}{1+\xi}, \quad \xi = \frac{\gamma_1}{\gamma_2}, \quad \gamma_1 = c_1 \mathring{\gamma}_1, \quad \gamma_2 = c_2 \mathring{\gamma}_2,$$

if

$$\tau_{n+1} = \frac{2}{\gamma_1 + \gamma_2}, \quad n = 0, 1, \ldots$$

In the case, if $\{\tau_{n+1}\}$ is the Chebyshev collection of parameters constructed by γ_1, γ_2 (see, [9, 10]) then instead of (19) we have

$$\|y_n - \overline{u}\| \leq q_n \|y_0 - \overline{u}\|,$$

where

$$q_n = \frac{2\rho_1^n}{1+\rho_1^{2n}}, \quad \tau_0 = \frac{2}{\gamma_1 + \gamma_2}, \quad \rho_0 = \frac{1-\xi}{1+\xi}, \quad \rho_1 = \frac{1-\sqrt{\xi}}{1+\sqrt{\xi}}, \quad \xi = \frac{\gamma_1}{\gamma_2},$$

Hence, the estimate of iteration numbers needed to achieve an approximation to the normal solution with the relative accuracy θ is follows:

- for the stationary process

$$n_d(\theta) = \frac{1}{2\xi} \ln \frac{1}{\theta}, \tag{20}$$

- for the Chebyshev process

$$n_c(\theta) = \frac{1}{2\sqrt{\xi}} \ln \frac{2}{\theta}. \tag{21}$$

Example 1. On the grid $\overline{\omega} = \{x = (ih_1, jh_2),\ 0 \le i \le N_1,\ 0 \le j \le N_2,\ N_1 h_1 = l_1,\ N_2 h_2 = l_2\}$ consider the system of equations

$$\begin{aligned} &\Lambda y = -f(x)\,, \quad x \in \overline{\omega}\,, \\ &\Lambda = \Lambda_1 + \Lambda_2\,, \quad f(x) = \varphi(x) + \frac{2}{h_1}\varphi_1(x) + \frac{2}{h_2}\varphi_2(x)\,. \end{aligned} \tag{22}$$

Here

$$\Lambda_\alpha y = \begin{cases} \frac{2}{h_\alpha} a_\alpha^{+1} y_{x_\alpha}\,, & x_\alpha = 0\,, \\ (a_\alpha y_{\overline{x}_\alpha})_{x_\alpha}\,, & h_\alpha \le x_\alpha \le l_\alpha - h_\alpha\,, \\ -\frac{2}{h_\alpha} a_\alpha y_{\overline{x}_\alpha}\,, & x_\alpha = l_\alpha\,, \end{cases}$$

$$\varphi_\alpha(x) = \begin{cases} g_{-\alpha}(x_\beta)\,, & x_\alpha = 0\,, \\ 0\,, & h_\alpha \le x_\alpha \le l_\alpha - h_\alpha\,, \\ g_{+\alpha}(x_\beta)\,, & x_\alpha = l_\alpha\,, \end{cases}$$

$$\beta = 3 - \alpha\,, \quad \alpha = 1,\, 2\,;$$

$$a_1^{+1}(x) = k_1(x_1 + h_1,\, x_2)\,, \quad a_2^{+1} = k_2(x_1,\, x_2 + h_2)\,.$$

The notations of difference deviratives are adopted as in [9, 10].

Assume that the coefficients $a_1(x)$, $a_2(x)$ satisfy the conditions

$$0 \le c_1 \le a_\alpha(x) \le c_2\,, \quad \alpha = 1,\, 2\,, \quad x \in \overline{\omega}\,. \tag{23}$$

The system of grid equations (22) is the difference scheme of the Neumann problem for elliptic equation

$$\frac{\partial}{\partial x_1}\left(k_1(x)\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(k_2(x)\frac{\partial u}{\partial x_2}\right) = -\varphi(x)\,, \quad x \in G = \{x : 0 < x_\alpha < l_\alpha\,,\ \alpha = 1,\, 2\}\,,$$

$$k_\alpha \frac{\partial u}{\partial x_\alpha} = g_{-\alpha}(x_\beta)\,, \quad x_\alpha = 0\,, \quad \beta = 3 - \alpha\,,$$

$$-k_\alpha \frac{\partial u}{\partial x_\alpha} = g_{+\alpha}(x_\beta)\,, \quad x_\alpha = l_\alpha\,, \quad \alpha = 1,\, 2\,.$$

In the space H of grid functions defined on ω we introduce scalar product

$$(u,\, v) = \sum_{x \in \overline{\omega}} u(x)\, v(x)\, \hbar_1(x_1)\, \hbar_2(x_2)\,,$$

$$\hbar_\alpha(x_\alpha) = \begin{cases} h_\alpha\,, & h_\alpha \le x_\alpha \le l_\alpha - h_\alpha \\ 0.5\, h_\alpha\,, & x_\alpha = 0,\, l_\alpha\,;\ \alpha = 1,\, 2 \end{cases}$$

and consider the operator $A = -\Lambda$.
Then $A = A^* \geq 0$. Ker$A$ consists of grid functions which are constants on the grid $\overline{\omega}$. We can assume that $f \in \text{Im}A$ because in the contrast case we replace f by $f_1 = f - (f, \mu_{00})\mu_{00}$, where $\mu_{00} = \mu_{00}(i, j) = 1/\sqrt{l_1 l_2}$.
The operators R and R_1, R_2 are defined as follows:

$$R = -\overset{\circ}{\Lambda}, \ \overset{\circ}{\Lambda} = \overset{\circ}{\Lambda}_1 + \overset{\circ}{\Lambda}_2, \ R_\alpha = -\overset{\circ}{\Lambda}_\alpha,$$

$$\overset{\circ}{\Lambda}_\alpha = \begin{cases} \frac{2}{h_\alpha} y_{x_\alpha}, & x_\alpha = 0, \\ y_{\overline{x}_\alpha x_\alpha}, & h_\alpha \leq x_\alpha \leq l_\alpha - h_\alpha, \\ -\frac{2}{h_\alpha} y_{\overline{x}_\alpha}, & x_\alpha = l_\alpha, \ \alpha = 1, 2, \end{cases}$$

The coeffcients of energetic equivalence of A and R in (9) are the same c_1, c_2 in (23). It is easy to verify that Ker$R$ = KerR_1 = KerR_2 = Ker$A$. The coefficients δ_α and Δ_α in (15) are given by

$$\delta_\alpha = \frac{4}{h_\alpha^2} \sin^2 \frac{\pi h_\alpha}{2l_\alpha} \geq \frac{8}{l_\alpha^2}, \ \Delta_\alpha = \frac{4}{h_\alpha^2}, \ \alpha = 1, 2.$$

By Theorem 1 the number of iterations needed to achieve the normal solution with the relative accuracy $\theta = h^2 = h_1^2 + h_2^2$ are

$$n_d(\theta) = O\Big(\frac{1}{h} \ln \frac{1}{h}\Big), \ n_c(\theta) = O\Big(\frac{1}{\sqrt{h}} \ln \frac{1}{h}\Big). \tag{24}$$

Case 2. Under Case 2 or 3D-case we shall understand the case, when the following conditions are satisfied:

$$\begin{aligned} &R = R_1 + R_2 + R_3, \ R_\alpha = R_\alpha^* \geq 0, \ R_\alpha R_\beta = R_\beta R_\alpha, \ \alpha, \beta = 1, 2, 3, \\ &\delta_\alpha E \leq R_\alpha \leq \Delta_\alpha E, \ \alpha = 1, 2, 3 \text{ in Im}A. \end{aligned} \tag{25}$$

In this case we take

$$B = (E + \omega R_1)(E + \omega R_2)(E + \omega R_3). \tag{26}$$

Denote

$$\delta = \min\{\delta_1, \delta_2, \delta_3\}, \ \Delta = \max\{\Delta_1, \Delta_2, \Delta_3\}, \ \eta = \frac{\delta}{\Delta}. \tag{27}$$

Then by [8, 9] with the choice

$$\omega = \frac{\sqrt[3]{\delta\Delta}}{\sqrt[3]{\delta} + \sqrt[3]{\Delta}} \tag{28}$$

we have

$$\mathring{\gamma}_1 B \leq R \leq \mathring{\gamma}_2 B \text{ in } \mathrm{Im}A,$$

where

$$\mathring{\gamma}_1 = 3\delta\left(\frac{1-\eta^{2/3}}{1-\eta}\right)^3, \quad \mathring{\gamma}_2 = \frac{\delta(1+2\eta)}{\eta^{2/3}}\left(\frac{1-\eta^{2/3}}{1-\eta}\right)^3.$$

Hence, the coefficients of energetic equivalence of A and B in $\mathrm{Im}A$ are

$$\gamma_1 = c_1\,\mathring{\gamma}_1, \quad \gamma_2 = c_2\,\mathring{\gamma}_2.$$

The results on convergence of the iterative scheme (12) in the case under consideration are the same as stated in Theorem 1.

Applying these results to the Neumann problem in unitary cub we obtain the estimate

$$n_d(\theta) = O\left(\frac{1}{h^{3/2}}\ln\frac{1}{\theta}\right), \quad n_c(\theta) = O\left(\frac{1}{h^{3/4}}\ln\frac{1}{\theta}\right). \tag{29}$$

IV. ACCELERATING CONVERGENCE RATE BY PARAMETRIC EXTRAPOLATION

In this section we shall accelerate the convergence rate of the iterative process (12) with the help of the extrapolation technique by a parameter, which will be introduced into the equation (1). The theoretical background of this accelerating method was elaborated in [3]. The applications of this method to the equation with positive definite operator is presented in [3,4] and with only positive operator in [1,2].

Below we only state the result on acceleration of convergence rate of the process (12) and point out the difference, which it should be drawn attention to when using known techniques to the case of degenerate operator.

Instead of (1) we consider the perturbed equation

$$A_\varepsilon u_\varepsilon = f, \tag{30}$$

where

$$A_\varepsilon = A + \varepsilon P, \quad \varepsilon > 0,$$
$$P = P^* \geq 0, \quad \mathrm{Ker}P = \mathrm{Ker}A.$$

By Lemma 3 we have $\mathrm{Ker}A_\varepsilon = \mathrm{Ker}A$, $\mathrm{Im}A_\varepsilon = \mathrm{Im}A$. Under the assumption $f \in \mathrm{Im}A$ the equation (30) has solutions and its normal solution $\overline{u}_\varepsilon \in \mathrm{Im}A$. Hence, like to (1), being considered in the subspace $\mathrm{Im}A$ (30) has unique solution $\overline{u}_\varepsilon$.

Theorem 2. *Assume that A and P satisfy the condition of Lemma 3 and let M be a natural number. Then the normal solution of (30) may be expanded in the form*

$$\overline{u}_\varepsilon = \overline{u}_0 + \sum_{k=1}^{M} \varepsilon^k v_k + \varepsilon^{M+1} w_\varepsilon ,$$

where $\overline{u}_0$ is the normal solution of (1), v_k ($k = 1, \ldots, M$) are elements (belonging to ImA) independent of ε and $w_\varepsilon \in ImA$ is uniformly bounded in ε, i.e. $\|w_\varepsilon\| \leq const$.

This theorem is proved in the same way as Theorem 2.1 in [3]. The only difference is that here all things occur in ImA. This is ensured by Lemma 3.

We now construct the approximation of the normal solution $\overline{u}$ of (1) in the form

$$\overline{U}^E = \sum_{k=1}^{M+1} \alpha_k \overline{u}_{\varepsilon/k} ,$$

where

$$\alpha_k = \frac{(-1)^{M+1-k} k^{M+1}}{k!\,(M+1-k)!} ,$$

$\overline{u}_{\varepsilon/k}$ is the normal solution of (30) with the parameter ε/k.

Since $\overline{u}_{\varepsilon/k} \in \text{Im}A$ ($k = 1, \ldots, M+1$) we have also $\overline{U}^E \in \text{Im}A$. Using Theorem 2 we obtain the following.

Theorem 3. *Suppose that A and P satisfy the conditions of Lemma 3. Then we have*

$$\|\overline{U}^E - \overline{u}\| \leq C\,\varepsilon^{M+1} ,$$

where C = const, independent of ε.

From the above results we see that in order to obtain an approximate normal solution of (1) with the prescribed accuracy θ we must choose $\varepsilon = \theta^{1/(M+1)}$ and solve (30) $M+1$ times with the parameters ε/k ($k = 1, \ldots, M+1$) at the same level of accuracy θ.

To solve (30) we use the iterative process

$$B\,\frac{y_\varepsilon^{(k+1)} - y_\varepsilon^{(k)}}{\tau_{k+1}} + A_\varepsilon\, y_\varepsilon^{(k)} = f , \quad k = 0, 1, \ldots \tag{31}$$
$$y_\varepsilon^{(0)} \in \text{Im}A .$$

Using the results of [3, 4] we obatin the following ones on convergene of (31) in the cases arising from two and three-dimensional problem as defined in Section 3.

In 2D-case (Case 1) we take $P = R_1R_2$. Then we have

Theorem 4. *If $B = \left(E + \sqrt{\frac{\varepsilon}{\delta}}R_1\right)\left(E + \sqrt{\frac{\varepsilon}{\delta}}R_2\right)$, where $\delta = \delta_1 + \delta_2$ and $\{\tau_k\}_{k=1}^n$ are the collection of chebyshev parameters constructed by the bounds $\gamma_1 = \frac{\delta}{2}$, $\gamma_2 = c_2\sqrt{\frac{\delta}{\varepsilon}}$, c_2 being coefficient in (9), then the iterative process (31) converges to the normal solution $\overline{u}_\varepsilon$ of (30) and there holds the estimate*

$$\|y_\varepsilon^{(n)} - \overline{u}_\varepsilon\| \le q_n \|y_\varepsilon^0 - \overline{u}_\varepsilon\|,$$

where

$$q_n = \frac{2\rho_1^n}{1+\rho_1^{2n}}, \quad \rho_1 = \frac{1-\sqrt{\mathring{\xi}}}{1+\sqrt{\mathring{\xi}}}, \quad \mathring{\xi} = \frac{\sqrt{\delta\varepsilon}}{2c_2}.$$

For the stationary iterative process with $\tau_k \equiv \tau = \frac{2}{\gamma_1^0+\gamma_2^0}$ there holds estimate

$$\|y_\varepsilon^{(k)} - \overline{u}_\varepsilon\| \le \rho^k \|y_\varepsilon^0 - \overline{u}_\varepsilon\|, \quad k = 1, 2, \dots$$

where $\rho = (1-\mathring{\xi})/(1+\mathring{\xi})$.
The estimates for number of iterations needed to get an approximate normal solution of (30) with relative accuracy θ are

$$n_c(\theta) = \frac{\ln(2/\theta)}{\sqrt{\frac{2}{c_2}\delta^{1/4}\varepsilon^{1/4}}}, \quad n_d(\theta) = \frac{\ln(2/\theta)}{\sqrt{\delta\varepsilon}/c_2}. \tag{32}$$

Example 2. Consider Example 1 in Section 3 again. The achieve accuracy $\theta = h^2$ when choosing $M = 1$ we take $\varepsilon = h$. Then it should solve (30) with parameters h and $h/2$.

According to Theorem 4 we have

$$n_c(\theta) = O\Big(\frac{1}{h^{1/4}}\ln\frac{1}{h}\Big), \quad n_d(\theta) = O\Big(\frac{1}{h^{1/2}}\ln\frac{1}{h}\Big).$$

From the above estimates and (24) we see that the iterative process for solving (30) requires iteration number less than for (1).

In 3D-case (Case) we take

$$P = R_1R_2 + R_2R_3 + R_1R_3 + \sqrt{h}R_1R_2R_3,$$
$$B = (E+\omega R_1)(E+\omega R_2)(E+\omega R_3), \quad \omega = a\sqrt{\varepsilon}.$$

Theorem 5. *Suppose* $A \geq \delta E$ *in* ImA *and* $\varepsilon \leq \min\{\frac{c_1^2\sqrt{h}}{c_2}, \frac{c_1^2}{\delta}, \delta h\}$, *where* c_1, c_2 *are the coefficients in (19). Then:*

i) If $h \leq c_2^2/\delta^2$ *and* $a = \overset{\circ}{a} = 1/\sqrt{\delta}$ *then the iterative process (31) with the Chebyshev parameters constructed by the bounds* $\gamma_1 = \overset{\circ}{\gamma}_1 = \delta/2$ *and* $\gamma_2 = \overset{\circ}{\gamma}_2 = c_2\sqrt{\delta}/\sqrt{\varepsilon}$ *converges to the normal solution* $\overline{u}_\varepsilon$ *of (30) and we have the estimate (32).*

ii) If $h > c_2^2/\delta^2$ *and* $a = \overline{a} = h^{1/4}/\sqrt{c_2}$ *then*

$$n_c(\theta = \frac{\ln(2/\theta)}{2\sqrt{\overline{\xi}}}, \quad n_d(\theta = \frac{\ln(2/\theta)}{2\overline{\xi}},$$

where

$$\overline{\xi} = \frac{\delta h^{1/4}\sqrt{\varepsilon}}{\sqrt{c_2}(c_2 + \delta\sqrt{h})}.$$

Applying the accelerated method to the Neumann problem in unitary cub we get the estimates for iterative number, which are better than (29).

V. SOME CONCLUDING REMARKS

1. Above we restrict ourselves to the case $f \in$ImA. If this condition is not satisfied we replace f by $f - \widetilde{f}$, where $\widetilde{f}$ is the projection of f onto Ker$A$. In applications it is realizable because we usually know Ker$A$, for example, when solving Neumann problem for elliptic equation or the Lame equation with stress given on whole boundary (see [7]).

2. In [5] Marchuk also studies iterative methods for solving degenerate system of linear algebraic equations, but there absent results on convergence rate of the methods. It should be emphasized that there the iterative process is convergent to some generalized solution of the system and not necessarily to the normal solution.

3. For solving degenerate system of linear algebraic equations with symmetric, nonnegative matrix some authors (c.f. [7]) used the method of shift for spectrum of operator, i.e. instead of (1) they considered the equation

$$(A + \varepsilon E)u_\varepsilon = f.$$

Then a problem has been set up. That is to choose the regularization parameter ε for obtaining the normal solution with a given accuracy on computers with a specified arithmetic. It means that we must draw attention to number of correct digits in floating-point arithmetic. This problem was involved in [7]. An further discussion of it and an study of application of parameter extrapolation technique will be presented elsewhere.

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