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AN ITERATIVE METHOD FOR SOLUTION OF NONLINEAR OPERATOR EQUATION

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Abstract. In the note, for finding a solution of nonlinear operator equation of Hammerstein's type an iterative process in infinite-dimensional Hilbert space is shown, where a new iteration is constructed basing on two last steps. An example in the theory of nonlinear integral equations is given for illustration.

1. INTRODUCTION

Let H be a real Hilbert space with the norm and scalar product denoted by $\|\ \|$ and $\langle .,. \rangle$, respectively.

Let F_i, $i = 1, 2$, be nonlinear monotone operators in H, i.e.

$$\langle F_i(x) - F_i(y), x - y \rangle \geq 0, \quad \forall x, y \in D(F_i) \equiv H, \ i = 1, 2.$$

The operator equation of Hammerstein's type

$$x + F_2F_1(x) = f_0, \quad f_0 \in H \tag{1.1}$$

was considered by several authors (see [1], [2], [4-7], [12-17] and bibliography there). In [10], an iterative process was given for solving (1.1) with the linear property of F_2. In [6], the author proposed an method of regularization for the solution of (1.1) in the case, where both the operators F_i are nonlinear and monotone.

In the note, basing on our result in [6] and the idea of iterative regularization proposed by A. Bakyshinski (see [3]), we give a two-step iteration method for solving (1.1) in infinite-dimensional Hilbert space H. The result is illustrated by an example in the theory of nonlinear integral equations.

Note that, recently, the problem of approximating a solution of (1.1) is investigated extensively because of its importance in applications (see [8], [9], [11], [16]).

2. MAIN RESULT

Let x^1 and x^2 be two arbitrary elements of H. The iteration procedure is defined by

$$x^{n+2} = \varphi_1^{n+1}(x^{n+1}) + \beta_{n+1}\left[\varphi_2^n\Big((x^{n+1} - \varphi_1^n(x^n))/\beta_n\Big) - \beta_n x^n\right], \quad (2.1)$$
$$n = 1, 2, \dots$$

where

$$\varphi_i^n(x) = x - \beta_n\Big(F_i(x) + \alpha_n x + a_i f_0\Big), \; i = 1,\, 2, \quad (2.2)$$
$$a_1 = 0, \quad a_2 = -1,$$

and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences of positive numbers. Later, we see, as in [3], that α_n plays the role of regularization and β_n, the role of iteration parameter.

Theorem. *If (1.1) has a solution and there exist the constants $L_i > 0$ such that*

$$\|F_i(x)\| \le L_i(1 + \|x\|), \; i = 1,\, 2, \; \forall x \in H,$$

then iteration process (2.1) converges to a solution of (1.1) under the condition

$$\alpha_n,\ \beta_n > 0, \quad \alpha_n \to 0, \quad lim\frac{|\beta_n - \beta_{n+1}|}{\beta_n \alpha_n^2} = 0, \quad \sum_1^\infty \alpha_n\beta_n = \infty.$$

Proof. Put

$$y^n = (x^{n+1} - \varphi_1^n(x^n))/\beta_n, \quad n = 1,\, 2, \dots \quad (2.3)$$

Then from (2.1) and (2.2) we have

$$\begin{aligned} y^{n+1} &= (x^{n+2} - \varphi_1^{n+1}(x^{n+1}))/\beta_{n+1} \\ &= \varphi_2^n(y^n) - \beta_n x^n \\ &= y^n - \beta_n(F_2(y^n) + x_n + \alpha_n y^n - f_0). \end{aligned}$$

On the other hand, from (2.3) and (2.2) we also obtain

$$\begin{aligned} x^{n+1} &= \varphi_1^n(x^n) + \beta_n y^n \\ &= x^n - \beta_n(F_1(x^n) - y_n + \alpha_n x^n), \quad n = 1,\, 2, \dots \end{aligned}$$

In the Hilbert space $H_1 = H \times H$ with the scalar product denoted by $\langle z_1, z_2 \rangle_1 = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle$, where $z_i = [x_i, y_i]$, $x_i, y_i \in H$, we can write

$$z^{n+1} = z^n - \beta_n \Big(\mathcal{F}(z^n) + \alpha_n z^n - \overline{f}_0 \Big), \tag{2.4}$$
$$\mathcal{F}(z^n) = [F_1(x^n), F_2(y^n)] + [-y^n, x^n],$$
$$z^n = [x^n, y^n], \quad \overline{f}_0 = [\theta, f_0],$$

where the θ denotes the zero element in H. It is easy to verify that in the Hilbert space H_1, $\mathcal{F}$ is a monotone operator. However, without any difficulty we can see that $\mathcal{F}$ satisfies the condition

$$\|\mathcal{F}(z)\| \leq \sqrt{2} \max L_i(1 + \|z\|_1),$$

where $\|.\|_1$ is the norm of H_1 generated by $\langle .,. \rangle_1$.

Applying Theorem 5.1 (p. 144) in [3] to the process (2.4), we can conclude that the sequence $\{z^n\}$ converges in H_1 to $z_0 = [x_0, F_1(x_0)]$, one solution of the equation

$$\mathcal{F}(z) = \overline{f}_0.$$

Therefore, the sequence $\{x^n\}$ converges in H to x_0, as $n \to \infty$. Theorem is proved.

Remarks. 1. The sequence $\beta_n = (1+n)^{-1/2}$ and $\alpha_n = (1+n)^{-p}$, $0 < p < 1/2$, satisfy all the conditions in the theorem.

2. If F_i are Lipschitz continuous with a Lipschitz constants L_i, then $\mathcal{F}$ also is Lipschitz continuous with Lipschitz constant $\mathcal{L} = 2\sqrt{\max\{1, L_1, L_2\}}$. Applying Theorem 5.2 in [3], we obtain the result that the iteration process (2.1) converges in H to a solution of (1.1), if

$$\overline{\lim}_{n \to \infty} \beta_n \frac{(1 + \alpha_n^2)}{\alpha_n} < \frac{2}{\mathcal{L}^2}.$$

In this case, we can chose the sequence $\beta_n = \theta \alpha_n$,

$$\alpha_n = (1+n)^{-p}, \quad 0 < p < 1/2, \quad \theta < \frac{2}{(1+\alpha_0)^2 \mathcal{L}^2}.$$

3. APPLICATION

Consider the nonlinear integral equation of Hammerstein's type

$$\varphi(t) + \int_0^1 k(t,s) f(\varphi(s)) ds = f_0(t), \quad t \in [0,1], \quad \varphi \in L_2[0,1], \tag{3.1}$$

where $f_0(t) \in L_2[0,1]$, $k(t,s) \geq 0$ is continuous and $f(t)$ is a nondecreasing and bounded function satisfying the condition $|f(t)| \leq a_0 + b_0|t|$, $t \in R$. Then,

$$(F_1\varphi)(t) = f(\varphi(t)), \quad \varphi(t) \in L_2[0,1],$$

$$(F_1\xi)(t) = \int_0^1 k(t,s)\xi(s)ds, \; \xi(t) \in L_2[0,1].$$

Since $k(t,s) \geq 0$ and $f(t)$ is nondeacreasing, then F_i, $i = 1, 2$, are monotone. The continuity of $k(t,s)$ implies that F_2 is bounded. It is not difficult to prove that F_i satisfy the conditions of the main theorem. Therefore, in order to obtain approximate solution for (3.1) we can apply the process (2.1) with defined above F_i and α_n, β_n in the remark 1.

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