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FINITE-DIMENSIONAL CHU SPACE

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Abstract. In this note we introduce the notion of finite-dimensional space and extend a result in [3] stating that, every fully complete Chu space is a fuzzy space. Finally, the general results are applied to a concrete example of theory of game.

1. INTRODUCTION

Rightly in 1970s, Hoang Tuy [5] suggested that the theory of category will be used widely in Computer Science. Actually, the general algebraic scheme, known as Chu's categories, becomes a useful tool today order to present ideas about Theoretical Computer Science. The best applications of Chu's spaces to Computer Science at this point in time are mainly investigated by Vaughan Pratt (see, e.g. [7], [8], [9], [10]). The notation of 2-dimensional Chu's space are studied in several papers (see, e.g. [3], [6]).

2. FINITE-DIMENSIONAL CHU SPACES IN GENERAL SETTINGS

By a $(n+m)$-*dimensional Chu space* we mean the set $\widetilde{C} = (X_1 \times X_2 \times \ldots \times X_n; f; A_1 \times A_2 \times \ldots \times A_m)$, where X_i, A_j $(i = 1, \ldots, n;\ j = 1, \ldots, m)$ are arbitrary sets and $f : X_1 \times \ldots \times X_n \times A_1 \times \ldots \times A_m \to [0,1]$ is a map, called the *probability function* of $\widetilde{C}$.

Example 1. Let X is a metric space, then $\widetilde{C} = (X_1 \times X_2 \times \ldots \times X_n; f; X_{n+1} \times X_{n+2} \times \ldots \times X_{n+m})$ is a Chu space, where $X_i = X$ for $i = 1, \ldots, n+m$ and $f : X_1 \times X_2 \times \ldots \times X_{n+m} \to [0,1]$ is defined by

$$f(x_1, x_2, \ldots, x_{n+m}) = \min\Big\{\sqrt{\sum_{i,j=1}^{n+m} d(x_i, x_j)^2}, 1\Big\}.$$

If $\widetilde{C} = (X_1, \times X_2 \times \ldots \times X_n; f; A_1 \times A_2 \times \ldots \times A_m)$ and $\widetilde{D} = (Y_1, \times Y_2 \times \ldots \times Y_n; g; B_1 \times B_2 \times \ldots \times B_m)$ are $(n+m)$ - Chu spaces, then a $(n+m)$ - *Chu morphism* $\Phi : \widetilde{C} \to \widetilde{D}$ is a $(n+m)$ - tuple of maps $\Phi = (\varphi_1, \varphi_2, \ldots, \varphi_n; \psi_1, \psi_2, \ldots, \psi_m)$, with $\varphi_i : X_i \to Y_i$ for $i = 1, \ldots, n$ and $\psi_j : B_j \to A_j$ for $j = 1, \ldots, m$ such that the diagram below commutes:

$$\begin{array}{ccc} \prod_{i=1}^n X_i \times \prod_{j=1}^m B_j & \xrightarrow{(\prod_{i=1}^n \varphi_i, 1_{\prod_{j=1}^m B_j})} & \prod_{i=1}^n Y_i \times \prod_{j=1}^m B_j \\ {\scriptstyle (1_{\prod_{i=1}^n X_i}, \prod_{j=1}^m \psi_j)} \downarrow & & \downarrow {\scriptstyle g} \\ \prod_{i=1}^n X_i \times \prod_{j=1}^m A_j & \xrightarrow[f]{} & [0,1] \end{array} \qquad (1)$$

where $1_{\prod_{i=1}^n X_i}, 1_{\prod_{i=1}^m B_j}$, denote identity maps. That is

$$f \circ (1_{\prod_{i=1}^n X_i}, \prod_{j=1}^m \psi_j) = g \circ (\prod_{i=1}^n \varphi_i, 1_{\prod_{j=1}^m B_j}),$$

or equivalently,

$$f(\prod_{i=1}^{n} x_i \times \prod_{j=1}^{m} \psi_j(b_j)) = g(\prod_{i=1}^{n} \varphi_i(x_i) \times \prod_{j=1}^{m} b_j) \text{ for } \prod_{i=1}^{n} x_i \in \prod_{i=1}^{n} X_i \text{ and } \prod_{j=1}^{m} b_j \in \prod_{j=1}^{m} B_j. \quad (2)$$

If $\Phi = (\varphi_1, \cdots, \varphi_n; \psi_1, \cdots, \psi_m) : \widetilde{C} = (X_1 \times \ldots \times X_n; f; A_1 \times \ldots \times A_m) \to \widetilde{D} = (Y_1 \times \ldots \times Y_n; g; B_1 \times \ldots \times B_m)$ is a $(n+m)$ - Chu morphism, then the $(n+m)$ - Chu space $(\prod_{i=1}^{n} X_i; f \times_\Phi g; \prod_{j=1}^{m} B_j)$, where

$$f \times_\Phi g) = f \circ (1_{\prod_{i=1}^{n} X_i}, \prod_{j=1}^{m} \psi_j) = g \circ (\prod_{i=1}^{n} \varphi_i, 1_{\prod_{j=1}^{m} B_j})$$

is called the *cross product of* $\widetilde{C}$ *and* $\widetilde{D}$ *over* Φ, denoted by $\widetilde{C} \times_\Phi \widetilde{D}$, see [4].

We say that the diagram (1) *upper-commutes* if instead of (2) we have

$$f(\prod_{i=1}^{n} x_i \times \prod_{j=1}^{m} \psi_j(b_j)) \leq g(\prod_{i=1}^{n} \varphi_i(x_i) \times \prod_{j=1}^{m} b_j) \quad (3)$$

for $\prod_{i=1}^{n} x_i \in \prod_{i=1}^{n} X_i$ and $\prod_{j=1}^{m} b_j \in \prod_{j=1}^{m} B_j$.

If (3) holds, then we say that $\Phi : \widetilde{C} = (X_1 \times X_2 \times \ldots \times X_n; f; A_1 \times A_2 \times \ldots \times A_m) \to \widetilde{D} = (Y_1 \times Y_2 \times \ldots \times Y_n; g; B_1 \times B_2 \times \ldots \times B_m)$ is a $(n+m)$ - *Chu upper-morphism.*

The *composition* of two $(n+m)$ - Chu morphisms $\Phi^{(1)} = (\varphi_1, \ldots, \varphi_n; \psi_1, \ldots, \psi_m)$ and $\Phi^{(2)} = (\varphi'_1, \ldots, \varphi'_n; \psi'_1, \ldots, \psi'_m)$ is given by $\Phi^{(2)}\Phi^{(1)} = (\varphi'_1\varphi_1, \ldots, \varphi'_n\varphi_n; \psi_1\psi'_1, \ldots, \psi_m\psi'_m)$. Cleraly $1_{\widetilde{C}} = (1_{X_1}, 1_{X_2}, \ldots, 1_{X_n}, 1_{A_1}, 1_{A_2}, \ldots, 1_{A_m})$ is the identity of $\widetilde{C} = (X_1 \times X_2 \times \ldots \times X_n; f; A_1 \times A_2 \times \ldots \times A_m)$.

Proposition 1. *$\Phi^{(1)}$ and $\Phi^{(2)}$ are $(n+m)$ - Chu morphisms (resp. $(n+m)$ - Chu upper-morphisms), then $\Phi^{(2)}\Phi^{(1)}$ is a $(n+m)$ - Chu morphism (resp. $(m+n)$ - Chu upper-morphirsm).*

Proof. We carry out the proof in case of $(n+m)$ - Chu morphisms. Let

$$\Phi^{(1)} = (\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m) : (X_1 \times \ldots \times X_n; f; A_1 \times \ldots \times A_m) \to (Y_1 \times \ldots \times Y_n; g; B_1 \times \ldots \times B_m)$$

$$\Phi^{(2)} = (\varphi'_1, \ldots, \varphi'_n, \psi'_1, \ldots, \psi'_m) : (Y_1 \times \ldots \times Y_n; g; B_1 \times \ldots \times B_m) \to (Z_1 \times \ldots \times Z_n; h; C_1 \times \ldots \times C_m)$$

be $(n+m)$ - Chu morphisms. Then

$$\Phi^{(2)}\Phi^{(1)} = (\varphi'_1\varphi_1, \ldots, \varphi'_n\varphi_n; \psi_1\psi'_1, \ldots, \psi_m\psi'_m).$$

Since $\Phi^{(1)}$ and $\Phi^{(2)}$ are $(n+m)$ - Chu morphisms, we have

$$f(\prod_{i=1}^{n} x_i \times \prod_{j=1}^{m} \psi_j(b_j)) = g(\prod_{i=1}^{n} \varphi_i(x_i) \times \prod_{j=1}^{m} b_j) \text{ for } \prod_{i=1}^{n} x_i \times \prod_{j=1}^{m} b_j \in \prod_{i=1}^{n} X_i \times \prod_{j=1}^{m} B_j,$$

$$g(\prod_{i=1}^{n} y_i \times \prod_{j=1}^{m} \psi'_j(c_j)) = h(\prod_{i=1}^{n} \varphi'_i(y_i) \times \prod_{j=1}^{m} c_j) \text{ for } \prod_{i=1}^{n} y_i \times \prod_{j=1}^{m} c_j \in \prod_{i=1}^{n} Y_i \times \prod_{j=1}^{m} C_j.$$

Hence

$$f(\prod_{i=1}^{n} x_i \times \prod_{j=1}^{m} \psi_j\psi'_j(c_j)) = g(\prod_{i=1}^{n} \varphi_i(x_i) \times \prod_{j=1}^{m} \psi'_j(c_j)) = h(\prod_{i=1}^{n} \varphi'_i\varphi_i(x_i) \times \prod_{j=1}^{m} c_j).$$

That is, the diagram below commutes:

$$\begin{array}{ccc}
\prod_{i=1}^n X_i \times \prod_{j=1}^m C_j & \xrightarrow{(\prod_{i=1}^n \varphi_i'\varphi_i, 1_{\prod_{j=1}^m C_j})} & \prod_{i=1}^n Z_i \times \prod_{j=1}^m C_j \\
{\scriptstyle (1_{\prod_{i=1}^n X_i}, \prod_{j=1}^m \psi_j\psi_j')}\downarrow & & \downarrow{\scriptstyle h} \\
\prod_{i=1}^n X_i \times \prod_{j=1}^m A_j & \xrightarrow[f]{} & [0,1]
\end{array}$$

Therefore, the assertion is proved.

By Proposition 1 we can define:

1. The $(n+m)$ - *Chu category*, denoted by $\mathcal{C}$, of $(n+m)$ - dimensional Chu spaces with $(n+m)$ - Chu morphisms.

2. The $(n+m)$ - *Chu upper-category*, denoted by $\mathcal{C}^*$, of $(n+m)$ - dimensional Chu spaces with $(n+m)$ - Chu upper-morphisms.

If $\widetilde{C} = (X_1 \times X_2 \times \ldots \times X_n; f; A_1 \times A_2 \times \ldots \times A_m)$ is an object in the $(n+m)$ - Chu category, then the products $\prod_{i=1}^m X_i$ and $\prod_{j=1}^m A_j$ are called the set of *events* (or *players*) and the set of *states* (or *situations*), respectively.

Let $\widetilde{C} = (X_1 \times X_2 \times \ldots \times X_n; f; A_1 \times A_2 \times \ldots \times A_m)$ be a $(n+m)$ - Chu space. For $\prod_{i=1}^n x_i \in \prod_{i=1}^n X_i$ and $\prod_{j=1}^m a_j \in \prod_{j=1}^m A_j$ we define the *support* of $\prod_{i=1}^n x_i$ and $\prod_{j=1}^m a_j$ by, respectively,

$$\mathrm{Supp}(\prod_{i=1}^n x_i) = \{\prod_{j=1}^m a_j \in \prod_{j=1}^m A_j : f(\prod_{i=1}^n x_i \times \prod_{j=1}^m a_j) > 0\}$$

and

$$\mathrm{Supp}(\prod_{j=1}^m a_j) = \{\prod_{i=1}^n x_i \in \prod_{i=1}^n X_j : f(\prod_{i=1}^n x_i \times \prod_{j=1}^m a_j) > 0\}.$$

For $(x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \ldots \times X_n$ we introduce the following notations:

1. The number $\| \prod_{i=1}^n x_i \|^* = \sup\{f(\prod_{i=1}^n x_i \times \prod_{j=1}^m a_j) : \prod_{j=1}^m a_j \in \prod_{j=1}^m A_j\}$ is called the *upper value* of $\prod_{i=1}^n x_i$.

2. The number $\| \prod_{i=1}^n x_i \|_* = \inf\{f(\prod_{i=1}^n x_i \times \prod_{j=1}^m a_j) : \prod_{j=1}^m a_j \in \prod_{j=1}^m A_j\}$ is called the *lower value* of $\prod_{i=1}^n x_i$.

3. The number $\| \prod_{i=1}^n x_i \| = \frac{1}{2}(\| \prod_{i=1}^n x_i \|^* + \| \prod_{i=1}^n x_i \|_*)$ is called the *valued* of $\prod_{i=1}^n x_i$.

4. The number $d(\prod_{i=1}^n x_i) = \| \prod_{i=1}^n x_i \|^* = -\| \prod_{i=1}^n x_i \|_*$ is called the *deviation* of $\prod_{i=1}^n x_i$.

We can also define the following notations for the whole space $\widetilde{C}$:

1. $M^*(\prod_{i=1}^n X_i, \widetilde{C}) = \sup\{\| \prod_{i=1}^n x_i \|^* : \prod_{i=1}^n x_i \in \prod_{i=1}^n X_i\}$.
The number $M^*(\prod_{i=1}^n X_i, \widetilde{C})$ is called the *upper event value* of $\widetilde{C}$.

2. $M_*(\prod_{i=1}^n X_i, \widetilde{C}) = \inf\{\| \prod_{i=1}^n x_i \|^* : \prod_{i=1}^n x_i \in \prod_{i=1}^n X_i\}$.
The number $M_*(\prod_{i=1}^n X_i, \widetilde{C})$ is called the *minimax event value* of $\widetilde{C}$.

3. $m^*(\prod_{i=1}^n X_i, \widetilde{C}) = \sup\{\| \prod_{i=1}^n x_i \|_* : \prod_{i=1}^n x_i \in \prod_{i=1}^n X_i\}$.

4. $m_*(\prod_{i=1}^n X_i, \widetilde{C}) = \inf\{\| \prod_{i=1}^n x_i \|_* : \prod_{i=1}^n x_i \in \prod_{i=1}^n X_i\}$.

Dually, we can defined values $\| \prod_{j=1}^m a_j \|^*$, $\| \prod_{j=1}^m a_j \|_*$, $\| \prod_{j=1}^m a_j \|$, $d(\prod_{j=1}^m a_j)$ for a state $\prod_{j=1}^m a_j \in \prod_{j=1}^m A_j$, and the numbers $M^*(\prod_{j=1}^m A_j, \widetilde{C})$, $M_*(\prod_{j=1}^m A_j, \widetilde{C})$, $m^*(\prod_{j=1}^m A_j, \widetilde{C})$, $m_*(\prod_{j=1}^m A_j, \widetilde{C})$ in the same way. For instance:

$$\| \prod_{j=1}^m a_j |^* = \sup\{f(\prod_{i=1}^n x_i \times \prod_{j=1}^m a_j) : \prod_{i=1}^n x_i \in \prod_{i=1}^n X_i\}.$$

Roughly speaking, for an event $(x_1, x_2, \dots, x_n) \in X_1 \times X_2 \times \dots \times X_n$ the upper value $\| \prod_{i=1}^n x_i \|^*$ measures the "skill" of $\prod_{i=1}^n x_i$ in the best situation and the lower value $\| \prod_{i=1}^n x_i \|_*$ measures the "skil" of $\prod_{i=1}^n x_i$ in the worst situation. An event $\prod_{i=1}^n x_i \in \prod_{i=1}^n X_i$ is called a *strong event* if $\| \prod_{i=1}^n x_i \| = 1$, and a *null event* if $\| \prod_{i=1}^n x_i \| = 0$, or equivalently $\text{supp}(\prod_{i=1}^n x_i) = \emptyset$.

Dually, for state $\prod_{j=1}^m a_j \in \prod_{j=1}^m A_j$ the upper value $\| \prod_{j=1}^m a_j \|^*$ describes the quality of the position $\prod_{j=1}^m a_j$ if a best player is staying there, and the lower value $\| \prod_{j=1}^m a_j \|_*$ describes the quality of the position $\prod_{j=1}^m a_j$ if a worst player is staying there. A state $\prod_{j=1}^m a_j \in \prod_{j=1}^m A_j$ is called a *winning state* if $\| \prod_{j=1}^m a_j \| = 1$, and a *dead state* if $\| \prod_{j=1}^m a_j \| = 0$, or equivalently $\text{supp}(\prod_{j=1}^m a_j) = \emptyset$.

We can define the Chu distances $\| \prod_{i=1}^n x_i - \prod_{i=1}^n y_i \|$ between two events $x = \prod_{i=1}^n x_i$, $y = \prod_{i=1}^n y_i$ and $\| \prod_{j=1}^m a_j - \prod_{j=1}^m b_j \|$ between two states $a = \prod_{j=1}^m a_j$ and $b = \prod_{j=1}^m b_j$ by, respectively,

$$\|x - y\| = \sup\{|f(\prod_{i=1}^n x_i \times \prod_{j=1}^m a_j) - f(\prod_{i=1}^n y_i \times \prod_{j=1}^m a_j)| : \prod_{j=1}^m a_j \in \prod_{j=1}^m A_j\}$$

and

$$\|a - b\| = \sup\{|f(\prod_{i=1}^n x_i \times \prod_{j=1}^m a_j) - f(\prod_{i=1}^n x_i \times \prod_{j=1}^m b_j)| : \prod_{i=1}^n x_i \in \prod_{i=1}^n X_i\}.$$

A $(n+m)$ - Chu space $\widetilde{C}$ is *separated*, see [2], if $\| \prod_{j=1}^m a_j - \prod_{j=1}^m b_j \| = 0$ implies $a_j = b_j$ for $j = 1, \dots, m$ and $\widetilde{C}$ is *extensional* if $\| \prod_{i=1}^n x_i - \prod_{i=1}^n y_i \| = 0$ implies $x_i = y_i$ for $i = 1, \dots, n$.

If $\widetilde{C}$ is both separated and extensional then we say that $\widetilde{C}$ is *biextensional*.

Clearly the Chu distances define pseudometrics on $\prod_{i=1}^n X_i$ and $\prod_{j=1}^m A_j$. Hence

Proposition 2. *If $\widetilde{C}$ is separated (resp. extensional) then $\prod_{j=1}^m A_j$ (resp. $\prod_{i=1}^n X_i$) is a metric space with the Chu distance. Therefore if $\widetilde{C}$ is biextensional then both $\prod_{j=1}^m A_j$ and $\prod_{i=1}^n X_i$ are metric spaces.*

We say that a $(n+m)$ - Chu space $\widetilde{C} = (X_1 \times \dots \times X_n; f; A_1 \times \dots \times A_m)$ in the $(n+m)$ - Chu catogory is *complete* if for any function $\varphi : \prod_{i=1}^n X_i \to [0,1]$ there exists a state $\prod_{j=1}^m a_j \in \prod_{j=1}^m A_j$ such that $\varphi(\prod_{i=1}^n x_i) = f(\prod_{i=1}^n x_i \times \prod_{j=1}^m a_j)$ for every $\prod_{i=1}^n x_i \in \prod_{i=1}^n X_i$. We say $\widetilde{C}$ *fully complete* if $\widetilde{C}$ is complete and separated.

For $(n+m)$ - Chu spaces $\widetilde{C} = (X_1 \times \dots \times X_n; f; A_1 \times \dots \times A_m)$ and $\widetilde{D} = (Y_1 \times \dots \times Y_n; g; B_1 \times \dots \times B_m)$ let $\mathcal{M}(\widetilde{C}, \widetilde{D})$ (resp. $\mathcal{M}^*(\widetilde{C}, \widetilde{D})$) denote the set of all $(n+m)$ - Chu morphisms (resp. $(n+m)$ - Chu upper-morphisms) from $\widetilde{C}$ into $\widetilde{D}$.

For $\mathcal{M}(\widetilde{C}, \widetilde{D}) \neq \emptyset$ we have the following proposition.

Proposition 3. *Let $\widetilde{C} = (X_1 \times \dots \times X_n; f; A_1 \times \dots \times A_m)$ and $\widetilde{D} = (Y_1 \times \dots \times Y_n; g; B_1 \times \dots \times B_m)$ be $(n+m)$ - Chu spaces. If $\mathcal{M}(\widetilde{C}, \widetilde{D}) \neq \emptyset$, then $M^*(\prod_{i=1}^n X_i, \widetilde{C}) \geq M_*(\prod_{i=1}^n Y_i, \widetilde{D})$.*

Proof. In fact, if it is not the case, then

$$\| \prod_{i=1}^n x_i \|^* < \| \prod_{i=1}^n y_i \|^* \text{ for any } \prod_{i=1}^n x_i \in \prod_{i=1}^n X_i \text{ and } \prod_{i=1}^n y_i \in \prod_{i=1}^n Y_i . \tag{4}$$

On the other hand, since $\mathcal{M}(\widetilde{C}, \widetilde{D}) \neq \emptyset$ there exists a morphism

$$\Phi = (\varphi_1, \dots, \varphi_n; \psi_1, \dots, \psi_m),$$

where $\prod_{i=1}^n \varphi_i : \prod_{i=1}^n X_i \to \prod_{i=1}^n Y_i$ and $\prod_{j=1}^m \psi_j : \prod_{j=1}^m B_j \to \prod_{j=1}^m A_j$ such that

$$f(\prod_{i=1}^{n} x_i \times \prod_{j=1}^{m} \psi_j(b_i)) = g(\prod_{i=1}^{n} \varphi_i(x_i) \times \prod_{j=1}^{m} b_j)$$

for $\prod_{i=1}^{n} x_i \in \prod_{i=1}^{n} X_i$ and $\prod_{j=1}^{m} b_j \in \prod_{j=1}^{m} B_j$.

It follows that

$$\begin{aligned} \| \prod_{i=1}^{n} x_i \|^* &= \sup\{f(\prod_{i=1}^{n} x_i \times \prod_{j=1}^{m} a_j) : \prod_{j=1}^{m} a_j \in \prod_{j=1}^{m} A_j\} \\ &\geq \sup\{f(\prod_{i=1}^{n} x_i \times \prod_{j=1}^{m} \varphi_j(b_j)) : \prod_{j=1}^{m} b_j \in \prod_{j=1}^{m} B_j\} \\ &= \sup\{g(\prod_{i=1}^{n} \varphi_i(x_i) \times \prod_{j=1}^{m} b_j) : \prod_{j=1}^{m} b_j \in \prod_{j=1}^{m} B_j\} \\ &= \| \prod_{i=1}^{n} \varphi_i(x_i) \|^*, \end{aligned}$$

which contradicts (4). Consequently $M^*(\prod_{i=1}^{n} X_i, \widetilde{C}) \geq M_*(\prod_{i=1}^{n} Y_i, \widetilde{D})$ and the proposition is proved.

Let $\widetilde{C}, \widetilde{D} \in \mathcal{C}$. If $\mathcal{M}(\widetilde{C}, \widetilde{D}) \neq \emptyset$, then way say that $\widetilde{C}$ is *dominated* by $\widetilde{D}$ and denote $\widetilde{C} \preceq \widetilde{D}$. We say that $\widetilde{C}$ and $\widetilde{D}$ are *equivalent*, denoted by $\widetilde{C} \approx \widetilde{D}$, if $\widetilde{C} \preceq \widetilde{D}$ and $\widetilde{D} \preceq \widetilde{C}$, and $\widetilde{C}$ and $\widetilde{D}$ are *connected* if either $\widetilde{C} \preceq \widetilde{D}$ or $\widetilde{D} \preceq \widetilde{C}$. A class of $(n+m)$ - Chu spaces $\mathcal{G}$ is called a *connected system* if any two members of $\mathcal{G}$ are connected. If $\widetilde{C} \approx \widetilde{D}$ for every $\widetilde{C}, \widetilde{D} \in \mathcal{G}$, then we say that $\mathcal{G}$ is an *equivalent system*. A connected system is called a *closed system* if $\mathcal{G}$ is closed under cross products. That is, $\widetilde{C} \times_\Phi \widetilde{D} \in \mathcal{G}$ for every $\widetilde{C}, \widetilde{D} \in \mathcal{G}$ and $\Phi \in \mathcal{M}(\widetilde{C}, \widetilde{D})$. A *complete system* is a closed equivalent system.

We say that $\widetilde{C}$ and $\widetilde{D}$ are *isomorphic*, denoted by $\widetilde{C} \cong \widetilde{D}$, if $\widetilde{C}$ and $\widetilde{D}$ are isomorphic objects in the category $\mathcal{C}$ of $(n+m)$ - Chu spaces. It is easy to see that a $(n+m)$ - Chu morphism $\Phi = (\varphi_1, \ldots, \varphi_n; \psi_1, \ldots, \psi_n) : (X_1 \times \ldots \times X_n; f; A_1 \times \ldots \times A_m) \to (Y_1 \times \ldots \times Y_n; g; B_1 \times \ldots \times B_m)$ is an isomorphism if and only if $\varphi_i : X_i \to Y_i$ for $i = 1, \ldots, n$ and $\psi_j : B_j \to A_j$ for $j = 1, \ldots, m$ are one-to-one and onto. If $\Phi = (\varphi_1, \ldots, \varphi_n; \psi_1, \ldots, \psi_m)$ is a $(n+m)$ - monomorphism, then we say that $\widetilde{C} = (X_1 \times \ldots \times X_n; f; A_1 \times \ldots \times A_m)$ is a *subspace* of $\widetilde{D} = (Y_1 \times \ldots \times Y_n; g; B_1 \times \ldots \times B_m)$, denoted by $\widetilde{C} \subseteq \widetilde{D}$. Finally, the $(n+m)$ - Chu space $\widetilde{C}^\perp = (X_1 \times \ldots \times X_n; 1-f; A_1 \times \ldots \times A_m)$ is called the *complement* of $\widetilde{C} = (X_1 \times \ldots \times X_n; f; A_1 \times \ldots \times A_m)$.

Proposition 4. *A $(n+m)$ - morphism $\Phi = (\varphi_1, \ldots, \varphi_n; \psi_1, \ldots, \psi_m) : \widetilde{C} \to \widetilde{D}$ is a $(n+m)$ - monomorphism if and only if for every $i = 1, \ldots, n$, $\varphi_i : X_i \to Y_i$ is one-to-one and for every $j = 1, \ldots, m$, $\psi_j : B_j \to A_j$ is onto.*

Proof. Assume that $\widetilde{C} = (X_1 \times \ldots \times X_n; f; A_1 \times \ldots \times A_m)$ is a subspace of $\widetilde{D} = (Y_1 \times \ldots \times Y_n; g; B_1 \times \ldots \times B_m)$, $\Phi^{(1)} = (\varphi'_1, \ldots, \varphi'_n; \psi'_1, \ldots, \psi'_m)$ and $\Phi^{(2)} = (\varphi''_1, \ldots, \varphi''_n, \psi''_1, \ldots, \psi''_m)$ are two $(n+m)$ - morphisms with the same target $\widetilde{C} = (X_1 \times \ldots \times X_n; f; A_1 \times \ldots \times A_m)$ such that $\Phi\Phi^{(1)} = \Phi\Phi^{(2)}$.

Then

$$(\varphi_1\varphi'_1, \ldots, \varphi_n\varphi'_n) = (\varphi_1\varphi''_1, \ldots, \varphi_n\varphi''_n) \tag{5}$$

and

$$(\psi_1\psi'_1, \ldots, \psi_m\psi'_m) = (\psi_1\psi''_1, \ldots, \psi_m\psi''_m). \tag{6}$$

Since φ_i is one-to-one for $i = 1, \ldots, n$ and ψ_j is onto for $j = 1, \ldots, m$, from (5) and (6) we infer that

$$\varphi'_i = \varphi''_i \text{ for } i = 1, \ldots, n \text{ and } \psi'_j = \psi''_j \text{ for } j = 1, \ldots, m.$$

Hence
$$\Phi^{(1)} = \Phi^{(2)},$$
that is, Φ is a $(n+m)$ - monomorphism.

Proposition 5. *For any* $(n+m)$ *- Chu space* $\widetilde{C} = (X_1 \times X_2 \times \ldots \times X_n; f; A_1 \times \ldots \times A_m)$ *we have*

1. $m_*(\prod_{i=1}^n X_i, \widetilde{C}) \leq m^*(\prod_{i=1}^n X_i, \widetilde{C})$, $M_*(\prod_{i=1}^n X_i, \widetilde{C}) \leq M^*(\prod_{i=1}^n X_i, \widetilde{C})$.

2. $m_*(\prod_{i=1}^n X_i, \widetilde{C}^{\perp}) = 1 - M^*(\prod_{i=1}^n X_i, \widetilde{C})$ *and* $m^*(\prod_{i=1}^n X_i, \widetilde{C}^{\perp}) = 1 - M_*(\prod_{i=1}^n X_i, \widetilde{C})$.

Proof. 1. An easy proof is omitted.

2.
$$\begin{aligned}
m_*(\prod_{i=1}^n X_i, \widetilde{C}^{\perp}) &= \inf\{\|\prod_{i=1}^n x_i\|_* : \prod_{i=1}^n x_i \in \prod_{i=1}^n X_i\}\\
&= \inf\{\inf\{1 - f(\prod_{i=1}^n x_i \times \prod_{j=1}^m a_j) : \prod_{j=1}^m a_j \in \prod_{j=1}^m A_j\} : \prod_{i=1}^n x_i \in \prod_{i=1}^n X_i\}\\
&= \inf\{1 - \sup\{f(\prod_{i=1}^n x_i \times \prod_{j=1}^m a_j) : \prod_{j=1}^m a_j \in \prod_{j=1}^m A_j\} : \prod_{i=1}^n x_i \in \prod_{i=1}^n X_i\}\\
&= 1 - \sup\{\sup\{f(\prod_{i=1}^n x_i \times \prod_{j=1}^m a_j) : \prod_{j=1}^m a_j \in \prod_{j=1}^m A_j\} : \prod_{i=1}^n x_i \in \prod_{i=1}^n X_i\}\\
&= 1 - \sup\{\|\prod_{i=1}^n x_i\|^* : \prod_{i=1}^n x_i \in \prod_{i=1}^n X_i\}\\
&= 1 - M^*(\prod_{i=1}^n X_i, \widetilde{C}).
\end{aligned}$$

$$\begin{aligned}
m^*(\prod_{i=1}^n X_i, \widetilde{C}^{\perp}) &= \sup\{\inf\{1 - f(\prod_{i=1}^n x_i \times \prod_{j=1}^m a_j) : \prod_{j=1}^m a_j \in \prod_{j=1}^m A_j\} : \prod_{i=1}^n x_i \in \prod_{i=1}^n X_i\}\\
&= \sup\{1 - \sup\{f(\prod_{i=1}^n x_i \times \prod_{j=1}^m a_j) : \prod_{j=1}^m a_j \in \prod_{j=1}^m A_j\} : \prod_{i=1}^n x_i \in \prod_{i=1}^n X_i\}\\
&= 1 - \inf\{\sup\{f(\prod_{i=1}^n x_i \times \prod_{j=1}^m a_j) : \prod_{j=1}^m a_j \in \prod_{j=1}^m A_j\} : \prod_{i=1}^n x_i \in \prod_{i=1}^n X_i\}\\
&= 1 - M_*(\prod_{i=1}^n X_i, \widetilde{C}).
\end{aligned}$$

Of course Proposition 5 still holds if the set $\prod_{i=1}^n X_i$ of events is replaced by the set $\prod_{j=1}^m A_j$ of states.

Observe that if $m^*(\prod_{i=1}^n X_i, \widetilde{C}) > M_*(\prod_{i=1}^n X_i, \widetilde{C})$ then $\|\prod_{i=1}^n x_i\|_* > \|\prod_{i=1}^n y_i\|^*$ for some $\prod_{i=1}^n x_i, \prod_{i=1}^n y_i \in \prod_{i=1}^n X_i$. This means that in the worst situation the players x_i $(i = 1, \ldots, n)$ can do better than the players y_i $(i = 1, \ldots, n)$ even when y_i are in the best situation. Clearly, in this situation the qualification of the set $\prod_{i=1}^n X_i$ is "every un-uniform". We say that $(n+m)$ - Chu space $\widetilde{C}$ is *event uniform* (resp. *state uniform*) if $m^*(\prod_{i=1}^n X_i, \widetilde{C}) \leq M_*(\prod_{i=1}^n X_i, \widetilde{C})$ (resp. $m^*(\prod_{j=1}^m A_j, \widetilde{C}) \leq M_*(\prod_{j=1}^m A_j, \widetilde{C})$), and $\widetilde{C}$ is *uniform* if it both event and state uniform. From the Proposition 5 we get

Proposition 6. *For any uniform* $(n+m)$ *- Chu space* $\widetilde{C} = (X_1 \times \ldots \times X_n; f; A_1 \times \ldots \times A_m)$*:*

1. $m_*(\prod_{i=1}^n X_i, \widetilde{C}) \leq m^*(\prod_{i=1}^n X_i, \widetilde{C}) \leq M_*(\prod_{i=1}^n X_i, \widetilde{C}) \leq M^*(\prod_{i=1}^n X_i, \widetilde{C})$.

2. $m_*(\prod_{j=1}^m A_j, \widetilde{C}) \leq m^*(\prod_{j=1}^m A_j, \widetilde{C}) \leq M_*(\prod_{j=1}^m A_j, \widetilde{C}) \leq M^*(\prod_{j=1}^m A_j, \widetilde{C})$.

We prove following theorem

Theorem 1. *Let $\widetilde{C} = (X_1 \times \ldots \times X_n; f; A_1 \times \ldots \times A_m)$ and $\widetilde{D} = (Y_1 \times \ldots \times Y_n; g; B_1 \times \ldots \times B_m)$ be $(n+m)$ - Chu spaces. If $\widetilde{C} \subseteq \widetilde{D}$ then*

1. $M^*(\prod_{i=1}^n X_i, \widetilde{C}) \leq M^*(\prod_{i=1}^n Y_i, \widetilde{D})$,

2. $M_*(\prod_{i=1}^n X_i, \widetilde{)} \geq M_*(\prod_{i=1}^n Y_i, \widetilde{D})$,

3. $m^*(\prod_{i=1}^n X_i, \widetilde{C}) \leq m^*(\prod_{i=1}^n Y_i, \widetilde{D})$,

4. $m_*(\prod_{i=1}^n X_i, \widetilde{C}) \geq m_*(\prod_{i=1}^n Y_i, \widetilde{D})$.

Therefore, if $\widetilde{C}$ and $\widetilde{D}$ are isomorphic, then

5. $M^*(\prod_{i=1}^n X_i, \widetilde{C}) = M^*(\prod_{i=1}^n Y_i, \widetilde{D})$,

6. $M_*(\prod_{i=1}^n X_i, \widetilde{C}) = M_*(\prod_{i=1}^n Y_i, \widetilde{D})$,

7. $m^*(\prod_{i=1}^n X_i, \widetilde{C}) = m^*(\prod_{i=1}^n Y_i, \widetilde{D})$,

8. $m_*(\prod_{i=1}^n X_i, \widetilde{C}) = m_*(\prod_{i=1}^n Y_i, \widetilde{D})$.

Proof.

1.
$$\begin{aligned}
M^*(\prod_{i=1}^n X_i, \widetilde{C}) &= \sup\{\sup\{f(\prod_{i=1}^n x_i \times \prod_{j=1}^m a_j) : \prod_{j=1}^m a_j \in \prod_{j=1}^m A_j\} : \prod_{i=1}^n x_i \in \prod_{i=1}^n X_i\} \\
&= \sup\{\sup\{f(\prod_{i=1}^n x_i \times \prod_{j=1}^m \psi_j(b_j)) : \prod_{j=1}^m b_j \in \prod_{j=1}^m B_j\} : \prod_{i=1}^n x_i \in \prod_{i=1}^n X_i\} \\
&= \sup\{\sup\{g(\prod_{i=1}^n \varphi_i(x_i) \times \prod_{j=1}^m b_j) : \prod_{j=1}^m b_j \in \prod_{j=1}^m B_j\} : \prod_{i=1}^n x_i \in \prod_{i=1}^n X_i\} \\
&\leq \sup\{\sup\{g(\prod_{i=1}^n y_i \times \prod_{j=1}^m b_j) : \prod_{j=1}^m b_j \in \prod_{j=1}^m B_j\} : \prod_{i=1}^n y_i \in \prod_{i=1}^n Y_i\} \\
&= \sup\{\| \prod_{i=1}^n y_i \|^* : \prod_{i=1}^n y_i \in \prod_{i=1}^n Y_i\} \\
&= M^*(\prod_{i=1}^n Y_i, \widetilde{D}).
\end{aligned}$$

2.
$$\begin{aligned}
M_*(\prod_{i=1}^n X_i, \widetilde{C}) &= \inf\{\sup\{f(\prod_{i=1}^n x_i \times \prod_{j=1}^m a_j) : \prod_{j=1}^m a_j \in \prod_{j=1}^m A_j\} : \prod_{i=1}^n x_i \in \prod_{i=1}^n X_i\} \\
&= \inf\{\sup\{f(\prod_{i=1}^n x_i \times \prod_{j=1}^m \psi_j(b_j)) : \prod_{j=1}^m b_j \in \prod_{j=1}^m B_j\} : \prod_{i=1}^n x_i \in \prod_{i=1}^n X_i\} \\
&= \inf\{\sup\{g(\prod_{i=1}^n \varphi_i(x_i) \times \prod_{j=1}^m b_j) : \prod_{j=1}^m b_j \in \prod_{j=1}^m B_j\} : \prod_{i=1}^n x_i \in \prod_{i=1}^n X_i\} \\
&= \inf\{\| \prod_{i=1}^n \varphi_i(x_i) \|^* : \prod_{i=1}^n x_i \in \prod_{i=1}^n X_i\} \\
&\geq \inf\{\| \prod_{i=1}^n y_i \|^* : \prod_{i=1}^n y_i \in \prod_{i=1}^n Y_i\} \\
&= M_*(\prod_{i=1}^n Y_i, \widetilde{D}).
\end{aligned}$$

3.
$$
\begin{aligned}
m^*(\prod_{i=1}^{n} X_i, \tilde{C}) &= \sup\{\inf\{f(\prod_{i=1}^{n} x_i \times \prod_{j=1}^{m} a_j) : \prod_{j=1}^{m} a_j \in \prod_{j=1}^{m} A_j\} : \prod_{i=1}^{n} x_i \in \prod_{i=1}^{n} X_i\} \\
&= \sup\{\inf\{f(\prod_{i=1}^{n} x_i \times \prod_{j=1}^{m} \psi_j(b_j)) : \prod_{j=1}^{m} b_j \in \prod_{j=1}^{m} B_j\} : \prod_{i=1}^{n} x_i \in \prod_{i=1}^{n} X_i\} \\
&= \sup\{\inf\{g(\prod_{i=1}^{n} \varphi_i(x_i) \times \prod_{j=1}^{m} b_j) : \prod_{j=1}^{m} b_j \in \prod_{j=1}^{m} B_j\} : \prod_{i=1}^{n} x_i \in \prod_{i=1}^{n} X_i\} \\
&= \sup\{\| \prod_{i=1}^{n} \varphi_i(x_i) \|_* : \prod_{i=1}^{n} x_i \in \prod_{i=1}^{n} X_i\} \\
&\leq \sup\{\| \prod_{i=1}^{n} y_i \|_* : \prod_{i=1}^{n} y_i \in \prod_{i=1}^{n} Y_i\} \\
&= m^*(\prod_{i=1}^{n} Y_i, \tilde{D}).
\end{aligned}
$$

4.
$$
\begin{aligned}
m_*(\prod_{i=1}^{n} X_i, \tilde{C}) &= \inf\{\inf\{f(\prod_{i=1}^{n} x_i \times \prod_{j=1}^{m} a_j) : \prod_{j=1}^{m} a_j \in \prod_{j=1}^{m} A_j\} : \prod_{i=1}^{n} x_i \in \prod_{i=1}^{n} X_i\} \\
&= \inf\{\inf\{f(\prod_{i=1}^{n} x_i \times \prod_{j=1}^{m} \psi_j(b_j)) : \prod_{j=1}^{m} b_j \in \prod_{j=1}^{m} B_j\} : \prod_{i=1}^{n} x_i \in \prod_{i=1}^{n} X_i\} \\
&= \inf\{\inf\{g(\prod_{i=1}^{n} \varphi_i(x_i) \times \prod_{j=1}^{m} b_j) : \prod_{j=1}^{m} b_j \in \prod_{j=1}^{m} B_j\} : \prod_{i=1}^{n} x_i \in \prod_{i=1}^{n} X_i\} \\
&= \inf\{\| \prod_{i=1}^{n} \varphi_i(x_i) \|_* : \prod_{i=1}^{n} x_i \in \prod_{i=1}^{n} X_i\} \\
&\geq \inf\{\| \prod_{i=1}^{n} y_i \|_* : \prod_{i=1}^{n} y_i \in \prod_{i=1}^{n} Y_i\} \\
&= m_*(\prod_{i=1}^{n} Y_i, \tilde{D}).
\end{aligned}
$$

3. FUZZY SPACES AND RELATIONS BETWEEN FUZZY SPACES AND CHU SPACES

In this section we introduce a special class of Chu spaces called *fuzzy spaces*, and similarly as [3] we show that a pre-fuzzy space is a fuzzy space if and only if it is fully complete.

By a *fuzzy subset* of a set $X = X_1 \times X_2 \times \ldots \times X_n$ we mean any function $f : X \to [0,1]$, see [6]. Observe that if $A = A_1 \times \ldots \times A_n$ is a subset of X, then the characteristic function χ_A of A is a fuzzy subset of X. So by identifying A with χ_A we can say that any subset of X is a fyzzy subset of X. A fuzzy subset of X is also simply called a *fuzzy set*.

Let S denote the category of sets. For a given set $X = X_1 \times X_2 \ldots \times X_n$, let $X^* = (X_1 \times X_2 \times \ldots \times X_n)^* = [0,1]^X$ denote the collection of all fuzzy sets of X.

For any set $A \subset X^*$ we define $f_A : X_1 \times \ldots \times X_n \times A \to [0,1]$ by

$$f_A(x_1, \ldots, x_n, a) = a(x_1, \ldots, x_n) \text{ for } (x_1, \ldots, x_n, a) \in X_1 \times \ldots \times X_n \times A. \quad (7)$$

Clearly $\widetilde{C} = (X_1 \times X_2 \times \ldots \times X_n; f_A; A)$ is a $(n+1)$-dimensional Chu space. This space is called a *pre-fuzzy space* on X. In the case $A = (X_1 \times X_2 \times \ldots \times X_n)$ the $(n+1)$-Chu space $F(X) = (X_1 \times \times \ldots \times X_n; f_{X^*}; X^*)$ is uniquely determined by X, and is called the *fuzzy space associate with* X, or shortly, *fuzzy space*.

We will show that

Proposition 7. *Any pre-fuzzy space is separated, but not necessarily extensional. However any fuzzy space* $F(X) = (X_1 \times X_2 \times \ldots \times X_n; f_{X^*}; X^*)$ *is dully complete and biextensional.*

Proof. We show firstly that the pre-fuzzy space is separated. Assume that

$$\|a-b\| = \sup\{|f(x_1, x_2, \ldots, x_n, a) - f(x_1, x_2, \ldots x_n, b)| : (x_1, x_2, \ldots, x_n) \in \prod_{i=1}^{n} X_i\} = 0.$$

Then

$$f(x_1, x_2, \ldots, x_n, a) = f(x_1, x_2, \ldots x_n, b) \text{ for every } (x_1, x_2, \ldots, x_n) \in (X_1 \times X_2 \times \ldots \times X_n).$$

Hence

$$a(x_1, x_2, \ldots, x_n) = b(x_1, x_2, \ldots x_n) \text{ for every } (x_1, x_2, \ldots, x_n) \in \prod_{i=1}^{n} X_i.$$

That is $a = b$ and the pre-fuzzy space is separated.

Now we are going to show that the pre-fuzzy space is not necessary extensional. In fact, let $A = \{a \in (X_1, \ldots, X_n)^* : a \equiv 1\}$. Then for $x = (x_1, x_2, \ldots, x_n) \neq y = (y_1, y_2, \ldots, y_n)$ we obtain

$$\|x-y\| = \sup\{|f(x_1, x_2, \ldots, x_n, a) - f(y_1, y_2, \ldots y_n, a)| : a \in A\} = 0,$$

that is, $\widetilde{C} = (X_1 \times X_2 \times \ldots \times X_n; f_A; A)$ is not extensional.

In the next, we show that $F(X) = (X_1 \times X_2 \times \ldots \times X_n; f_{X^*}; X^*)$ is biextensional. Since $F(X)$ is separated, it is sufficient to show that $F(X)$ is extensional.

Assume that

$$\|x-y\| = \sup\{|f(x_1, x_2, \ldots, x_n, a) - f(y_1, y_2, \ldots y_n, a)| : a \in X^*\} = 0,$$

thus, $f(x_1, x_2, \ldots, x_n, a) = f(y_1, y_2, \ldots, y_n, a)$ for all $a \in X^*$.

Hence

$$a(x_1, x_2, \ldots, x_n) - a(y_1, y_2, \ldots y_n) \text{ for all } a \in X^*.$$

Let $a = \chi_{\{x\}}$, then $a(x_1, x_2, \ldots, x_n) = 1$. It implies that $a(y) = \chi_{\{x\}}(y) = 1$, therefore $x = y$.

Finnaly, we claim that $F(X) = (X_1 \times X_2 \times \ldots \times X_n; f_{X^*}; X^*)$ is complete.

Let $\varphi : X_1 \times X_2 \times \ldots \times X_n \to [0,1]$ be a map of set $X = X_1 \times X_2 \times \ldots \times X_n$ into the interval $[0,1]$. Then $\varphi \in X^*$. Thus, with $a = \varphi \in X^*$ we have

$$f(x_1, x_2, \ldots, x_n, a) = \varphi(x_1, x_2, \ldots x_n) \text{ for every } (x_1, x_2, \ldots, x_n) \in X.$$

The theorem is peoved.

Since the fuzzy space $F(X) = (X_1 \times X_2 \times \ldots \times X_n; F_{X^*}; X^*)$ is biextensional, by Proposition 1 the Chu distance on $X = X_1 \times X_2 \times \ldots \times X_n$ defines a metric. It is easy to see that it is a *discrete metric*.

The category of pre-fuzzy spaces with $(n+1)$ - Chu morphisms is called the pre-fuzzy category and denoted by $\mathcal{F}_P$. The fuzzy category, denoted by $\mathcal{F}$, is the subcategory of $\mathcal{F}_P$ consisting of fuzzy spaces.

For any map $\alpha : X = X_1 \times X_2 \times \ldots \times X_n \to Y = Y_1 \times Y_2 \times \ldots \times Y_n$ we define the conjugate $\alpha^* : Y^* \to X^*$ of α by the formula

$$\alpha^*(a)(x) = a(\alpha(x)) \text{ for every } x \in X \text{ and } a \in Y^*.$$

It is easy to see that

$$(\beta\alpha)^* = \alpha^*\beta^* \text{ for every } \alpha : X \to Y \text{ and } \beta : Y \to Z.$$

Observe that a $(n+1)$ - Chu morphism $\Phi : \tilde{C} = (X_1 \times X_2 \times \ldots \times X_n; f_A; A) \to \tilde{D} = (Y_1 \times Y_2 \times \ldots \times Y_n; f_B; B)$ in the pre-fuzzy category is a collection of maps $\Phi = (\varphi_1, \varphi_2, \ldots, \varphi_n; \psi)$, where

$$\prod_{i=1}^n \varphi_i : \prod_{i=1}^n X_i \to \prod_{i=1}^n Y_i \text{ with } (\prod_{i=1}^n \varphi_i)(\prod_{i=1}^n x_i) = \prod_{i=1}^n \varphi_i(x_i) \in \prod_{i=1}^n Y_i,$$

and $\psi : B \to A$ satisfy the condition

$$\psi(b)(\prod_{i=1}^n x_i) = b(\prod_{i=1}^n \varphi_i(x_i)) \text{ for } (x_1, \ldots, x_n, b) \in X \times B. \tag{8}$$

In fact, since bellow diagram commutes

$$\begin{array}{ccc} \prod_{i=1}^n X_i \times B & \xrightarrow{(\prod_{i=1}^n \varphi_i, 1_B)} & \prod_{i=1}^n Y_i \times B \\ {\scriptstyle (1_{\prod_{i=1}^n X_i}, \psi)} \downarrow & & \downarrow {\scriptstyle f_B} \\ \prod_{i=1}^n X_i \times A & \xrightarrow[f_A]{} & [0,1] \end{array}$$

we have for every $\prod_{i=1}^n x_i \in \prod_{i=1}^n X_i$ and $b \in B$

$$\begin{aligned} b(\prod_{i=1}^n \varphi_i(x_i)) &= f_B(\prod_{i=1}^n \varphi_i(x_i), b) \\ &= f_A(\prod_{i=1}^n x_i, \psi(b)) \\ &= \psi(b)(\prod_{i=1}^n x_i). \end{aligned}$$

That is (8) is claimed.

Thus, Proposition 7 shows that, any fuzzy space $F(X) = (X_1 \times X_2 \times \ldots \times X_n; f_{X^*}; X^*)$ associated with a set X is fully complete. Conversely, we have the following theorem.

Theorem 2. *A $(n+1)$ - Chu space $\tilde{C} = (X_1 \times X_2 \times \ldots \times X_n; f; A)$ is a fuzzy space if $\tilde{C}$ is fully complete.*

Proof. Let $\tilde{C} = (X_1 \times X_2 \times \ldots \times X_n; f; A)$ be a fully complete $(n+1)$ - Chu space. Then $F(X) = (X_1 \times X_2 \times \ldots \times X_n; f_{X^*}; X^*)$ is a fuzzy space. We will show that $\tilde{C}$ and $F(X)$ are isomorphic. To see this we first define $T : A \to X^*$ by

$$T(a)f(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots x_n, a) \text{ for every } (x_1, x_2, \ldots, x_n) \in (X_1 \times X_2 \times \ldots \times X_n). \tag{9}$$

We claim that T is one-to-one. In fact, assume that $a, b \in A$ and $a \neq b$. Since $\widetilde{C}$ is separated, there exists $x \in X$ such that $f(x_1, x_2, \ldots, x_n, a) \neq f(x_1, x_2, \ldots, x_n, b)$. Hence $T(a) \neq T(b)$.

To see that T is onto, let $\varphi \in X^*$. Then $\varphi : X \to [0,1]$. Since $\widetilde{C}$ is state complete there exists a state $a \in A$ such that

$$\varphi(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots x_n, a) \text{ for every } (x_1, x_2, \ldots, x_n) \in (X_1 \times X_2 \times \ldots \times X_n).$$

It follows that $T(a) = \varphi$.

Now we define

$$\Phi = (1_X, T^{-1}) : \widetilde{C} = (X_1 \times X_2 \times \ldots \times X_n; f; A) \to F(X) = (X_1 \times X_2 \times \ldots \times X_n; f_{X^*}; X^*),$$

$$\Psi = (1_X, T) : F(X) = (X_1 \times X_2 \times \ldots \times X_n; f_{X^*}; X^*) \to \widetilde{C} = (X_1 \times X_2 \times \ldots \times X_n; f; A).$$

From (7) and (9) we get

$$f(x, T^{-1}(a)) = TT^{-1}(a)(x) = a(x) = f_{X^*}(x, a)$$

for every $x \in X$ and $a \in X^*$, and by (9)

$$f_{X^*}(x, T(a)) = T(a)(x) = f(x, a)$$

for every $x \in X$ and $a \in A$. Therefore $\Phi = (1_X, T^{-1})$ and $\Psi = (1_X, T)$ are $(n+1)$ - Chu morphisms.

It is easy to see that $\Psi\Phi = 1_{\widetilde{C}}$ and $\Phi\Psi = 1_{F(X)}$. Consequently $\widetilde{C}$ and $F(\widetilde{C})$ are isomorphic, and the theorem is proved.

Example 2 (In forming a National Football Team to win a match). Given the set $A = A_1 \times A_2 \times A_3 \times A_4$, by a *game space over* $A = A_1 \times A_2 \times A_3 \times A_4$ we mean a 15-dimensional Chu space $\widetilde{C} = (X_1 \times X_2 \times \ldots \times X_{11}; f; A_1 \times \ldots \times A_4)$, where

1. X_i, $i = 1, \ldots, 11$ is a set of all players selected from Football Clubs in Serie A, they can play at the position i (Goal-keeper, right Back, left Back, right Advanced Guard, left Advanced Guard, Forward,...).
2. A_1 is the set consisting of two possibilities: home field or other field.
3. A_2 is the attitude of Refree (prejudice, not prejudice,...).
4. A_3 is weather condition (hot, cold, warm,...).
5. A_4 is condition of Football field (dry, wet, slippery,...).
6. $f(x_1, \ldots, x_{11}, a_1, \ldots, a_4)$ is the winning probability of Soccer Team at the situation (a_1, a_2, a_3, a_4).

In this example the function $f : X_1 \times \ldots \times X_{11} \times A_1 \times \ldots \times A_4 \to [0,1]$ takes only three values: $L = 0$, $D = \frac{1}{2}$, $W = 1$.

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