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CONVERGENCE RATES IN REGULARIZATION FOR EQUATIONS INVOLVING ACCRETIVE OPERATORS

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Abstract. The aim of this paper is to give a theoretical analysis of convergence rates of the regularized solutions for operator equations involving m-accretive operators and of their convergence rates in combination with finite-dimensional approximations. An application to nonlinear integral equations is considered for illustration.

1. INTRODUCTION

Let X be a real reflexive Banach space and X^* be its dual space. For the sake of simplicity, norms of X and X^* will be denoted by one symbol $\|\cdot\|$. We write $\langle x^*, x\rangle$ instead of $x^*(x)$ for $x^* \in X^*$ and $x \in X$. Let A be a m-accretive operator in X, i.e. [17]

i) $\langle A(x+h) - A(x), J(h)\rangle \geq 0$, $\forall x, h \in X$, where J is the dual mapping of X, i.e. the mapping from X onto X^* satisfies the condition

$$\langle J(x),\ x\rangle = \|x\|^2 = \|J(x)\|^2,\ \forall x \in X,$$

ii) $R(A + \lambda I) = X$ for each $\lambda > 0$, where $R(A)$ denotes the range of A and I is the identity operator in X.

We are interested in solving the ill-posed operator equation [15]

$$A(x) = f,\ f \in X. \tag{1.1}$$

Without additional conditions on the structure of A, as strongly or uniformly accretive property, problem (1.1) is, in general, an ill-posed one [1, 13, 14]. In order to solve it we have to use stable methods. A widely used and effective method is the variational version of Tikhonov regularization [11] that consists of minimizing the functional

$$\|A(x) - f_\delta\|^2 + \alpha\|x\|^2, \text{ over } X, \tag{1.2}$$

where $\alpha > 0$ is a parameter of regularization, and f_δ are approximations for f:

$$\|f_\delta - f\| < \delta,\ \delta \to 0.$$

The aspects of existence, convergence and stability for the solutions of (1.2) have been established in [4, 15].

For given equation (1.1) involving m-accretive operators there is another version of Tikhonov regularization that consists of solving the regularized equation (see [1], [14])

$$A(x) + \alpha(x - x_*) = f_\delta, \tag{1.3}$$

where x_* is some fixed element of X. For finding the solution x_α^δ of (1.3) one can use the iterative methods (see [3, 11, 12]). Actually, in order to use them first we have to approximate (1.3) by the sequence of finite-dimensional problems

$$A_n(x) + \alpha(x - x_*^n) = f_\delta^n,\ x \in X_n, \tag{1.4}$$

where $f_\delta^n = P_n f_\delta$, $x_*^n = P_n x_*$, $A_n = P_n A P_n$, P_n is a sequence of linear projections from X onto X_n, $P_n x \to x$, $\forall x \in X$, $\|P_n\| \le c$, c - some positive constant, and X_n is the sequence of finite-dimensional subspaces of X such that

$$X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X.$$

It's easy to see that if A is m-accretive, A_n are m-accretive, too. The aspects of existence and convergence of the solutions $x_{\alpha n}^\delta$ of (1.4) to the solution x_α^δ of (1.3), for each $\alpha > 0$, has been studied in [17]. The question under which conditions the sequence $x_{\alpha n}^\delta$ converges to a solution x_0 of (1.1), as α, $\delta \to 0$ and $n \to \infty$, was considered in [7]. The main purpose of this paper is the theoretical analysis of convergence rates of the sequences $\{x_\alpha^\delta\}$ and $\{x_{\alpha n}^\delta\}$ in Section 2. In Section 3 an application in the theory of nonlinear integral equations is considered for illustration.

Note that for the variational version of Tikhonov regularization this question has been studied in [4]. It was showed in [4] and [5] that the solutions of the variational version of the Tikhonov regularization converge to a solution x_0 of (1.1) if $\gamma_n = \|(I - P_n)x_0\| = o(\alpha(n))$ for the case when A is linear and bounded. When A is a nonlinear monotone operator these problems are investigated in [8].

Here and below, the symbols $\to$ and $\rightharpoonup$ denote the strong convergence and the weak convergence, respectively. The symbol $a \sim b$ means $a = O(b)$ and $b = O(a)$.

2. MAIN RESULTS

Theorem 2.1. *Assume that the following conditions hold:*
(i) A is twice-Fréchet differentiable with $\|A''(x)\| \le M$, M is some positive constant,
(ii) there exists an element $v \in X$ such that

$$A'(x_0)v = x_0 - x_*,$$

(iii) $\|v\|M < 2$.
Then, if α is chosen such that $\alpha \sim \delta^\theta$, $0 < \theta < 1$, we have

$$\|x_\alpha^\delta - x_0\| = O(\delta^\mu), \quad \mu = \min\{\theta, 1 - \theta\}.$$

Proof. From (1.1) and (1.3) it follows

$$A(x_\alpha^\delta) - A(x_0) + \alpha(x_\alpha^\delta - x_0) = f_\delta - f - \alpha(x_0 - x_*).$$

Put

$$P_{\alpha(\delta)} = \int_0^1 A'(x_0 + t(x_\alpha^\delta - x_0))dt + \alpha I.$$

Since Fréchet derivative of any accretive operators is also accretive, then $P_{\alpha(\delta)}$ has the inversion $P_{\alpha(\delta)}^{-1}$ with $\|P_{\alpha(\delta)}^{-1}\| \le 1/\alpha$. Consequently, we have

$$\begin{aligned}\|x_\alpha^\delta - x_0\| &\le \delta/\alpha + \alpha\|P_{\alpha(\delta)}^{-1}(x_0 - x_*)\| \\ &\le \delta/\alpha + \alpha\Big[\|P_{\alpha(\delta)}^{-1}(P_{\alpha(\delta)} + A'(x_0) - P_{\alpha(\delta)})v\|\Big] \\ &\le \delta/\alpha + \alpha\|v\| + \|(P_{\alpha(\delta)} - A'(x_0))v\| \\ &\le \delta/\alpha + 2\alpha\|v\| + \|v\|M\|x_\alpha^\delta - x_0\|/2.\end{aligned}$$

Therefore,

$$(1 - \frac{\|v\|M}{2})\|x_\alpha^\delta - x_0\| \le \delta/\alpha + 2\alpha\|v\|.$$

Hence,

$$\|x_\alpha^\delta - x_0\| \le O(\delta^\mu).$$

Then, $\|x_\alpha^\delta - x_0\| = O(\delta^\mu)$ (see [2]).

Remark. We can consider the case, when instead of A we know its approximations A_h which are also m-accretive and such that

$$\|A_h(x) - A(x)\| \leq hg(\|x\|),\ \forall x \in X,$$

where $g(t)$ is some positive nondecreasing real function with $g(0) = 0$ (see [14]). It is not difficult to prove that if conditions (i) - (iii) of Theorem 2.1 hold, and if α is chosen such that $\alpha \sim (h+\delta)^\theta$, then the solution $x_{\alpha h}^\delta$ of the equation

$$A_h(x) + \alpha(x - x_*) = f_\delta \tag{2.1}$$

converges to the solution x_0 of (1.1) and

$$\|x_{\alpha h}^\delta - x_0\| = O((\delta + h)^\mu).$$

Theorem 2.2. *Suppose that conditions (i) - (ii) of Theorem 2.1 hold,* $c\|v\|M < 2$, *and* α *is chosen such that* $\alpha \sim \delta^{\theta_1} + \gamma_n^{\theta_2}$, $0 < \theta_i < 1$. *Then we have*

$$\|x_{\alpha n}^\delta - x_0\| = O(\delta^{\mu_1} + \gamma_n^{\mu_2}),\ \mu_i = \min\{\theta_i, 1 - \theta_i\}.$$

Proof. First, we estimate the value $\|x_{\alpha n}^\delta - x_0^n\|$, where $x_0^n = P_n x_0$. From (1.1) and (1.4) it implies that

$$A^n(x_{\alpha n}^\delta) - A^n(x_0^n) + \alpha(x_{\alpha n}^\delta - x_0^n) = f_\delta^n - f^n$$
$$-\alpha P_n(x_0 - x_*) + P_n(A(x_0) - A(x_0^n)),\ f^n = P_n f.$$

Put

$$P_{\alpha(\delta)}^n = \int_0^1 P_n A'(x_0^n + t(x_{\alpha n}^\delta - x_0^n))dt + \alpha I_n,$$

where I_n denotes the identity operator in X_n. Clearly, the operator $P_{\alpha(\delta)}^n$ is linear, bounded, and has the inversion $P_{\alpha(\delta)}^{n(-1)} : X_n \to X_n$ with $\|P_{\alpha(\delta)}^{n(-1)}\| \leq 1/\alpha$. Since

$$\|P_{\alpha(\delta)}^{n(-1)} P_n(f_\delta - f)\| \leq c\delta/\alpha,$$
$$\|P_{\alpha(\delta)}^{n(-1)} P_n(A(x_0) - A(x_0^n))\| = \|P_{\alpha(\delta)}^{n(-1)} P_n(A'(x_0)(P_n - I)x_0$$
$$+\frac{1}{2}A''(x_0 + \tau(P_n - I)x_0)(P_n - I)x_0(P_n - I)x_0)\|,\ 0 < \tau < 1,$$
$$\leq c\|A'(x_0)\|\gamma_n/\alpha + cM\gamma_n^2/(2\alpha) \leq O(\gamma_n/\alpha)$$
$$\alpha\|P_{\alpha(\delta)}^{n(-1)} P_n(x_0 - x_*)\| = \alpha\|P_{\alpha(\delta)}^{n(-1)}(P_{\alpha(\delta)}^n + P_n A'(x_0) - P_{\alpha(\delta)}^n)v\|$$
$$\leq \alpha\|v\|(1 + c) + \|\int_0^1 P_n(A'(x_0) - A'(x_0^n + t(x_{\alpha n}^\delta - x_0^n))vdt\|$$
$$\alpha\|v\|(1 + c) + c\|v\|M\gamma_n + \frac{c\|v\|M}{2}\|x_{\alpha n}^\delta - x_0^n\|.$$

Therefore,

$$\|x_{\alpha n}^\delta - x_0^n\| \leq O((\delta + \gamma_n)/\alpha + \alpha).$$

Then,

$$\|x_{\alpha n}^\delta - x_0^n\| = O(\delta^{\mu_1} + \gamma_n^{\mu_2}).$$

Hence,

$$\|x_\alpha^\delta - x_0\| = O(\delta^{\mu_1} + \gamma_n^{\mu_2}).$$

Now, equation (2.1) can be approximated by the sequence finite-dimensional problems

$$A_h^n(x) + \alpha(x - x_*^n) = f_\delta^n, \tag{2.2}$$

where $A_h^n = P_n A_h P_n$, are also m-accretive, and equation (2.2) has a unique solution $x_{\alpha\delta}^{hn} \in X_n$. We have the following result.

Theorem 2.3. *Assume that conditions (i) - (ii) of Theorem 2.1 hold, $c\|v\|M < 2$, and α is chosen such that $\alpha \sim (\delta + h)^{\theta_1} + \gamma_n^{\theta_2}$. Then*

$$\|x_{\alpha\delta}^{hn} - x_0\| = O((\delta + h)^{\mu_1} + \gamma_n^{\mu_2}).$$

The proof of the theorem is completely similar as the proof of Theorem 2.2. Therefore, we omit it here.

3. APPLICATION

We can use the results obtained in Section 2 to solve the nonlinear integral equations of Hammerstein's type

$$x(s) - \int_{\Omega} k(s,t)F(x(t))dt = f(s), \tag{3.1}$$

where $F(t)$ is a real nonlinear function satisfying the condition

$$|F(t)| \leq a + b|t|, \; a, b > 0,$$

$f(s) \in L_p[\Omega]$, the space of p-summable functions in σ-finite measure set $\Omega \subseteq R^n$, the kernel function $k(s,t)$ is such that the operator K in $L_p[\Omega]$ defined by

$$(Kx)(s) = \int_{\Omega} k(s,t)x(t)dt$$

has an eigenvalue $\lambda = 1$. If the operator G defined by

$$G(x)(s) = \int_{\Omega} k(s,t)F(x(t))dt$$

maps $L_p[\Omega]$ into $L_p[\Omega]$ [12], and $F(t)$ is Lipschitz continuous with Lipschitz constant $\|K\|^{-1}$, then $I - G$ is accretive (see [7]). If we want to solve (3.1) by the collocation-method (see [6]), then the important condition which needs to be satisfied is that $KG'(x_0)$ does not have 1 as an eigenvalue, where $x_0(t)$ is a solution of (3.1). This fact is equivalent to that $I - KG'(x_0)$ has to have the bounded inversion. In applying our result, we can obtain the convergence rates under the weaker condition: the range of $I - KG'(x_0)$ contains the element $x_0 - x_*$. Let the subsets Ω_j, $j = 1, 2, ..., m$, be such that $\cup_{j=1}^m \Omega_j = \Omega$. Denote by $f_j(t)$ the characteristic function of Ω_j. Then the linear combination of $\{f_1, f_2, ..., f_m\}$ is the subspace $L_p(m)$ of the space $L_p(\Omega)$. We can choose

$$P_m f = \sum_{j=1}^{m} \frac{1}{(m\mu(\Omega_j))^{1/q}} \int_{\Omega_j} f(t)dt,$$

where $p^{-1} + q^{-1} = 1$. Then $\|P_m\| = 1$ (see, [17]), and $\|(I - P_m)\phi\| = O(1/m)$, $\forall \phi \in L_p(\Omega)$ (see [10]).

Thus, the finite-dimensional regularized equation (1.4) in this case has a form

$$(1+\alpha)\int_{\Omega_j} x(s)ds - \int_{\Omega_j}\int_{\Omega} k(s,t)F\Big(\sum_{j=1}^{m} \frac{1}{(m\mu(\Omega_j))^{1/q}} \int_{\Omega_j} x(t)dt\Big)dtds$$
$$= \int_{\Omega_j} f(s)ds, \; j = 1, ..., m. \tag{3.2}$$

Put

$$y_j = \int_{\Omega_j} y(s)ds.$$

We have the following system of nonlinear algebraic equations with unknowns y_j, $j = 1, 2, ...m$.

$$(1+\alpha)y_j - b_jF(\sum_{j=1}^{m} c_jy_j) = f_j,$$

$$b_j = \int_{\Omega_j}(\int_{\Omega} k(s,t)dt)ds,\ c_j = \frac{1}{(m\mu(\Omega_j))^{1/q}},\ f_j = \int_{\Omega_j} f(s)ds.$$

This system of equations can be solved by the methods presented in [9].

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