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DIFFERENCE SCHEMES FOR GENERALIZED SOLUTIONS OF SOME ELLIPTIC DIFFERENTIAL EQUATIONS, I

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Abstract. It is known that many applied problems are reduced to boundary value problems for differential equations with non-regular data. There are some works devoted to the construction of difference schemes and the estimation of approximate solutions for these problems [1-3]. In this work the difference schemes for generalized solutions of some elliptic differential equations are constructed. Here we first consider the partial differential equations with the right-hand side defined by a linear functional, for example, by the Dirac distribution δ.

1. INTRODUCTION

In the environment problems the initial condition and the right-hand side of differential equations are often the point and surface distributions of masses, charges, forces,... [4, 5]. Thus, these data cannot be described within the framework of classical concept of a function, to describe it requires using a more general mathematical nature, linear functionals. To investigate these problems, for an illustration, we first consider the difference schemes for the Dirichlet problem of Poisson equation in the unit square:

$$\begin{aligned} \Delta u &= -f(x), \quad x \in G, \\ u(x) &= 0 \quad \text{on } \partial G, \end{aligned} \tag{1}$$

where $G = \{x = (x_1, x_2) \, : \, 0 < x_i < 1, \ i = 1, 2\}$.

The case of spatial variables is considered similarly.

2. DIFFERENCE SCHEME FOR GENERALIZED SOLUTION

The generalized solutions, satisfying an integral identity, of the problem (1) are considered in the spaces $W_2^m(G)$, m is an integer number ≥ 0, where $W_2^m(G)$ is a Hilbert space of $L_2(G)$ functions whose generalized derivatives up to and including $m-th$ order are square integrable in G.

2.1. Construction of difference schemes

For deriving finite-difference methods, let us introduce in the region G a grid $\overline{\omega}$:

$$\overline{\omega} = \left\{ (x_1, x_2) \, : \, x_i \equiv x_{ij_i} = j_i h_i; \ j_i = 0, 1, \ldots, N_i; \ h_i = \frac{1}{N_i}; \ i = 1, 2 \right\},$$

where N_i are positive integer numbers. For the steplengths h_i, suppose that there exists positive constants C_i, $i = 1, 2$, such that $C_1 \leq \dfrac{h_1}{h_2} \leq C_2$ uniformly as $h_1 \to 0$, $h_2 \to 0$.

Denote the set of interior and boundary netpoints of the region G by ω and γ respectively.

We shall consider the generalized solution of the problem (1) $u(x) \in W_2^m(G) \cap \overset{\circ}{W}{}_2^1(G)$, satisfying the integral equation (see [3, chap. I, §2]):

$$Pu \equiv \iint\limits_G \Delta u(x)\, v(x)\, dx = -\iint\limits_G f(x)\, v(x)\, dx, \quad \forall v(x) \in L_2(G), \tag{2}$$

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and boundary condition

$$u(x) = 0, \ x \in \partial G. \tag{3}$$

To obtain a net problem we may take the test function $v(x)$ in the form

$$v(x) = \begin{cases} \dfrac{1}{4\pi h_1^m h_2^m} \exp\Big(-\dfrac{|x|^2}{4h_1^m h_2^m}\Big), & x \in e, \\ 0, & x \in \overline{G} \setminus e, \end{cases} \tag{4}$$

where

$$|x|^2 = x_1^2 + x_2^2, \ e = e(x) \equiv \{\varsigma = (\varsigma_1, \varsigma_2) \ : \ |x_i - \varsigma_i| < 0.5\, h_i, \ i = 1, 2\},$$

m is a natural number.

Then, the generalized solution $u(x)$ (denoted by the NSR $u(x)$) satisfies the following integral equality

$$P^e u \equiv \frac{1}{h_1 h_2} \int\limits_{x_1 - 0.5h_1}^{x_1 + 0.5h_1} \int\limits_{x_2 - 0.5h_2}^{x_2 + 0.5h_2} \alpha(\varsigma)\, \Delta u(\varsigma)\, d\varsigma_1\, d\varsigma_2 = -Rf, \ x \in \omega, \tag{5}$$

where

$$Rf = \frac{1}{h_1 h_2} \iint\limits_e \alpha(\varsigma)\, f(\varsigma)\, d\varsigma \tag{6}$$

$$\alpha(x) \equiv \alpha(x_1, x_2) = \begin{cases} \dfrac{1}{4\pi h_1^{m-1} h_2^{m-1}} \exp\Big\{-\dfrac{x_1^2 + x_2^2}{4h_1^m h_2^m}\Big\}, & x \in e, \\ 0, & x \in \overline{G} \setminus e. \end{cases} \tag{7}$$

We may rewrite the equation (5) as follows

$$P^e u = \frac{1}{h_1 h_2} \iint\limits_e \Big[\sum_{i=1}^{2} \frac{\partial}{\partial \varsigma_i}\Big(\alpha \frac{\partial u}{\partial \varsigma_i}\Big) - \sum_{i=1}^{2} \frac{\partial \alpha}{\partial \varsigma_i} \frac{\partial u}{\partial \varsigma_i}\Big]\, d\varsigma = -Rf, \ x \in \omega.$$

It is clear that

$$\frac{1}{h_1 h_2} \int\limits_{x_1 - 0.5h_1}^{x_1 + 0.5h_1} \int\limits_{x_2 - 0.5h_2}^{x_2 + 0.5h_2} \frac{\partial}{\partial \varsigma_i}\Big(\alpha \frac{\partial u}{\partial \varsigma_i}\Big) d\varsigma_1\, d\varsigma_2 = \frac{1}{h_i} S_{3-i}\Big[\Big(\alpha \frac{\partial u}{\partial x_i}\Big)^{(+0.5_i)} - \Big(\alpha \frac{\partial u}{\partial x_i}\Big)^{(-0.5_i)}\Big], \ i = 1, 2,$$

where S_i is the one-dimensional mean operator:

$$S_i\, u(x) \equiv \frac{1}{h_i} \int\limits_{x_i - 0.5h_i}^{x_i + 0.5h_i} u(x_1, \ldots, \varsigma_i, \ldots, x_n)\, d\varsigma_i\,,$$

$$u^{(\pm 0.5_i)}(x) \equiv u^{(\pm 0.5_i)}(x_1, \ldots, x_n) = u(x_1, \ldots, x_i \pm 0.5h_i, \ldots, x_n).$$

Hence,

$$P^e u = \sum_{i=1}^{2} \frac{1}{h_i} S_{3-i}\Big[\Big(\alpha \frac{\partial u}{\partial x_i}\Big)^{(+0.5_i)} - \Big(\alpha \frac{\partial u}{\partial x_i}\Big)^{(-0.5_i)}\Big] - S_1 S_2\Big(\sum_{i=1}^{2} \frac{\partial \alpha}{\partial \varsigma_i} \frac{\partial u}{\partial \varsigma_i}\Big) = -Rf.$$

Thus, for every netpoint $x = (x_1, x_2)$, we have the following net problem for the NSR $u(x)$:

$$P^e u = \sum_{i=1}^{2} \Big[S_{3-i}\Big(\alpha \frac{\partial u}{\partial x_i}\Big)^{(-0.5_i)}\Big]_{x_i} - S_1 S_2\Big(\sum_{i=1}^{2} \frac{\partial \alpha}{\partial \varsigma_i} \frac{\partial u}{\partial \varsigma_i}\Big) = -Rf, \; x \in \omega,$$

$$u(x) = 0, \; x \in \gamma, \tag{8}$$

where

$$u_{x_i} \equiv u_{x_i}(x) = \frac{1}{h_i}\Big[u^{(+1_i)} - u\Big], \; u_{\overline{x}_i} \equiv u_{\overline{x}_i}(x) = \frac{1}{h_i}\Big[u - u^{(-1_i)}\Big]$$

$$u^{(\pm 1_i)} \equiv u^{(\pm 1_i)}(x) = u(x_1, ..., x_i \pm h_i, ..., x_n), \; i \geq 1.$$

Now, to obtain a difference scheme of the operator $P^e u$ (8) one may approximate the mean integral operator S_i by the quadrature formula of average rectangles and the partial derivatives by difference quotients as

$$\begin{aligned}\frac{1}{h_1} S_2\Big(\alpha \frac{\partial u}{\partial x_1}\Big)^{(+0.5_1)} &= \frac{1}{h_1}\Big\{\frac{1}{h_2}\int_{x_2-0.5h_2}^{x_2+0.5h_2} \alpha(x_1 + 0.5h_1, \varsigma_2)\frac{\partial u}{\partial x_1}(x_1 + 0.5h_1, \varsigma_2)d\varsigma_2\Big\}\\ &\approx \frac{1}{h_1}\Big\{\alpha_1(x_1 + 0.5h_1, x_2)\frac{\partial u}{\partial x_1}(x_1 + 0.5h_1, x_2)\Big\}\\ &\approx \frac{1}{h_1}\Big\{\alpha(x_1 + 0.5h_1, x_2)u_{\overline{x}_1}(x_1 + h_1, x_2)\,.\end{aligned}$$

Then,

$$\frac{1}{h_1} S_2\Big[\Big(\alpha \frac{\partial u}{\partial x_1}\Big)^{(+0.5_1)} - \Big(\alpha \frac{\partial u}{\partial x_1}\Big)^{(-0.5_1)}\Big] \approx \big(\alpha^{(-0.5_1)}(x)\, u_{\overline{x}_1}(x)\big)_{x_1}.$$

Therefore, we get the following difference approximation of the problem (8):

$${}^1P_h^e u \equiv K\breve{y} = -\big(\alpha^{(-0.5_1)}\breve{y}_{\overline{x}_1}\big)_{x_1} - \big(\alpha^{(-0.5_2)}\breve{y}_{\overline{x}_2}\big)_{x_2} + S_1 S_2 \sum_{i=1}^{2} \alpha_{\overline{x}_1}(x)\, \breve{y}_{\overline{x}_1}(x) = \varphi, \; x \in \omega\,,$$

$$\breve{y}(x) = 0, \; x \in \gamma, \tag{9}$$

where

$$\varphi(x) = R\, f(x).$$

Further, consider an other approximation of the problem (8). First, it can be verified that

$$\lim_{h_1,h_2 \to 0} \iint_{R^2} \Delta u(\varsigma)\tilde{\nu}(\varsigma)\, d\varsigma = \Delta u(x), \tag{10}$$

where

$$\tilde{\nu}(x) = \frac{1}{4\pi h_1^m h_2^m} \exp\Big\{-\frac{|x|^2}{4h_1^m h_2^m}\Big\}.$$

Note that $\tilde{\nu}(x)$ is an infinitely differentiable function in G.
On the other hand, one has

$$\lim_{h_1,h_2 \to 0} \iint_{e} \Delta u(\varsigma)\, \omega(\varsigma)\, d\varsigma = \Delta u(x), \tag{11}$$

where

$$\omega(x) = \begin{cases} (h_1 h_2)^{-1}, & x \in e,\\ 0, & x \in \overline{G} \setminus e.\end{cases}$$

By (5), (8) - (11) one may approximate the problem (8) by the following difference scheme (cf. [3, chap. III, §1]):

$$\begin{aligned} {}^2P_h^e u \equiv L\vartheta &= -\vartheta_{\bar{x}_1 x_1} - \vartheta_{\bar{x}_2 x_2} = \varphi, \quad x \in \omega, \\ \vartheta(x) &= 0, \quad x \in \gamma. \end{aligned} \tag{12}$$

2.2. Estimation of the convergence rate

For our purpose we will estimate the method error and approximation error of the schemes (9) and (12).

2.2.1. Consider the difference scheme (12). We see that the left-hand side of the difference equation (12) coincides with a standard fivepoints approximation for the one of the differential equation (1). Denote the method error by $z = \vartheta - u$, where ϑ is the solution of the problem (12) and u is the NSR of the problem (1). It follows from (12) that

$$Lz = \Psi(x), \ x \in \omega; \ z(x) = 0, \ x \in \gamma, \tag{13}$$

where $\Psi(x)$ is the approximation error of the scheme (12): $\Psi(x) = \varphi(x) - Lu$. Then, using the expression (9) of φ we get

$$\Psi = \sum_{i=1}^{2}\Big[u_{\bar{x}_i} - S_{3-i}\big(\alpha\frac{\partial u}{\partial x_i}\big)^{(-0.5_i)}\Big]_{x_i} + S_1 S_2\Big(\sum_{i=1}^{2}\frac{\partial \alpha}{\partial x_i}\frac{\partial u}{\partial x_i}\Big) \equiv \sum_{i=1}^{2}(\eta_i)_{x_i} + \eta_0, \tag{14}$$

where

$$\eta_i = u_{\bar{x}_i} - S_{3-i}\Big(\alpha\frac{\partial u}{\partial x_i}\Big)^{(-0.5_i)}, \quad i = 1, 2,$$

$$\eta_0 = S_1 S_2\Big(\sum_{i=1}^{2}\frac{\partial \alpha}{\partial \varsigma_i}\frac{\partial u}{\partial \varsigma_i}\Big), \quad x \in \omega, \tag{15}$$

$\alpha(x)$ is defined by (7).

To obtain a priori estimate for the problem (13) - (15) we use the method of energy inequalities. For this purpose let us take the scalar product of $z(x)$ and the equation (13):

$$-\sum_{i=1}^{2}(z_{\bar{x}_i x_i}, z) = \sum_{i=1}^{2}(\eta_{i_{x_i}}, z) + (\eta_0, z),$$

where (u, v) is the scalar product on the set of net functions.

Since $z(x) = 0$ for $x \in \gamma$, one has

$$-(z_{\bar{x}_i x_i}, z) = (z_{\bar{x}_i}, z_{\bar{x}_i}]_i = \|z_{\bar{x}_i}]|_i^2,$$

$$(\eta_{i_{x_i}}, z) = -(\eta_i, z_{\bar{x}_i}]_i, \quad i = 1, 2,$$

where

$$(\omega, z]_1 = \sum_{j_1=1}^{N_1}\sum_{j_2=1}^{N_2-1}\omega(j_1h_1, j_2h_2)z(j_1h_1, j_2h_2)h_1h_2,$$

$$(\omega, z]_2 = \sum_{j_1=1}^{N_1-1}\sum_{j_2=1}^{N_2}\omega(j_1h_1, j_2h_2)z(j_1h_1, j_2h_2)h_1h_2.$$

Hence

$$\sum_{i=1}^{2}\|z_{x_i}\|_i^2 \equiv \|\nabla z\|_{0,\omega}^2 \le \sum_{i=1}^{2}\|\eta_i\|_i\|z_{\bar{x}_i}\| + \|\eta_0\|\,\|z\|, \tag{16}$$

where

$$\|z\|^2 = \sum_{j_1=1}^{N_1} \sum_{j_2=1}^{N_2} z^2(j_1 h_1, j_2 h_2) h_1 h_2 .$$

Let H be the space of the functions defined on the net $\overline{\omega}$. $\overset{\circ}{H}$ be its subset of the functions satisfying the condition: $u(x) = 0$, $x \in \gamma$.

By the embedding theorems of net functions (see [3, chap. I, §3]), for a function $u \in \overset{\circ}{H}$ one has

$$\frac{2}{3}\|u\|^2_{1,\omega} \le \|\nabla u\|^2_{0,\omega} \le \|u\|^2_{1,\omega} , \tag{17}$$

where

$$\|u\|^2_{1,\omega} \equiv \|u\|^2_{0,\omega} + \|\nabla u\|^2_{0,\omega} \equiv \|u\|^2_{W_2^1(\omega)} ,$$

$$\|u\|_{0,\omega} \equiv \|u\| .$$

Combining (16) and (17) yields

$$\|z\|_{1,\omega} \le M(\|\eta_1\|_1 + \|\eta_2\|_2 + \|\eta_0\|) . \tag{18}$$

where M is a constant, independent of $h(|h|^2 = h_1^2 + h_2^2)$ and $z(x)$.

Now, to estimate the rate of convergence for $\tilde{y}$, we first consider the functional $\eta_1(x)$ defined by (15):

$$\eta_1(x) = u_{\overline{x}_1}(x) - \frac{1}{h_2} \int\limits_{x_2-0.5h_2}^{x_2+0.5h_2} \alpha(x_1 - 0.5h_1, \varsigma_2) \frac{\partial u}{\partial \varsigma_1}(x_1 - 0.5h_1, \varsigma_2) d\varsigma_2. \tag{19}$$

Let us denote by e^i the following mesh of the net ω:

$$e^i \equiv e^i(x) = \{\varsigma = (\varsigma_1, \varsigma_2) : x_i - h_i < \varsigma_i < x_i, \ |x_{3-i} - \varsigma_{3-i}| < 0.5h_{3-i}\} .$$

We introduce a transformation of the variable ς as follows: $\varsigma_i = x_i + h_i s_i$, $i = 1, 2$. Then, the region $e^i(x)$ will be transformed into the rectangle $E^1 = \{(s_1, s_2) : -1 < s_1 < 0, |s_2| < 0.5\}$. Setting $\tilde{u}(s_1, s_2) \equiv u(x_1 + h_1 s_1, x_2 + h_2 s_2)$, one has from (19)

$$\eta_1(x) = \frac{1}{h_1}[\tilde{u}(0,0) - \tilde{u}(-1,0)] - \frac{1}{h_1} \int\limits_{-0.5}^{0.5} \tilde{\alpha}(-0.5; s_2) \frac{\partial \tilde{u}}{\partial s_1}(-0.5; s_2) ds_2 \equiv I_1 + I_2 , \tag{20}$$

where

$$I_1 = \frac{1}{h_2}\Big[\tilde{u}(0,0) - \tilde{u}(-1,0) - \int\limits_{-0.5}^{0.5} \frac{\partial \tilde{u}}{\partial s_1}(-0.5; s_2)\, ds_2\Big] ,$$

$$I_2 = \frac{1}{h_1} \int\limits_{-0.5}^{0.5} \Big[1 - \tilde{\alpha}\big(-\frac{1}{2}, s_2\big)\Big] \frac{\partial \tilde{u}}{\partial s_1}(-0.5; s_2) ds_2 .$$

It is clear that if $u \in W_2^m(e^1)$ then $\tilde{u} \in W_2^m(E^1)$.I_1 is a bounded linear functional of $\tilde{u} \in W_2^m(E^1)$, $m \ge 2$ and is equal to zero at the polynomials of second order ($\tilde{u} = 1, s_1, s_2, s_1 s_2, s_1^2, s_2^2$). Then, by Lemma Bramble-Hilbert (see [3, chap. I, §1]) one has

$$|I_1| \le \frac{M}{h_1}|\tilde{u}|_{m,E^1} , \ m = 2, 3 .$$

Further,

$$|\tilde{u}|_{m,E^1} \equiv \Big(\sum_{|\alpha|=m} \int_{E^1} |D^\alpha \tilde{u}|^2 ds \Big)^{1/2} \le |h|^m (h_1 h_2)^{-1/2} |u|_{m,e^1} ,$$

where $h^\alpha = h_1^{\alpha_1} h_2^{\alpha_2}, \alpha = (\alpha_1, \alpha_2)$.
Hence,

$$|I_1| \le M |h|^{m-1} (h_1\, h_2)^{-1/2} |u|_{m,e^1} , \tag{21}$$

where M is a constant, independent of h and $u(x)$, $m = 2, 3$.

We rewrite I_2 in the form:

$$I_2 = \frac{1}{h_1} \int_{-1}^{0} \int_{-0.5}^{0.5} [1 - \tilde{\alpha}(-0.5; s_2)] \frac{\partial \tilde{u}}{\partial s_1}(s_1, s_2)\, ds_1\, ds_2 .$$

Since $\tilde{\alpha}(-0.5; s_2)$ tends to zero as h_1, $h_2 \to 0$, there exists a value $s_2 = \tilde{s}_2$, $-0.5 < \tilde{s}_2 < 0.5$, such that $\tilde{\alpha}(-0.5; s_2) = 1$ if h_1 and h_2 are sufficiently small. Then one has

$$1 - \tilde{\alpha}(-0.5; s_2) = \int_{s_2}^{\tilde{s}_2} \frac{\partial \tilde{\alpha}(-0.5; r)}{\partial r}\, dr .$$

Hence,

$$\begin{aligned} |I_2| &\le \frac{M}{h_1} \sup_{s \in E^1} \Big| \frac{\partial \tilde{\alpha}(s)}{\partial s_2} \Big| \Big(\iint_{E^1} \Big(\frac{\partial \tilde{u}}{\partial s} \Big)^2 ds \Big)^{1/2} \\ &\le M\, |\alpha|_{1,\infty,G}\, |h|\, (h_1\, h_2)^{-1/2}\, |u|_{1,e^1} \end{aligned} \tag{22}$$

where

$$|\alpha|_{p,\infty,G} \equiv \sum_{|\lambda|=p} \|D^\lambda \alpha\|_{0,\infty,G} = \sum_{|\lambda|=p} \|D^\lambda \alpha\|_{L_\infty(G)} .$$

By (20), (21) ($m = 2$) and (22) we get

$$|\eta_1(x) \le M\, |h|\, (h_1\, h_2)^{-1/2}\, \|u\|_{2,e^1} , \tag{23}$$

where

$$\|u\|_{m,e^1} = \|u\|_{W_2^m(e^1)} = \Big(\sum_{|\alpha| \le m} \int_{e^1} |D^\alpha u|^2\, dx \Big)^{1/2}.$$

The functional $\eta_2(x)$ is estimated similarly.

Consider now the summand $\eta_0(x)$:

$$\eta_0(x) = \int_{-0.5}^{0.5} \int_{-0.5}^{0.5} \sum_{i=1}^{2} \frac{1}{h_i^2} \frac{\partial \tilde{\alpha}}{\partial r_i} \frac{\partial \tilde{u}}{\partial r_i}\, dr_1\, dr_2 \equiv K_1 + K_2 . \tag{24}$$

One has

$$|K_1| \le \frac{1}{h_1} \Big| \int_{-0.5}^{0.5} \frac{1}{h_1} \frac{\partial \tilde{\alpha}}{\partial r_1} \frac{\partial \tilde{u}}{\partial r_1} dr_2 \Big| .$$

Since $\dfrac{1}{h_1} \dfrac{\partial \tilde{\alpha}}{\partial r_i} \to 0$ as $h_1, h_2 \to 0$, by the same way as we did for I_2, we obtain

$$|\eta_0(x)| \le M\, |\beta(h)|\, |h|\, (h_1\, h_2)^{-1/2}\, |u|_{1,e^1} , \tag{25}$$

where $\beta(h) \to 0$ when h_1, $h_2 \to 0$.

It follows from (23) and (25) that

$$\|\eta_i\|_i = \Big(\sum_{x} |\eta_i|^2 h_1 h_2\Big)^{1/2} \le M\,|h|\Big(\sum_{x} \|u\|^2_{2,e^i}\Big)^{1/2} \le M\,|h|\,\|u\|_{2,G}, \qquad \|\eta_0\| \le M\,|h|\,|\beta(h)|\,\|u\|_{1,G}. \tag{26}$$

Finally, combining (18) and (26) we derive

$$\|z\|_{1,\omega} = \|\hat{y} - u\|_{1,\omega} \le M\,|h|\,\|u\|_{W_2^2(G)}.$$

The case $m = 3$ is considered in the same manner.

Thus, we obtain the following result.

Theorem 1. *Let the NSR $u(x)$ of the problem (8) belong to the space $W_2^m(G)$, $m = 2, 3$. Then the solution of the difference scheme (12) converges to the NSR (8) in the net norm $W_2^1(\omega)$, with the rate $O(|h|^{m-1})$ such that one has the following error estimation*

$$\|\hat{y} - u\|_{1,\omega} \le M\,|h|^{m-1}\,\|u\|_{m,G}, \tag{27}$$

where the constant M is independent of h and $u(x)$.

2.2.2. We now consider the following difference scheme

$$Ty = \frac{1}{2}(K + L)\,y = \varphi, \quad x \in \omega,$$
$$y(x) = 0, \quad x \in \gamma,$$

where $y = \frac{1}{2}(\breve{y} + \hat{y})$, $\breve{y}$ and $\hat{y}$ are defined by (9) and (12) respectively.

Hence,

$$Ty = -\frac{1}{2}\sum_{i=1}^{2} y_{\bar{x}_i x_i} - \frac{1}{2}\sum_{i=1}^{2}\big(\alpha^{(-0.5_i)} y_{\bar{x}_i}\big)_{x_i} + \frac{1}{2} S_1 S_2 \sum_{i=1}^{2} \alpha_{\bar{x}_i} y_{\bar{x}_i} = \varphi, \quad x \in \omega, \qquad y(x) = 0, \ x \in \gamma. \tag{28}$$

The difference operator (28) satisfies the maximum principle (see [6, chap. 4, §2, 4]), then there exists uniquely a solution of the difference problem (28). Now, consider the method error $z = y - u$. One has

$$Tz = \Psi(x), \quad x \in \omega, \qquad z(x) = 0, \quad x \in \omega, \tag{29}$$

where $\Psi(x) = \varphi - \frac{1}{2}(K + L)\,u$,

$$2\Psi = \sum_{i=1}^{2} (\eta_i)_{x_i} + \sum_{i=1}^{2} (\gamma_i)_{x_i} + \gamma_0, \tag{30}$$

$$\eta_i = u_{\bar{x}_i} - S_{3-i}\Big(\alpha\,\frac{\partial u}{\partial x_i}\Big)^{(-0.5_i)}, \qquad \gamma_i = \alpha^{(-0.5)}\,u_{\bar{x}_i} - S_{3-i}\Big(\alpha\,\frac{\partial u}{\partial x_i}\Big)^{(-0.5_i)}, \qquad \gamma_0 = S_1 S_2 \sum_{i=1}^{2}\Big[2\frac{\partial \alpha}{\partial \varsigma_i}\frac{\partial u}{\partial \varsigma_i} - \alpha_{\bar{x}_i}(x)\,u_{\bar{x}_i}(x)\Big]. \tag{31}$$

From (29) - (31) one has

$$T^0 z \equiv \sum_{i=1}^{2} z_{\bar{x}_i x_i} + \sum_{i=1}^{2} \left(\alpha^{(-0.5_i)} z_{\bar{x}_i}\right)_{x_i} = -\Psi_0(x), \quad x \in \omega,$$

$$z(x) = 0, \quad x \in \gamma, \tag{32}$$

where

$$\Psi_0(x) = \sum_{i=1}^{2} (\eta_i + \gamma_i)_{x_i} + \tilde{\gamma}_0, \tag{33}$$

$$\tilde{\gamma}_0 = S_1 S_2 \sum_{i=1}^{2} \left[2 \frac{\partial \alpha}{\partial \varsigma_i} \frac{\partial u}{\partial \varsigma_i} - \alpha_{\bar{x}_i}(x)\, y_{\bar{x}_i}(x) \right].$$

Since $y_{\bar{x}_i}(x) \approx \dfrac{\partial u(x)}{\partial x_i} \approx u_{\bar{x}_i}$, $\tilde{\gamma}_0$ may be written as

$$\tilde{\gamma}_0 = S_1 S_2 \sum_{i=1}^{2} \left[2 \frac{\partial \alpha}{\partial \varsigma_i} \frac{\partial u}{\partial \varsigma_i} - \alpha_{\bar{x}_i}(x)\, u_{\bar{x}_i}(x) \right]. \tag{34}$$

From (32), (33), arguing as in the proof of (18) we get

$$\|z\|_{1,\omega} \le M \left(\sum_{i=1}^{2} (\|\eta_i\|_i + \|\gamma_i\|_i) + \|\tilde{\gamma}_0\| \right). \tag{35}$$

By (15) and (31) one has the estimation (23) for η_i. Consider now the $\gamma_i(x)$ in (31):

$$\gamma_1 = \frac{1}{h_1} \alpha(x_1 - 0.5h_1; x_2)[u(x_1, x_2) - u(x_1 - h_1, x_2)]$$

$$- \frac{1}{h_2} \int_{x_2-0.5h_2}^{x_2+0.5h_2} \alpha(x_1 - 0.5h_1, \varsigma_2) \frac{\partial u}{\partial \varsigma_1}(x_1 - 0.5h_1, \varsigma_2)\, d\varsigma_2,$$

or,

$$\gamma_1 = \frac{1}{h_1} \tilde{\alpha}(-0.5; 0) \left[\tilde{u}(0,0) - \tilde{u}(-1,0) - \int_{-0.5}^{0.5} \frac{\partial \tilde{u}}{\partial s_1}(-0.5; s_2)\, ds_2 \right]$$

$$+ \frac{1}{h_1} \int_{-0.5}^{0.5} [\tilde{\alpha}(-0.5; 0) - \tilde{\alpha}(-0.5; s_2)] \frac{\partial \tilde{u}}{\partial s_1}(-0.5; s_2)\, ds_2$$

$$\equiv H_1 + H_2. \tag{36}$$

The estimates of H_1 and H_2 are analogous to I_1 and I_2 in (20) respectively, then,

$$|H_1| \le M \max_{x \in G} |\alpha(x)|\, |h|^{m-1} (h_1 h_2)^{-1/2} |u|_{m,e^1}, \quad m = 2, 3, \tag{37}$$

$$|H_2| \le M |\alpha|_{m-1,\infty,G}\, |h|^{m-1} (h_1 h_2)^{-1/2} |u|_{1,e^1}, \quad m = 2, 3. \tag{38}$$

It follows from (37) and (38) that

$$|\gamma_1| \le M |\beta_1(h)|\, |h|^{m-1} (h_1 h_2)^{-1/2} \|u\|_{m,e^1}, \quad m = 2, 3.$$

The function $\gamma_2(x)$ is estimated similarly. Thus,

$$\|\gamma_i\| \le M\,|h|^{m-1}\,|\beta_i(h)|\,\|u\|_{m,G}, \quad m = 2, 3\,, \tag{39}$$

where $\beta_i(h)$, $i = 1, 2$, tend to zero as h_1, $h_2 \to 0$.

Consider the last summand $\tilde{\gamma}_0$ in (35):

$$\begin{aligned}\tilde{\gamma}_0 &= \frac{1}{h_1 h_2} \int\limits_{x_1-0.5h_1}^{x_1+0.5h_1} \int\limits_{x_2-0.5h_2}^{x_2+0.5h_2} \sum_{i=1}^{2} \left[2\frac{\partial \alpha}{\partial \varsigma_i}\frac{\partial u(\varsigma)}{\partial \varsigma_i} - \alpha_{\bar{x}_i}(x)\, u_{\bar{x}_i}(x)\right] d\varsigma \\ &= \frac{1}{h_1 h_2} \iint\limits_{e} \sum_{i=1}^{2} \left[\frac{\partial \alpha}{\partial \varsigma_i}\left(\frac{\partial u}{\partial \varsigma_i} - u_{\bar{x}_i}(x)\right) + \left(\frac{\partial \alpha}{\partial \varsigma_i} - \alpha_{\bar{x}_i}(x)\right) u_{\bar{x}_i}(x) + \frac{\partial \alpha}{\partial \varsigma_i}\frac{\partial u}{\partial \varsigma_i}\right] d\varsigma \\ &\equiv \sum_{j=1}^{4} A_j\,. \end{aligned} \tag{40}$$

By the Cauchy - Buniakopski inequality one has

$$\begin{aligned}|A_i| &= \frac{1}{h_1 h_2}\left|\iint\limits_{e} \frac{\partial \alpha}{\partial \varsigma_i}\left[\frac{\partial u}{\partial \varsigma_i} - u_{\bar{x}_i}(x)\right] d\varsigma\right| \\ &\le (h_1 h_2)^{-1/2} \sup_{\varsigma \in e}\left|\frac{\partial \alpha}{\partial \varsigma_i}\right| \left(\iint\limits_{e}\left[\frac{\partial u}{\partial \varsigma_i}(\varsigma) - u_{\bar{x}_i}(x)\right]^2 d\varsigma\right)^{1/2}, \quad i = 1, 2.\end{aligned}$$

Therefore,

$$|A_i| \le M\,|h|\,|\lambda_i(h)|\,(h_1 h_2)^{-1/2}\left(|u|_{2,e} + |u|_{2,e^i}\right), \quad i = 1, 2\,, \tag{41}$$

where $\lambda_i(h) \to 0$ as h_1, $h_2 \to 0$.

Since

$$u_{\bar{x}_i} = \frac{1}{h_i}\int\limits_{x_i-h_i}^{x_i} \frac{\partial u}{\partial \varsigma_i}(x_1, \ldots, \varsigma_i, \ldots, x_n)\, d\varsigma_i,$$

$$A_3 = \frac{1}{h_1 h_2}\iint\limits_{e}\sum_{i=1}^{2}\left[\frac{\partial \alpha}{\partial \varsigma_i} - \alpha_{\bar{x}_i}(x)\right]\left(\frac{1}{h_i}\int\limits_{x_i-h_i}^{x_i}\frac{\partial u}{\partial r_i}\, dr_i\right) d\varsigma\,.$$

Thus,

$$|A_3| \le M\,|h|\,|\lambda_3(h)|(h_1 h_2)^{-1/2}\,|u|_{1,e^1}, \tag{42}$$

where $\lambda_3(h) \to 0$ as h_1, $h_2 \to 0$.

The estimation of A_4 is analogous to η_0 in (15), then, by (25) we have

$$|A_4| \le M\,|h|\,|\lambda_4(h)|\,(h_1 h_2)^{-1/2}\,|u|_{1,e}\,, \tag{43}$$

Combining (40) - (43) yields

$$\|\tilde{\gamma}_0\| \le M\,|h|\,|\lambda(h)|\,\|u\|_{2,G}\,, \tag{44}$$

where $\lambda(h) \to 0$ when h_1, $h_2 \to 0$. The case $m = 3$ is considered analogously.

Finally, we get from (35), (23), (39) and (44)

$$\|\tilde{y} + \hat{y} - 2u\|_{1,\omega} \le M\,|h|^{m-1}\,\|u\|_{m,G}, \quad m = 2, 3. \tag{45}$$

Hence, there holds the following theorem:

Theorem 2. *Suppose that $\alpha(x)\, f(x) \in L_2(G)$. Then the solution $\breve{y}$ of the difference scheme (9) converges to the NSR (8) $u(x)$ of the problem (1) in the net norm $W_2^1(\omega)$ with the rate $O(|h|^{m-1})$, $m = 2, 3$, such that one had the following error estimation*

$$\|\breve{y} - u\|_{1,\omega} \le M\, |h|^{m-1}\, \|u\|_{m,G}\,, \tag{46}$$

where the constant M is independent of h and $u(x)$.

Indeed, by assumption $\alpha(x)\, f(x) \in L_2(G)$ and from the formulas (2), 3), (5) and (6) it follows that $u(x) \in W_2^m(G)$, $m = 2, 3$. Then, by the forms (27) and (45) one has

$$\|\breve{y} - u\|_{1,\omega} \le \|\breve{y} + \hat{y} - 2u\|_{1,\omega} + \|\hat{y} - u\|_{1,\omega} \le M\, |h|^{m-1}\, \|u\|_{m,G},$$

that proves Theorem 2.

Remark. For the sake of simplicity, the homogeneous Dirichlet condition (1) was considered. In the case where $u(x) = g(x)$, $x \in \partial G$, the assertions of the theorems 1 and 2 are also true.

2.3. Some generalizations

2.3.1. In a manner analogous to the proof of the theorem 1 and 2, one may verify that these theorems are also valid, if in the formulas (2) and (4) of the generalized solution $v(x)$ is any test function in the space of Schwartz basic functions $D(G)$ [5].

2.3.2. Let in the differential equation (1) $f(x) \in W_2^{(-l)}(G)$, l is a nonnegative integer (see [3, §1]). We consider the generalized solution for (1) of following form:

$$\langle \Delta u, v\rangle = \langle -f, v\rangle, \quad u = 0 \text{ on } \partial G, \tag{47}$$

where $\langle\cdot\,,\,\cdot\rangle$ is a continuous linear functional on the space $W_2^l(G)$.
Then, one may represent $f(x)$ in the form [7]:

$$f = \sum_{|\alpha| \le l} D^\alpha f_{|\alpha|},$$

where $f_{|\alpha|} \in L_2(G)$, α is a multi-index of nonnegative integers:

$$\alpha = (\alpha_1, \dots, \alpha_n), \quad |\alpha| = \sum_{j=1}^{n} \alpha_j\,,$$

$$D^\alpha \equiv \frac{D^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}}, \quad x \in R^n.$$

Then, let $v \in \mathring{W}_2^l(G)$ one has

$$\langle \Delta u, v\rangle = \sum_{|\alpha| \le l} (-1)^{\alpha+1} \langle f_{|\alpha|},\, D^\alpha v\rangle. \tag{48}$$

Furthermore, let $v \in D(G)$. Because $D(G)$ is dense in $\mathring{W}_2^l(G)$ and $f_{|\alpha|} \in L_2(G)$, by (48) we may consider the following generalized solution u_r of the problem (1):

$$\iint_G \Delta u_r(x)\, v(x)\, dx = \sum_{|\alpha| \le l} (-1)^{|\alpha|+1} \iint_G f_{|\alpha|}(x)\, D^\alpha v\, dx\,. \tag{49}$$

Since the function $v(x)$ is infinitely differentiable, by (49) one has $u_r(x) \in W_2^m(G)$, $m = 2, 3$, and we see that the equality (49) is analogous to the formula (2). Then, we can use the difference scheme (9) for the problem (1) in the case where $f(x) \in W_2^{(-l)}(G)$, and thus, the convergence Theorem 2 is also valid in this case.

For example, let in (1) the right-hand side $f(x) = \delta(x - x^0)$, $x^0 \in G$, δ is the Dirac delta function. It is known that $\delta(x - x^0) = D\theta(x - x^0)$, $\theta(x)$ being the Heaviside function. Then, the form (49) may be written in the mesh $e(x^0)$ (with the netpoint x^0) as

$$\iint_{e(x^0)} \Delta u_r(\varsigma)\, v(\varsigma)\, d\varsigma = -v(x^0)\,. \tag{50}$$

Remark. For simplicity of presentation, let us suppose that $G \subset R^1$, $x = 0 \in G$ and $f(x) = \delta(x)$. Let in (4) $m = 1$, then in (50)

$$v(x^0) = v(0) = \frac{1}{4\pi h}\,. \tag{51}$$

By (50), (51) we showed that, in the finite-difference problems, one may approximate the Dirac delta function (being a distribution) $\delta(x)$ by a rational fraction of h (see the first formula in [5, Chap. I, 1]; the formula (37), [3, chap. II, §3]; and the other formulas in classical bibliographies).

3. ELLIPTIC DIFFERENTIAL EQUATION OF THE SECOND ORDER WITH VARIABLE COEFFICIENTS

Consider the following elliptic problem

$$\begin{aligned} Pu &= \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(k_i(x)\, \frac{\partial u}{\partial x_i}\right) = -f(x), \quad x \in G, \\ u(x) &= 0, \quad x \in G, \end{aligned} \tag{52}$$

where G is the unit square as in *n*. 1, $k_i(x) \in C(\overline{G})$,

$$0 < C_1 \le k_i(x) \le C_2, \quad x \in \overline{G}, \tag{53}$$

C_i, $i = 1, 2$, are the constants.

3.1. Construction of difference schemes

Consider the NSR of the problem (52) $u \in W_2^m(G) \cap \overset{\circ}{W}{}_2^1(G)$, satisfying the integral equality

$$\iint_G \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(k_i(x)\, \frac{\partial u}{\partial x_i}\right) v(x)\, dx = -\iint_G f(x)\, v(x)\, dx,$$

where the test function $v(x)$ has the form (4).

In every mesh $e(x)$ of a net point $x \in \omega$, the last equation may be written as

$$\begin{aligned} P^e u \equiv \frac{1}{h_1 h_2} &\iint_e \sum_{i=1}^{2} \frac{\partial}{\partial \varsigma_i}\left[\alpha(\varsigma)\, k_i(\varsigma)\, \frac{\partial u}{\partial \varsigma_i}\right] d\varsigma \\ - \frac{1}{h_1 h_2} &\iint_e \sum_{i=1}^{2} k_i(\varsigma)\, \frac{\partial \alpha}{\partial \varsigma_i}\, \frac{\partial u}{\partial \varsigma_i}\, d\varsigma = -Rf, \quad x \in \omega, \end{aligned}$$

where Rf has the form (5).
Then, one has the following net problem similar to the one (8)

$$P^e u = \sum_{i=1}^{2} \left[S_{3-i}\left(\alpha k_i \frac{\partial u}{\partial x_i} \right)^{(-0.5_i)} \right]_{x_i} - S_1 S_2 \Big(\sum_{i=1}^{2} k_i(\varsigma) \frac{\partial \alpha}{\partial \varsigma_i} \frac{\partial u}{\partial \varsigma_i} \Big) = -Rf, \ x \in \omega,$$
$$u(x) = 0, \ x \in \gamma. \tag{54}$$

Arguing as in the proof of the form (9) and (12), n. 2.1., we obtain the following difference schemes for the net problem (54):

$$\tilde{K} \breve{y} \equiv \sum_{i=1}^{2} (a_i \breve{y}_{\bar{x}_i})_{x_i} - S_1 S_2 \sum_{i=1}^{2} k_i(x) \alpha_{\bar{x}_i}(x) \breve{y}_{\bar{x}_i}(x) = -\varphi(x), \ x \in \omega,$$
$$\breve{y}(x) = 0, \ x \in \gamma, \tag{55}$$

$$\tilde{L} \hat{y} \equiv \sum_{i=1}^{2} (b_i \hat{y}_{\bar{x}_i})_{x_i} = -\varphi(x), \ x \in \omega, \ \ \hat{y}(x) = 0, \ x \in \gamma, \tag{56}$$

where $a_i(x) = \alpha^{(-0.5_i)} k_i^{(-0.5_i)}(x), \ b_i(x) = k_i^{(-0.5_i)}, \ \varphi(x) = Rf(x).$

3.2. Estimate of convergence rate

We first consider the difference scheme (56). Denote the method error by $z = \hat{y} - u$, where u is the NSR of the problem (54), one has

$$\tilde{L}z - \Psi(x), \ x \in \omega$$
$$z(x) = 0, \ x \in \gamma, \tag{57}$$

where

$$\Psi(x) = \sum_{i=1}^{2} \left[b_i u_{\bar{x}_i} - S_{3-i}\left(\alpha k_i \frac{\partial u}{\partial x_i} \right)^{(-0.5_i)} \right]_{x_i} + S_1 S_2 \Big(\sum_{i=1}^{2} k_i \frac{\partial \alpha}{\partial \varsigma_i} \frac{\partial u}{\partial \varsigma_i} \Big)$$
$$\equiv \sum_{i=1}^{2} (\gamma_i)_{x_i} + \gamma_0 \tag{58}$$

$$\gamma_i = b_i u_{\bar{x}_i} - S_{3-i}\left(k_i \alpha \frac{\partial u}{\partial x_i} \right)^{(-0.5_i)}, \quad \gamma_0 = S_1 S_2 \Big(\sum_{i=1}^{2} k_i \frac{\partial \alpha}{\partial \varsigma_i} \frac{\partial u}{\partial \varsigma_i} \Big), \ x \in \omega. \tag{59}$$

By (57), (59) one has the following inequality analogous to (18)

$$\|z\|_{1,\omega} \le M \left(\|\gamma_1\|_1 + \|\gamma_2\|_2 + \|\gamma_0\| \right). \tag{60}$$

The estimation of γ_i and γ_0 are analogous to that of γ_i and γ_0 in (31) and (15) respectively. Hence, as in the previous section we have

Theorem 3. *Let $k_i(x) \in W_\infty^{m-1}(G)$, $m = 2, 3$, satisfying the condition (53) $\alpha(x) f(x) \in L_2(G)$. Then the solution of difference scheme (56) converges to the NSR (54) of the problem (52) with the rate $O(|h|^{m-1})$ such that*

$$\|\hat{y} - u\|_{1,\omega} \le M |h|^{m-1} \|u\|_{m,G}, \tag{61}$$

where the constant M is independent of h and $u(x)$.

For the difference scheme (55), in the same way as we did for the scheme (9) in the section 2.2, one has the following

Theorem 4. *Let* $k_i \in W_\infty^{m-1}(G) \cap C^{m-2}(\overline{G})$, *satisfying the condition (53),* $m = 2, 3$; $\alpha(x) f(x) \in L_2(G)$. *Then the solution of the scheme (55) converges to the NSR (54) of the problem (52) in the net norm* $W_2^1(\omega)$ *with the rate* $O(|h|^{m-1})$ *such that*

$$\|\vartheta - u\|_{1,\omega} \leq M\,|h|^{m-1}\,\|u\|_{m,G}\,, \tag{61}$$

where the constant M *is independent of* h *and* $u(x)$.

Finally, note that some generalization in sec. 2.3. are also valid for the problem (52).

In the part II of this work we will consider the difference schemes of the problems (1) and (52) in the case where G is a region of arbitrary form.

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