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ON THE APPROXIMATION OF CONTINUOUS FUNCTION BY SPLINE FUNCTION HIERARCHICAL FUZZY SYSTEMS

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Abstract. In this paper we introduce a class of hierarchical fuzzy systems, in which spline functions of degree two are used as membership functions of input variables. Then we investigate the capabilities of hierarchical fuzzy systems to approximate function on compact input space. By constructive proof, we prove that any continuous function on a compact space can be approximated to any degree of accuracy by hierarchical fuzzy systems.

Tóm tắt. Bài báo giới thiệu một lớp các hệ mờ phân cấp trong đó hàm Spline bậc hai được sử dụng làm hàm thành viên của các biến vào. Tiếp đó, chứng minh lớp hệ mờ phân cấp đó có thể xấp xỉ một hàm liên tục bất kỳ với độ chính xác tùy ý. Phương pháp chứng minh này có tính kiến thiết, từ đó xây dựng được một thuật toán để xấp xỉ các hàm liên tục. Và cuối cùng là ví dụ minh họa cho các kết quả.

1. INTRODUCTION

Fuzzy systems including Takagi-Sugeno systems and Mamdani systems are capable of approximating a wide class of functions, such as continuous functions and integrable functions [2, 3, 5, 14, 18], and so on. Like artificial neural networks [14], the approximation research is of much theoretical importance as it can enrich approximation theory [12, 15, 17], also of practical importance as it has found very useful in many applied areas, e.g., system identification [6], pattern recognition[10], adaptive control [11], etc.

In most rule base fuzzy systems, the fuzzy rule base consisting of a number of inference rules defined as IF. . .THEN. . . is a key part and it is often assumed to be complete. Thus, the rule base is valid for all possible conditions. When the number of the system input variables increases, the number of rules in the complete fuzzy rule base increases exponentially. Specifically, suppose that there are n input variables and m fuzzy sets are defined for each variable, then the number of rules in the standard fuzzy systems is m^n. That is the 'Rule explosion' problem, which will not only generate the complicated system structures, but also cause long computational time, even memory overload of the computer.

An efficient tool to deal with the 'rule explosion' problem is the hierarchical system by which a fuzzy systems can be decomposed into a number of hierarchically connected low-dimensional systems (see in [12], [13] and [14] for more details). Instead of applying a fuzzy systems with higher-dimensional input, a number of lower-dimensional fuzzy systems are linked in a hierarchical fashion. By such a hierarchy, the number of the fuzzy rules will increase linearly with the number of the input variables. We can see in [7] that, if we define m fuzzy sets for each variable, including the internal variables $y_1, \ldots, y_{n-2}$, the total number of rules is $(n-1)m^2$, which is a linear function of the number of input variables n. So the hierarchical fuzzy systems can be efficiently used in some large scale systems. And the application of fuzzy systems is undoubtedly extended and expanded. Our paper focuses on the hierarchical fuzzy systems in input - output (I/O) relationship representation and approximating properties. Naturally, we may put forward an important problem, how can the approximating capability of a hierarchical fuzzy systems be analyzed? So we have to analyze the approximate representation of a function by the hierarchical fuzzy systems. In order to analyze this problem, we show that the I/O relationship of a hierarchical fuzzy systems can be represented as a standard fuzzy system. Based on the fact that the universal approximation of hierarchical fuzzy systems is studied systematically, we see that the objective fuzzy system can be constructed directly. Since with a given degree of accuracy, the constructive representation of approximation fuzzy system may be set up, the achievements related the subjects are operative. Therefore, this method has useful applications in practice.

This paper is organized as follows. In section 2, we define a class of piecewise linear functions for the study of approximation, and introduce a class of hierarchical fuzzy systems with spline functions of degree two as membership functions of the input variables. The corresponding I/O relationship is represented in this section as one of a standard fuzzy system. The universal approximation of the hierarchical systems is shown in Section 3, and the number of the fuzzy rules in the rule base of the hierarchical system is estimated. In Section 4, we give an example to show that the hierarchical fuzzy systems can be uniformly approximate continuous functions defined in a high dimensional space, and the fuzzy rules in their rule base are strikingly less than those in the standard fuzzy systems that ensure the same approximating accuracy.

2. FORMULATION OF HIERARCHICAL FUZZY SYSTEMS

A. Piecewise linear function

Let $\mathbb{N}$ and $\mathbb{R}$ be the set of natural numbers and real numbers, respectively, $\mathbb{R}^n$ be the n dimensional Euclidean space. Let $U = [-1, 1]^n \subset \mathbb{R}^n$, by $\|\cdot\|_U$ we denote the maximum norm on U, i.e., if $f : U \to \mathbb{R}$, $\|f\|_U = sup\{|f(x)|\ |x \in U\}$.

To analyze the approximation capability of hierarchical fuzzy systems, we introduce piecewise linear functions as intermediate functions.

Definition 2.1. A continuous function $S : \mathbb{U} \to \mathbb{R}$ is said to be a piecewise linear function, if there are polyhedrons $\Delta_1, \ldots, \Delta_{N(S)} \subset U$, $U = \bigcup_{j=1}^{N(S)} \Delta_j$ such that S is linear on each

Δ_j $(j = 1, 2, \ldots, N(S))$, i.e., the function S has the following form

$$S(x_1, \ldots, x_n) = \begin{cases} \lambda_{10} + \lambda_{11}x_1 + \cdots + \lambda_{1n}x_n, & \text{if } (x_1, \ldots, x_n) \in \Delta_1, \\ \ldots\ldots\ldots\ldots\ldots & \\ \lambda_{N(S)0} + \lambda_{N(S)1}x_1 + \cdots + \lambda_{N(S)n}x_n, & \text{if } (x_1, \ldots, x_n) \in \Delta_{N(S)}, \end{cases}$$

where λ_{ij} $(i = 1, 2, \ldots, N(S), j = 0, 1, \ldots, n)$ are appropriate real numbers.

B. Spline functions as fuzzy numbers

For given $N \in \mathbb{N}$, we partition $[-1, 1]$ into 2N equal parts: $[\frac{j-1}{N}, \frac{j}{N}]$, $j \in \{-N+1, \ldots, N\}$. Setting $I = \{0, \pm 1, \ldots, \pm N\}$. Let A be a fuzzy number on $\mathbb{R}$ with membership function is a spline function of degree two as follows:

$$A(x) = \begin{cases} 2(Nx+1)^2 & \text{if } -\frac{1}{N} \le x \le -\frac{1}{2N}, \\ 1 - 2(Nx)^2 & \text{if } -\frac{1}{2N} \le x \le \frac{1}{2N}, \\ 2(Nx-1)^2 & \text{if } \frac{1}{2N} \le x \le \frac{1}{N}, \\ 0 & \text{otherwise.} \end{cases}$$

On the i–th axis (i=1,2,...,n), we consider the fuzzy partition of $[-1, 1]$ by spline fuzzy numbers A_{ij} $(j \in I)$ with membership functions as follows:

$$A_{i(-N)}(x) = \begin{cases} A(x+1) & \text{if } x \ge -1, \\ 0 & \text{otherwise.} \end{cases} \quad A_{iN}(x) = \begin{cases} A(x-1) & \text{if } x \le 1, \\ 0 & \text{otherwise} \end{cases}$$

and $A_{ij}(x) = A(x - j/N)$ $(j = 0, \pm 1, \ldots, \pm(N-1))$ are obtained by parallel translation of A.

Remark 2.1. For each $i \in \{1, 2, \ldots, n\}$, we have $\sum_{j=-N}^{N} A_{ij}(x) \equiv 1$, for all $x \in [-1, 1]$.

C. Hierarchical fuzzy systems

In the following part, we assume that $x_1, x_2, \ldots, x_n$ are input variables of the hierarchical fuzzy system. Since we can always scale the input variables so that they all belong to $[-1, 1]$, so, without loss of generality, we suppose that $x = (x_1, \ldots, x_n) \in U$.

The main idea of a hierarchical fuzzy system is to put the input variables into collection of low-dimensional fuzzy systems, instead of a single high-dimensional fuzzy system as in the usual case. The low-dimensional fuzzy systems are Takagi-Sugeno (TS) fuzzy systems, which were proposed in [9] as an alternative to the standard fuzzy systems. The hierarchical fuzzy system with M level is constructed as follows:

• The first level is a TS fuzzy system with n_1 $(n_1 < n)$ input variables $x_1, \ldots, x_{n_1}$, which are constructed by the following rules:

IF x_1 is A_{1j_1} AND ... AND x_{n_1} is $A_{n_1 j_{n_1}}$ THEN $y_1 = a_{j_1 \ldots j_{n_1}}$, $(j_1, \ldots, j_{n_1} \in I)$.

The output $f_1(x_1, \ldots, x_{n_1})$ of this fuzzy system is computed by

$$y_1 = f_1(x_1, \ldots, x_{n_1}) = \frac{\sum\limits_{j_1,\ldots,j_{n_1}=-N}^{N} \prod\limits_{i=1}^{n_1} A_{ij_i}(x_i).a_{j_1\ldots j_{n_1}}}{\sum\limits_{j_1,\ldots,j_{n_1}=-N}^{N} \prod\limits_{i=1}^{n_1} A_{ij_i}(x_i)}. \tag{1}$$

From Remark 2.1 we can rewrite (1) in the following reduced form

$$y_1 = f_1(x_1, \ldots, x_{n_1}) = \sum_{j_1,\ldots,j_{n_1}=-N}^{N} \prod_{i=1}^{n_1} A_{ij_i}(x_i).a_{j_1\ldots j_{n_1}}. \tag{2}$$

• The second level is a TS fuzzy system with $n_2 + 1$ input variables $x_{n_1+1}, \ldots, x_{n_1+n_2}$, and the intermediate variable y_1, which are constructed from the following rules:

IF x_{n_1+1} is $A_{n_1+1j_{n_1+1}}$ AND ... AND $x_{n_1+n_2}$ is $A_{n_1+n_2j_{n_1+n_2}}$ AND y_1 is B_{1k_1} THEN $y_2 = a_{j_{n_1+1}\ldots j_{n_1+n_2}} + c^2_{k_1} y_1$.

The output $y_2 = f_2(x_{n_1+1}, \ldots, x_{n_1+n_2}, y_1)$ of this fuzzy system is computed by

$$y_2 = \frac{\sum\limits_{j_{n_1+1},\ldots,j_{n_1+n_2},k_1=-N}^{N} \prod\limits_{i=n_1+1}^{n_1+n_2} A_{ij_i}(x_i) B_{1k_1}(y_1).(a_{j_{n_1+1}\ldots j_{n_1+n_2}} + c^2_{k_1} y_1)}{\sum\limits_{k_1=-N}^{N} B_{1k_1}(y_1)}. \tag{3}$$

Substituting y_1 from (2) into (3), we have

$$y_2 = \frac{\sum\limits_{j_1,\ldots,j_{n_1+n_2},k_1=-N}^{N} \prod\limits_{i=1}^{n_1+n_2} A_{ij_i}(x_i) B_{1k_1}(y_1).a_{k_1 j_1\ldots j_{n_1+n_2}}}{\sum\limits_{k_1=-N}^{N} B_{1k_1}(y_1)}, \tag{4}$$

where

$$a_{k_1 j_1\ldots j_{n_1+n_2}} = a_{j_{n_1+1}\ldots j_{n_1+n_2}} + c^2_{k_1} a_{j_1\ldots j_{n_1}}. \tag{5}$$

• The construction continues until $m-$th level $(m = 2, \ldots, M)$ with $n_m + 1$ input variables $x_{n_1+\cdots+n_{m-1}+1}, \ldots, x_{n_1+\cdots+n_m}, y_{m-1}$. For convenient, we denote $d_k = n_1 + n_2 + \cdots + n_k$ $(k = 2, \ldots, M)$. By an argument analogous to the previous one, we get

$$y_m = \frac{\sum\limits_{\substack{k_1,\ldots,k_{m-1}\\ j_1,\ldots,j_{d_m}=-N}}^{N} \prod\limits_{i=1}^{d_m} A_{ij_i}(x_i) \prod\limits_{i=1}^{m-1} B_{ik_i}(y_i) a_{k_1\ldots k_{m-1} j_1\ldots j_{d_m}}}{\sum\limits_{k_1,\ldots,k_{m-1}=-N}^{N} \prod\limits_{i=1}^{m-1} B_{ik_i}(y_i)}, \tag{6}$$

where

$$a_{k_1\ldots k_{m-1}j_1\ldots j_{d_m}} = a_{j_{d_{m-1}+1}\ldots j_{d_m}} + c^m_{k_{m-1}} a_{k_1\ldots k_{m-2}j_1\ldots j_{d_{m-1}}} \tag{7}$$

for all $m = 2, 3, \ldots, M$. When $m = M$ we have $d_M = n_1 + \cdots + n_M = n$. Thus all the input variables are used in one of the levels.

We call $y_1, y_2, \ldots, y_M$, defined by equations (2), (4) and (6) with the parameters $a_{k_1\ldots k_{m-1}j_1\ldots j_{d_m}}$, $c^m_{k_{m-1}}$ $(m = 2, \ldots, M)$ satisfying iterative scheme (5) and (7), are spline functions of hierarchical fuzzy system.

3. UNIVERSAL APPROXIMATION PROPERTIES OF SPLINE HIERARCHICAL FUZZY SYSTEMS

In this section, we prove that the spline function hierarchical fuzzy systems are universal approximators. Our method is directly based on the I/O relationship of the functions to be approximated, no intermediate step needed. Specially, for given error bounds $\varepsilon > 0$, we can estimate the minimum number of fuzzy sets for each variables.

Problem. Let f be a continuous function on the compact set U. Suppose that the analytic formula of f is unknown, but for arbitrary $x \in U$, we can obtain the value $f(x)$. Our objective is to design a spline function hierarchical fuzzy system, which can approximate the function f with arbitrary accuracy.

Theorem 3.1. *Suppose that $S : U \to \mathbb{R}$ is a piecewise linear function. Then for arbitrary $\varepsilon > 0$, there exists spline function hierarchical fuzzy system $y_1, y_2, \ldots, y_M$ such that $\|S - y_M\|_U < \varepsilon$.*

Proof. Let

$$S(x_1, \ldots, x_n) = \begin{cases} \lambda_{10} + \lambda_{11}x_1 + \cdots + \lambda_{1n}x_n, & \text{if } (x_1, \ldots, x_n) \in \Delta_1, \\ \ldots\ldots\ldots\ldots\ldots & \\ \lambda_{N(S)0} + \lambda_{N(S)1}x_1 + \cdots + \lambda_{N(S)n}x_n, & \text{if } (x_1, \ldots, x_n) \in \Delta_{N(S)}, \end{cases}$$

be a piecewise linear function on U. Given $\varepsilon > 0$, we will show that there exists a natural number N large enough, such that the spline function of hierarchical fuzzy systems, which is constructed in Section 2, satisfies $\|S - y_M\|_U < \varepsilon$.

In fact, we define $a_{k_1\ldots k_{M-1}j_1\ldots j_n}$ for $k_1, \ldots, k_{M-1}, j_1, \ldots, j_M \in I$ respectively as follows:

$$a_{k_1,\ldots,k_{M-1}j_1,\ldots,j_n} = S(j_1/N, \ldots, j_n/N). \tag{8}$$

Let $x = (x_1, \ldots, x_n) \in U$ and $j_1, \ldots, j_n \in I$ such that $\prod\limits_{i=1}^{n} A_{ij_i}(x_i) \neq 0$. From the definition of A_{ij_i} we have $|x_i - \frac{j_i}{N}| \leq \frac{2}{N}$ $(i = 1, \ldots, N)$. Hence, applying the mean value theorem we have

$$\left| S(x_1, \ldots, x_n) - S(\frac{j_1}{N}, \ldots, \frac{j_n}{N}) \right| \leq \sum_{i=1}^{n} |x_i - \frac{j_i}{N}| D_i(S) \leq \frac{2}{N} \sum_{i=1}^{n} D_i(S), \tag{9}$$

where $D_i(S) = \sup\limits_{x\in U}\{\frac{\partial S}{\partial x_i}(x)\}$. Combining (8) and (9), we get

$$\|S - y_M\|_U =$$

$$= \sup_{x\in U}\left| \frac{\sum\limits_{\substack{k_1,\dots,k_{m-1}\\ j_1,\dots,j_{d_m}=-N}}^{N} \prod\limits_{i=1}^{d_m} A_{ij_i}(x_i) \prod\limits_{i=1}^{m-1} B_{ik_i}(y_i)\big(a_{k_1\dots k_{m-1}j_1\dots j_{d_m}} - S(x)\big)}{\sum\limits_{k_1,\dots,k_{m-1}=-N}^{N} \prod\limits_{i=1}^{m-1} B_{ik_i}(y_i)} \right|$$

$$\le \sup_{x\in U}\left| \frac{\sum\limits_{\substack{k_1,\dots,k_{m-1}\\ j_1,\dots,j_{d_m}=-N}}^{N} \prod\limits_{i=1}^{d_m} A_{ij_i}(x_i) \prod\limits_{i=1}^{m-1} B_{ik_i}(y_i)\left|S(\frac{j_1}{N},\dots,\frac{j_n}{N}) - S(x)\right|}{\sum\limits_{k_1,\dots,k_{m-1}=-N}^{N} \prod\limits_{i=1}^{m-1} B_{ik_i}(y_i)}\right.$$

$$\le \frac{2}{N}\sum_{i=1}^{n} D_i(S). \tag{10}$$

Finally, by choosing $N > \frac{2\sum\limits_{i=1}^{n} D_i(S)}{\varepsilon}$, we have $\|S - y_M\|_U < \varepsilon$.

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Theorem 3.2. *Suppose that $f : U \to \mathbb{R}$ is a continuous function. Then for arbitrary $\varepsilon > 0$, there exists a piecewise linear function S such that $\|S - f\|_U < \varepsilon$.*

Proof. For simplicity, let $n = 2$ (the case when $n > 2$ can be proved similarly). For any $N \in \mathbb{N}$, we partition every side of U into $2N$ equal parts, we obtain $4N^2$ subparallelepiped $U_{ij} = \{(x_1, x_2) | \frac{i-1}{N} \le x_1 \le \frac{i}{N},\ \frac{j-1}{N} \le x_1 \le \frac{j}{N}\}$ $(i, j = -N+1, \dots, N)$.

Because f is uniformly continuous on compact set U, we have for all $\varepsilon > 0$, there exists N large enough such that $|f(x_1, x_2) - f(x'_1, x'_2)| \le \frac{\varepsilon}{4}$, $\forall (x_1, x_2),\ (x'_1, x'_2) \in U_{ij}$. Setting $\overline{\delta}_{ij} = \sup\limits_{x\in U_{i,j}}\{f(x)\}$ and $\underline{\delta}_{ij} = \inf\limits_{x\in U_{i,j}}\{f(x)\}$, we see that $|\overline{\delta}_{ij} - \underline{\delta}_{ij}| \le \frac{\varepsilon}{4}$.

Respectively linking a pair of opposed vertices of U_{ij}, we obtain $8N^2$ equicrurally rectangular triangles Δ_k $(k = 1, \dots, 8N^2)$. Suppose that 3 vertices of one rectangular triangles Δ_k are $(x_1^k, y_1^k), (x_2^k, y_2^k)$ and (x_3^k, y_3^k). The plane equation $z = S(x, y)$ determined by $(x_1^k, y_1^k, f(x_1^k, y_1^k)), (x_2^k, y_2^k, f(x_2^k, y_2^k))$ and $(x_3^k, y_3^k, f(x_3^k, y_3^k))$ of the vertices of Δ_k is given as follows:

$$\Delta = \begin{vmatrix} x & y & z & 1 \\ x_1^k & y_1^k & f(x_1^k, y_1^k) & 1 \\ x_2^k & y_2^k & f(x_2^k, y_2^k) & 1 \\ x_3^k & y_3^k & f(x_3^k, y_3^k) & 1 \end{vmatrix} = 0$$

Without loss of generality, we suppose $x_1^k = x_3^k$, $x_2^k = x_1^k + 1/N$, $y_1^k = y_2^k$ and $y_3^k = y_1^k + 1/N$. Adding -1 times the second row to the rest of rows of Δ. After that, by expanding the determinant on the first row we obtain

$$\begin{aligned} S^k(x,y) &= f(x_1^k, y_1^k) \\ &+ \frac{(x-x_1^k)(f(x_2^k,y_2^k)-f(x_1^k,y_1^k)) + (y-y_1^k)(f(x_3^k,y_3^k)-f(x_1^k,y_1^k))}{1/N}, \end{aligned} \tag{11}$$

with $(x,y) \in \Delta_k$. From (11) the following inequality holds

$$\begin{aligned} |f(x,y) - S^k(x,y)| &\le |f(x,y) - f(x_1^k,y_1^k)| + \frac{|x-x_1^k|}{1/N}|f(x_2^k,y_2^k) - f(x_1^k,y_1^k)| \\ &+ \frac{|y-y_1^k|}{1/N}|f(x_3^k,y_3^k) - f(x_1^k,y_1^k)| \le \frac{\varepsilon}{4} + \frac{\varepsilon}{4}\sqrt{2} + \frac{\varepsilon}{4}\sqrt{2} < \varepsilon. \end{aligned} \tag{12}$$

Let S be the connection function of S^k $(k = 1, 2, \dots, 8N^2)$, i.e., with $k \in \{1, 2, \dots, 8N^2\}$ we have $S(x,y) = S^k(x,y)$ for all $(x,y) \in \Delta_k$. Then S is a piecewise linear function on U and $|S(x,y) - f(x,y)| < \varepsilon,\ \forall (x,y) \in U$. Hence, $\|S - f\|_U < \varepsilon$. This completes the proof.

■

Remark 3.1. From equation (11), we have

$$\frac{\partial S^k}{\partial x} = \frac{f(x_2^k,y_2^k) - f(x_1^k,y_2^k)}{1/N} \text{ and } \frac{\partial S^k}{\partial y} = \frac{f(x_3^k,y_3^k) - f(x_3^k,y_1^k)}{1/N}, \tag{13}$$

Hence, we have

$$D_1(S) = \sup\{\frac{f(x+h,y) - f(x,y)}{h} | x, x+h \in [-1,1]\}, \tag{14}$$

$$D_2(S) = \sup\{\frac{f(x,y+h) - f(x,y)}{h} | y, y+h \in [-1,1]\}. \tag{15}$$

Thus, we can estimate the supremum $\max\{D_1(S), D_2(S)\}$ from the value of f.

Theorem 3.3. *Let $f : U \to \mathbb{R}$ be a continuous function. Then for arbitrary $\varepsilon > 0$, there exist hierarchical fuzzy systems $y_1, y_2, \dots, y_M$ such that $\|f - y_L\|_U < \varepsilon$.*

Proof. By the proof of Theorem 3.2, for $\varepsilon > 0$, if we partition $[-1,1]^n$ into identical cubes and divide further the cubes into $n-$dimension polyhedrons $\Delta_1, \dots, \Delta_N$ respectively, then a piecewise linear function S is defined. Moreover, $\|f - S\|_U < \varepsilon/2$. By (14) and (15) we see that

$$\begin{aligned} D &= \max_{i=1,\dots,n} D_i(S) \\ &= \max_{i=1,\dots,n} \left\{ \sup_{x_i, x_i+h \in [-1,1]} \{\frac{f(x_1,\dots,x_i+h,\dots,x_n) - f(x_1,\dots,x_n)}{h}\} \right\} \end{aligned} \tag{16}$$

Thus, by Theorem 3.1, if $N > \frac{4nD}{\varepsilon}$, then we have $\|S - y_M\| < \frac{\varepsilon}{2}$. Hence

$$\|f - y_M\|_U \le \|f - S\|_U + \|S - y_M\|_U < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\blacksquare$

From the constructive proof of Theorems $3.1 - 3.3$, we can establish the approximation representation of f by a hierarchical fuzzy system as follows:

• Step 1. By (16) we approximately calculate the supremum value of D. Hence, the minimum value of N can be estimated.

• Step 2. With a given function f, we define the suitable antecedent fuzzy sets B_{ik_i} $(i = 1, \ldots, M-1;\ k_i \in I)$.

• Step 3. From (8), the parameters of hierarchical fuzzy systems can be backwards determined by the following systems

$$\begin{cases} a_{k_1\ldots k_{M-1}j_1\ldots j_n} = a_{j_{d_{M-1}+1}\ldots j_{d_M}} + c^M_{k_{M-1}} a_{k_1\ldots k_{M-2}j_1\ldots j_{d_{M-1}}} = S(\frac{j_1}{N}, \ldots, \frac{j_n}{N}) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{k_1\ldots k_{m-1}j_1\ldots j_{d_m}} = a_{j_{d_{m-1}+1}\ldots j_{d_m}} + c^m_{k_{m-1}} a_{k_1\ldots k_{m-2}j_1\ldots j_{d_{m-1}}} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{k_1j_1\ldots j_{n_1+n_2}} = a_{j_{n_1+1}\ldots j_{n_1+n_2}} + c^2_{k_1} a_{j_1\ldots j_{n_1}} \end{cases} \tag{17}$$

• Step 4. By (2), (4) and (6) we establish the spline function hierarchical fuzzy systems $y_1, y_2, \ldots, y_M$.

4. NUMERICAL EXAMPLE

We consider a continuous function $f : [-1, 1]^3 \to \mathbb{R}$ of the form:

$$f(x_1, x_2, x_3) = \sin(-\frac{x_1^2 + x_2^2 + x_3^2}{100}).$$

For given error bound $\varepsilon = 0.1$, we estimate the supremum of D by $D \le 1/50$. Hence, the minimum of N can be calculated by $N = 3$, then by Theorem 3.3, we have $\|y_2 - f\|_U < \varepsilon$, with the spline function hierarchical fuzzy systems y_1, y_2 are constructed as:

$$\begin{cases} y_1 = \sum\limits_{j_1,j_2=-N}^{N} A_{1j_1}(x_1)A_{2j_2}(x_2)a_{j_1j_2}, \\ y_2 = \dfrac{\sum\limits_{j_3,k_1=-N}^{N} A_{3j_3}(x_3)B_{1k_1}(y_1)(a_{j_3}+c^2_{k_1}y_1)}{\sum\limits_{k_1=-N}^{N} B_{1k_1}(y_1)}. \end{cases}$$

By (17) and the fact $f(j_1/N, j_2/N, j_3/N) = S(j_1/N, j_2/N, j_3/N)$, we backwards determine the parameters $a_{j_1j_2}$, a_{j_3} and $c^2_{k_1}$ by the following systems

$$\begin{cases} a_{j_1j_2j_3} = f(j_1/N, j_2/N, j_3/N), \\ a_{j_3} + c_{k_1}^2 a_{j_1j_2} = a_{j_1j_2j_3}. \end{cases}$$

We choose some sample points randomly in $[-1,1]^3$ as shown in Table , and the corresponding approximating errors of the hierarchical fuzzy systems at these points are also given in Table .

Table 1. Approximating errors at randomly choosen sample points

Sample points	Values of function f	Values of y_2	Error $\delta = \|f - y_2\|$
(1,0.9,-0.8)	-1.216969958800586e-002	-1.226318028973910e-002	9.348070173323571e-005
(0,-0.95,0.45)	7.662425017669018e-003	7.545668315662267e-003	1.167567020067513e-004
(-0.2,0.5,-1)	8.829885256216155e-003	8.831145163406079e-003	1.259907189923859e-006
(0.9,0.9,-0.9)	-7.289935430090077e-003	-7.548800621191955e-003	2.588651911018779e-004
(0.1,0.2,-0.1)	-7.999999991466668e-005	-8.148082744390174e-005	1.480827529235057e-006
(1,0.9,0)	-1.728913855562812e-002	-1.754824412618336e-002	2.591055705552375e-004

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