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LYAPUNOV-BASED SYNCHRONIZATION OF TWO COUPLED CHAOTIC HINDMARSH-ROSE NEURONS

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Abstract. This paper addresses the synchronization of coupled chaotic Hindmarsh-Rose neurons. A sufficient condition for self-synchronization is first attained by using Lyapunov method. Also, to achieve the synchronization between two coupled Hindmarsh-Rose neurons when the self-synchronization condition not satisfied, a Lyapunov-based nonlinear control law is proposed and its asymptotic stability is proved. To verify the effectiveness of the proposed method, numerical simulations are performed.

Keywords. Chaos, Hindmarsh-Rose neurons, Lyapunov function, nonlinear control, synchronization.

1. INTRODUCTION

Neurons play an important role in processing the information in the brain. To understand the behaviour of individual neurons and further comprehend the biological information processing of neural networks, various neuronal models have been proposed [1–4]. One of the most important models is the Hodgkin-Huxley model [1]. This model describes how action potentials are initiated and propagated in the squid giant axon in term of time- and voltage-dependent conductance of sodium and potassium. However, the Hodgkin-Huxley model consists of a large number of nonlinear equations as well as parameters that makes it difficult to study the behaviour of neuronal networks. The Hindmarsh-Rose (HR) model, a simplification of the Hodgkin-Huxley and the Fitzhugh models, provides very realistic descriptions on various dynamic features of biological neurons such as rapid firing, bursting, and adaptation [4]. Therefore, the HR model is getting more attention in the study of many features of the brain activity. Individual neurons can exhibit irregular behaviour, whereas ensembles of different neurons might synchronize in order to process biological information or to produce regular and rhythmical activities [5–7]. Therefore, the study of synchronization processes for populations of interacting chaotic neurons is basic to the understanding of some key issues in neuroscience.

Since the discovery of chaotic synchronization [8], various modern control methods have been proposed for achieving the synchronization of chaotic systems in recent years [8–11]. In neuroscience, most investigations have focused on the synchronization of two coupled neurons, whose resolution aids in the understanding of the synchronization processes in neural networks [12-31, and reference therein]. The synchronization between interacting neurons can be classified into two types: the first pertains to natural coupling, in which the effects of the synapse types and internal noises on synchronization (self-synchronization) are issues [12–18]; the second pertains to artificial coupling, in which an explicit control signal is applied in order to archive synchronization [18–31]. Following the first approach, many studies have confirmed that when the intensity of an internal noise exceeds a critical value, the self-synchronization can be achieved [12–14]. Other numerical results have shown that strong coupling can also synchronize a system of two FitzHugh-Nagumo neurons [15–17]. Also, the effect of difference in coupling strengths caused by the unidirectional gap junctions and the impact of time delay due to separation of neurons on the coupled FitzHugh-Nagumo neurons has been investigated [18]. For the second approach, various methods using modern control theories have been proposed to synchronize two chaotic neurons. In [15, 16, 19, 20], different Lyapunov based-nonlinear feedback control laws were developed to synchronize two coupled chaotic FitzHugh-Nagumo neurons under external electrical stimulation. The backstepping control technique was utilized to achieve the synchronization in coupled FitzHugh-Nagomo neuron system [21] and in coupled Hindmarsh-Rose neuron system [22]. Various sliding mode control laws were also proposed to synchronize the coupled neuron system [23–26]. In order to synchronize coupled chaotic neuron system with unknown or uncertain parameters, many adaptive and robust control laws were also proposed [27–31]. Despite many control methods have been proposed to synchronize coupled chaotic neurons, much detailed work remains to be done.

In this paper, the synchronization of two coupled chaotic HR neurons is studied. First, the dynamic behaviour of a single HR neuron model is reviewed. Then, from the Lyapunov stability theory, the author derives a sufficient condition of the coupling coefficient that guarantees the self-synchronization. Lastly, for the case that the coupling coefficient does not satisfy the self-synchronization condition, a Lyapunov-based nonlinear control law, which guarantees the synchronization of two coupled HR neurons, is designed. The proposed control law can be extended to cover the case that the external electrical signals applied to each neuron are different. The main contributions of this paper are to:

(1) Provide a sufficient condition for self-synchronization of coupled chaotic HR neurons; and

(2) Propose a new nonlinear control law for achieving the synchronization of coupled chaotic HR neurons.

The paper is organized as follows: In Section 2, the dynamic behaviour of a single HR neuron model under various applied currents is reviewed. In Section 3, a sufficient condition of the coupling coefficient for self-synchronization of two coupled HR neurons is proposed. The details of the design procedure of the nonlinear controller based on a Lyapunov function are also provided in this section. Finally, conclusions are drawn in Section 4.

2. DYNAMICS OF A SINGLE HR NEURON

2.1. Time responses of a single HR neuron

The HR neuron model, a modification of the Hodgkin-Huxley and the FitzHugh models, is a genetic model of the membrane potential which enables to simulate spiking, bursting and chaos phenomena in biological neurons. This model is described by the following three-dimensional

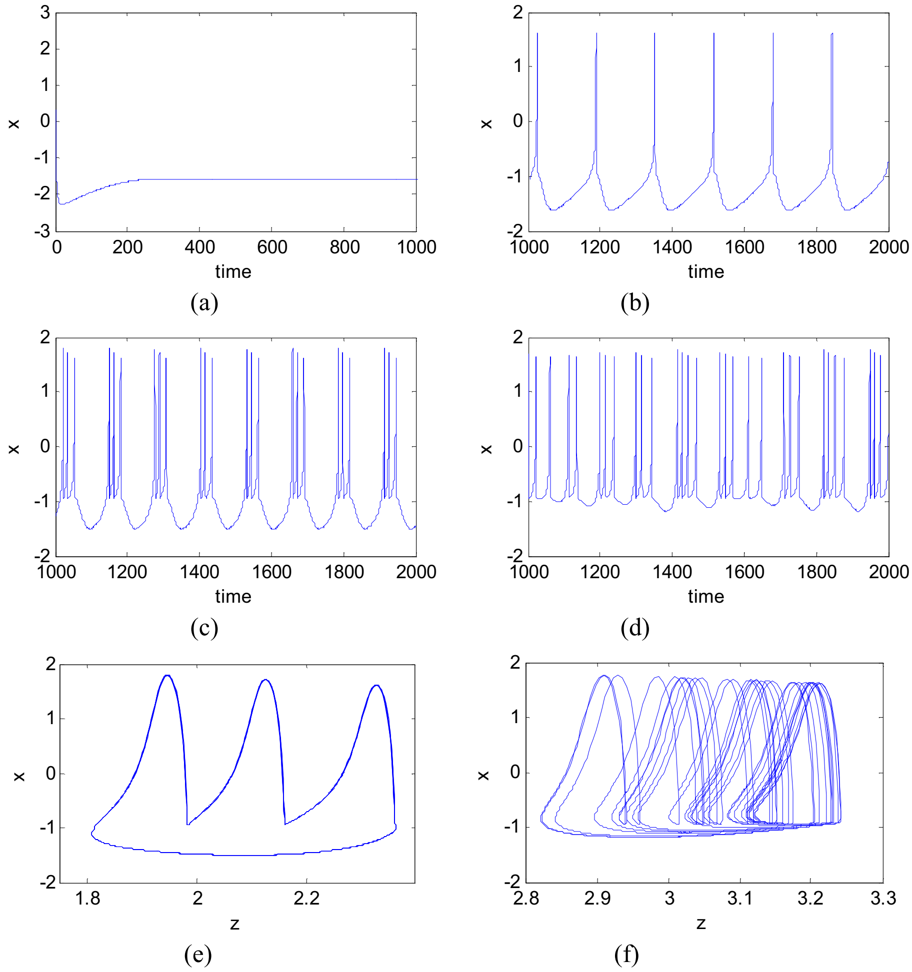


Figure 1: Time responses of the membrane potential for various value of the stimulated current: (a) resting state when $I = 0$, (b) tonic spiking when $I = 1.2$, (c) regular bursting when $I = 2.2$, (d) chaotic bursting when $I = 3.1$, (e) the $x - z$ phase portrait when $I = 2.2$, (f) the $x - z$ phase portrait when $I = 3.1$.

system of nonlinear first order differential equations.

$$\begin{aligned} \dot{x} &= a x^2 - x^3 + y - z + I, \\ \dot{y} &= c - d x^2 - y, \\ \dot{z} &= r[b(x - k) - z], \end{aligned} \tag{1}$$

where x represents the membrane potential, y is the recovery variable associated with the fast current of Na^+ or K^+ ions, z is the adaptation current associated with the slow current of, for instance, Ca^+ ions, I is the applied current that mimics the membrane input current for biological neurons, and a, b, c, d, r, and k are constants. The values of these constant parameters are chosen in such a way that the response of (1) is similar to that obtained experimentally from the identified cell in Lymnaea visceral ganglion as reported by Hindmarsh and Rose [4]. In this paper, the same values of these parameters are used; they are $a = 3.0$,

$b = 4.0$, $c = 1.0$, $d = 5.0$, $r = 0.006$, and $k = -1.56$. By varying the amplitude of the applied current I, various firing patterns can be observed as shown in Figure 1. When $I = 0$, the membrane potential is constant, the neuron is in a resting state (Figure 1a). When $I = 1.2$, the neuron exhibits tonic spiking (Figure 1b). A regular bursting appears when the amplitude of the applied current is increased to $I = 2.2$ as shown in Figure 1c. Finally, a chaotic bursting of the HR neuron can be observed at $I = 3.1$ (Figure 1d). The $x - z$ phase portraits for the cases $I = 2.2$ and $I = 3.1$ are plotted in Figure 1e and Figure 1f, respectively.

2.2. Bifurcation analysis of a single HR neuron

In order to convey more information about dynamic behaviours of a single HR neuron under varying amplitude of the applied current, the bifurcation of the inter-spike intervals as a function of the applied current I is investigated, as shown in Figure 2. Figure 2 reveals that for small values of the applied current$I < 1.15$, the neuron is in the quiescent state. When the applied current is increased out of $I = 1.15$, the period-one firing patterns appear and this behaviour is maintained for the current up to $I \approx 1.41$. The period-two, -three, and -four firing patterns can be determined in the regions of $1.41 \leqslant I < 1.98$, $1.98 \leqslant I < 2.49$, and $2.49 \leqslant I < 2.75$ respectively. It is obvious from Figure 2 that the HR neuron exhibits chaotic bursting for the values of the applied current in the region of $2.75 \leqslant I < 3.25$. After that, the HR neuron exhibits again the period-two and -one firings with $3.25 \leqslant I < 3.32$ and $I \geqslant 3.32$ respectively.

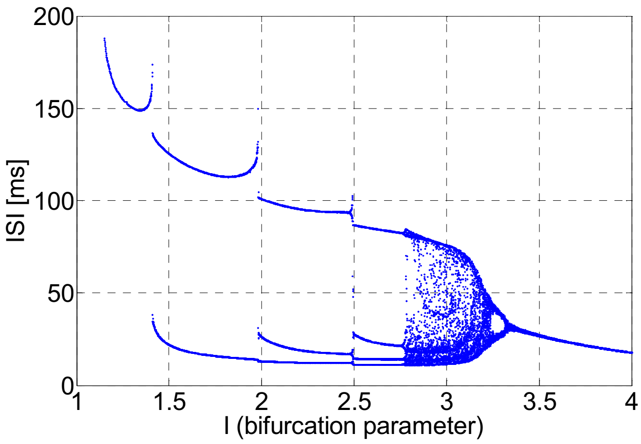


Figure 2: Bifurcation diagram of the inter-spike intervals versus the stimulated current I in a single HR neuron model.

3. SYNCHRONIZATION OF TWO COUPLED HR NEURONS

3.1. Sufficient condition for self-synchronization

Self-synchronization of two neurons due to the external noise has been investigated in [12–14]. Here, the author proposes a theoretical condition of the coupling coefficient for asymptotic self-synchronization of two coupled HR neurons. Based on (1), a coupled HR neuron system can be described as

$$\begin{aligned}
\dot{x}_1 &= a x_1^2 - x_1^3 + y_1 - z_1 - g(x_1 - x_2) + I,\\
\dot{y}_1 &= c - d x_1^2 - y_1,\\
\dot{z}_1 &= r[b(x_1 - k) - z_1],\\
\dot{x}_2 &= a x_2^2 - x_2^3 + y_2 - z_2 - g(x_2 - x_1) + I,\\
\dot{y}_2 &= c - d x_2^2 - y_2,\\
\dot{z}_2 &= r[b(x_2 - k) - z_2],
\end{aligned} \tag{2}$$

where x_i, y_i, and z_i $(i = 1, 2)$ are the state variables and $g$is the positive coupling coefficient.

Definition 3.1. The two coupled HR neurons (2) are said to be globally asymptotically synchronized if, for all initial conditions $x_1(0)$, $y_1(0)$, $z_1(0)$ and $x_2(0)$, $y_2(0)$, $z_2(0)$, $\lim_{t\to\infty} \|x_1(t) - x_2(t)\| = 0$, $\lim_{t\to\infty} \|y_1(t) - y_2(t)\| = 0$, and $\lim_{t\to\infty} \|z_1(t) - z_2(t)\| = 0$

Let the error signals be defined as

$$e_x = x_2 - x_1, \quad e_y = y_2 - y_1, \quad e_z = z_2 - z_1, \tag{3}$$

based on (2), the error dynamics, results in

$$\dot{e}_x = [-2g + a(x_1 + x_2) - (x_1^2 + x_1 x_2 + x_2^2)]e_x + e_y - e_z \tag{4}$$

$$\dot{e}_y = -d(x_2 + x_1)e_x - e_y \tag{5}$$

$$\dot{e}_z = r b e_x - r e_z \tag{6}$$

Equations (4)-(6) can be rewritten in a matrix form as follows.

$$\dot{\mathbf{e}} = (\mathbf{A} + \mathbf{M} + \mathbf{P})\mathrm{e} \tag{7}$$

where $\mathbf{e} = [\ e_x \quad e_y \quad e_z\ ]^T$ and

$$\begin{aligned}
\mathbf{A} &= \begin{bmatrix} -2g & 1 & -1 \\ 0 & -1 & 0 \\ rb & 0 & -r \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} a(x_1 + x_2) - (x_1^2 + x_1 x_2 + x_2^2) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\\
\mathbf{P} &= \begin{bmatrix} 0 & 0 & 0 \\ -d(x_1 + x_2) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{8}$$

Next, let us define the following matrices

$$\mathbf{B} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) = \begin{bmatrix} -2g & \frac{1}{2} & \frac{rb-1}{2} \\ \frac{1}{2} & -1 & 0 \\ \frac{rb-1}{2} & 0 & -r \end{bmatrix} \tag{9}$$

$$\mathbf{N} = \frac{1}{2}(\mathbf{M} + \mathbf{M}^T) = \begin{bmatrix} a(x_1 + x_2) - (x_1^2 + x_1 x_2 + x_2^2) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{10}$$

$$\mathbf{Q} = \frac{1}{2}(\mathbf{P} + \mathbf{P}^T) = \begin{bmatrix} 0 & \frac{-d(x_1+x_2)}{2} & 0 \\ \frac{-d(x_1+x_2)}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{11}$$

Fact 1 *For a symmetric matrix* **S** *and any vector* ***x****, the following inequality holds:*

$$\mathbf{x}^T\mathbf{S}\mathbf{x} \leqslant \lambda_{\max}(\mathbf{S})\mathrm{x}^T\mathbf{x} \tag{12}$$

where $\lambda_{\max}(\mathbf{S})$ *is the maximum eigenvalue of the matrix* ***S.***

Theorem 3.1. *The two coupled HR neurons will achieve self-synchronization with any initial condition ($x_1(0)$, $y_1(0)$, $z_1(0)$, $x_2(0)$, $y_2(0)$, $z_2(0)$) as $t \to \infty$, if there exists a positive scalar g such that the following linear matrix inequality holds:*

$$\begin{bmatrix} -2g+\xi & \frac{1}{2} & \frac{rb-1}{2} \\ \frac{1}{2} & -1+\xi & 0 \\ \frac{rb-1}{2} & 0 & -r+\xi \end{bmatrix} < 0 \tag{13}$$

where $\xi = [(2a+d)+3\kappa]\kappa$ *and* $\kappa = \max_{\{}|x_1|, |x_2|\}$

Proof. Choose a Lyapunov function

$$V_1 = \frac{1}{2}\mathbf{e}^T\mathbf{e} \geqslant 0 \tag{14}$$

The derivative of V_1 along the trajectory of the system represented by (7) is

$$\begin{aligned} \dot{V}_1 =& \frac{1}{2}(\mathbf{e}^T\dot{\mathbf{e}} + \dot{\mathbf{e}}^T\mathbf{e}) = \frac{1}{2}\mathbf{e}^T(\mathbf{A}+\mathbf{A}^T)\mathbf{e} + \frac{1}{2}\mathbf{e}^T(\mathbf{M}+\mathbf{M}^T)\mathbf{e} + \frac{1}{2}\mathbf{e}^T(\mathbf{P}+\mathbf{P}^T)\mathbf{e} \\ =& \mathbf{e}^T\mathbf{B}\mathbf{e} + \mathbf{e}^T\mathbf{N}\mathbf{e} + \mathbf{e}^T\mathbf{Q}\mathbf{e}. \end{aligned} \tag{15}$$

Using Fact 1 yields

$$\dot{V}_1 \leqslant \mathbf{e}^T\mathbf{B}\mathbf{e} + \lambda_{\max}(\mathrm{N})\mathbf{e}^T\mathbf{e} + \lambda_{\max}(\mathbf{Q})\mathrm{e}^T\mathbf{e}. \tag{16}$$

Here, note that

$$\lambda_{\max}(\mathbf{N}) = a(x_1+x_2) - (x_1^2 + x_1x_2 + x_2^2) \tag{17}$$

$$\lambda_{\max}(\mathbf{Q}) = \left|\frac{-d(x_1+x_2)}{2}\right| \tag{18}$$

Since the system described by (2) has bounded trajectories, there exists a positive constant κ, such that are $|x_1| \leqslant \kappa$ and $|x_2| \leqslant \kappa$. Therefore, we have

$$\lambda_{\max}(\mathbf{N}) \leqslant 2a\kappa + 3\kappa^2 \tag{19}$$

$$\lambda_{\max}(\mathbf{Q}) \leqslant d\kappa \tag{20}$$

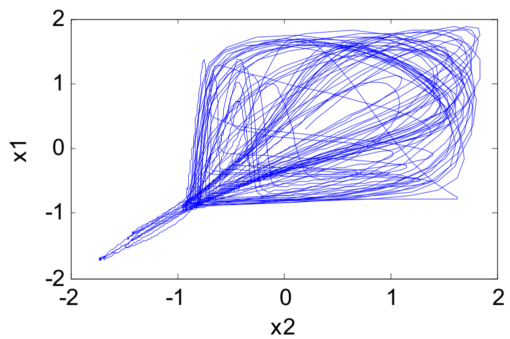
Using the inequalities (19) and (20) results in a new bound of $\dot{V}_1$ as follows.

$$\dot{V}_1 \leqslant \mathbf{e}^T[\mathbf{B} + (2a\kappa + 3\kappa^2)\mathbf{I} + d\kappa\mathbf{I}]\mathbf{e} \tag{21}$$

where **I** denotes the identity matrix. Therefore, if $\mathbf{B} + (2a\kappa + 3\kappa^2)\mathbf{I} + d\kappa\mathbf{I} < 0$, which is equivalent to (13), the error dynamics described in (7) is asymptotically stable according to the Lyapunov stability theory [32]. The proof is completed.

In order to demonstrate the above theorem, the coupled HR neuron system (2) is simulated for different values of the coupling coefficient g. The applied current for each neuron is $I = 3.1$. As shown in Section 2 (for$I = 3.1$), individual neurons exhibit chaotic bursting behaviours. The initial conditions for the first and second neurons are $(x_1(0),\ y_1(0),\ z_1(0)) = (0.3,\ 0.3,\ 3.0)$ and $(x_2(0),\ y_2(0),\ z_2(0)) = (-0.3,\ 0.4,\ 3.2)$ respectively. When the coupling coefficient g = 0.2, self-synchronization between two coupled chaotic HR neurons cannot occur (see Figure 3). However, when we chose g sufficiently large to satisfy the above theorem (for example, g = 3.0), self-synchronization can be achieved as shown in Figure 4.

Remark 3.1. The theorem gives only a sufficient condition for self-synchronization of two coupled HR neurons. In other words, if the coupling coefficient g does not satisfy the above condition, it does not mean that two coupled HR neurons cannot achieve self-synchronization. In fact, numerical simulations revealed that there exists coupling coefficients that, whereas not satisfying the above sufficient condition, are adequate for self-synchronization of two coupled HR neurons.



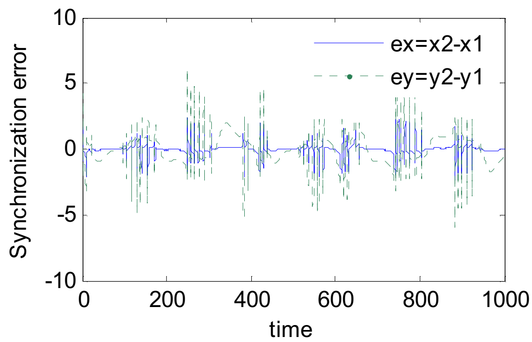


Figure 3: The responses of two coupled HR neurons with the coupling coefficient $g = 0.2$: (a) the x_2x_1 phase portrait, (b) the synchronization errors $e_x = x_2x_1$ and $e_y = y_2y_1$

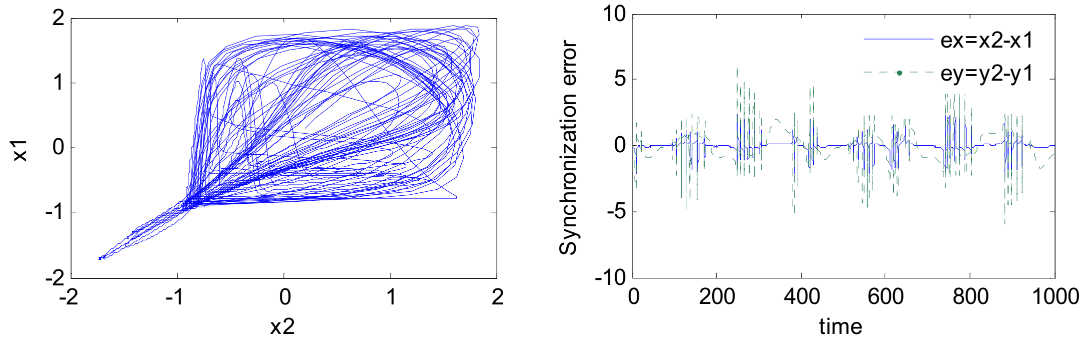
3.2. Synchronization via nonlinear control

The coupled HR neuron system under control can be determined as

$$\begin{cases} \dot{x}_1 = ax_1^2 - x_1^3 + y_1 - z_1 - g(x_1 - x_2) + I, \\ \dot{y}_1 = c - dx_1^2 - y_1, \\ \dot{z}_1 = r[b(x_1 - k) - z_1], \end{cases} \tag{22}$$

$$\begin{cases} \dot{x}_2 = ax_2^2 - x_2^3 + y_2 - z_2 - g(x_2 - x_1) + I + u, \\ \dot{y}_2 = c - dx_2^2 - y_2, \\ \dot{z}_2 = r[b(x_2 - k) - z_2], \end{cases} \tag{23}$$

Equations (22) and (23) are considered as the master and the slave neuron descriptions respectively. Aiming at designing a control law u where the master neuron (22) and the slave



(a) (b)

Figure 4: The responses of two coupled HR neurons with the coupling coefficient $g = 3.0$: (a) the $x_2 x_1$ phase portrait, (b) the synchronization errors $e_x = x_2 x_1$ and $e_y = y_2 y_1$

neuron (23) can be synchronized. With the error signals defined in (3), the error dynamics become

$$\dot{e}_x = [-2g + a(x_1 + x_2) - (x_1^2 + x_1 x_2 + x_2^2)]e_x + e_y - e_z + u, \tag{24}$$

$$\dot{e}_y = -d(x_2 + x_1)e_x - e_y, \tag{25}$$

$$\dot{e}_z = rbe_x - re_z. \tag{26}$$

Define state-dependent terms in (24) and (25) as follows.

$$h_1(x_1, x_2, e_x) = [a(x_2 + x_1) - (x_2^2 + x_1 x_2 + x_1^2)]e_x, \tag{27}$$

$$h_2(x_1, x_2) = -d(x_1 + x_2) \tag{28}$$

Then, (24)-(26) are reduced to

$$\dot{e}_x = -2ge_x + e_y - e_z + h_1(x_1, x_2, e_x) + u, \tag{29}$$

$$\dot{e}_y = h_2(x_1, x_2)e_x - e_y, \tag{30}$$

$$\dot{e}_z = rbe_x - re_z. \tag{31}$$

The synchronization problem is replaced by determining a suitable control law u such that the asymptotic stability of the error dynamics described in (29)-(31) at the origin can be guaranteed.

Theorem 3.2. *Two coupled HR neurons described in (22) and (23) will achieve the synchronization for any initial condition ($x_1(0)$, $y_1(0)$, $z_1(0)$, $x_2(0)$, $y_2(0)$, $z_2(0)$) as $t \to \infty$, if the following control law is used:*

$$u = -h_1(x_1, x_2, e_x) - [h_2(x_1, x_2) + 1]e_y - (rb - 1)e_z \tag{32}$$

Proof. Chose a Lyapunov function

$$V_2 = \frac{1}{2}\left(e_x^2 + e_y^2 + e_z^2\right) \geqslant 0 \tag{33}$$

The derivative of V_2 is given by

$$\dot{V}_2 = -2ge_x^2 - e_y^2 - re_z^2 + e_x e_y + (rb-1)e_x e_z + h_2(x_1,x_2)e_x e_y + h_1(x_1,x_2,e_x)e_x + ue_x. \quad (34)$$

Substituting (32) into (34) yields

$$\dot{V}_2 = -2ge_x^2 - e_y^2 - re_z^2 \leqslant 0 \quad (35)$$

According to Lyapunov theory [32], the asymptotic stability at the origin of (29)-(31) holds, which is equivalent to the fact that two coupled HR neurons described in (22) and (23) are synchronized. The proof is completed.

To demonstrate the effectiveness of the proposed control law, numerical simulations are performed. Here, we choose the applied current $I = 3.1$ and the coupling coefficient $g = 0.2$ such that individual neurons are chaotic bursting and the self-synchronization of two coupled HR neurons cannot occur (as shown in Figure 3). The initial conditions of the master and the slave neurons were chosen as $(x_1(0),\ y_1(0),\ z_1(0)) = (0.3,\ 0.3,\ 3.0)$ and $(x_2(0),\ y_2(0),\ z_2(0)) =$ (-0.3, 0.4, 3.2) respectively. The total simulation time is set to $t = 1000$. The control law in (32) is applied at time $t = 500$. As shown in Figure 5b, the synchronization errors between the master and the slave neurons, $e_x = x_2 - x_1$ and $e_y = y_2 - y_1$, converge asymptotically to zero within a finite period of time after applying the control law. The phase portraits $x_2 - x_1$ before (dashed line) and after (solid line) the application of the control law are plotted in Figure 5a.

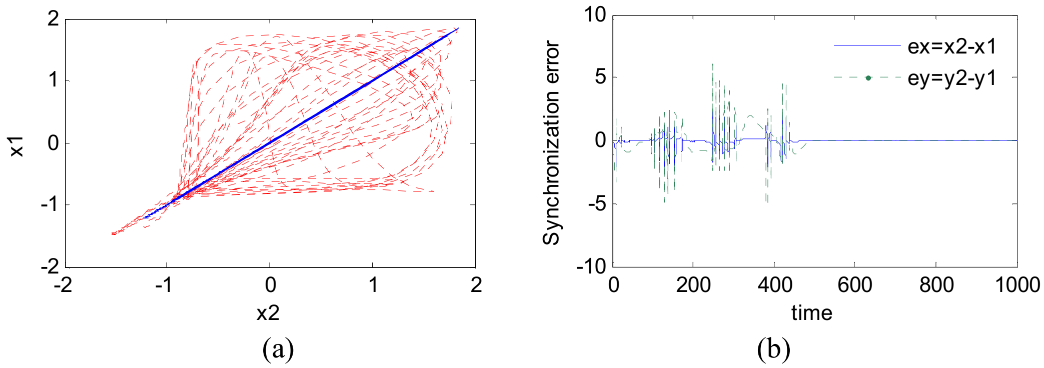
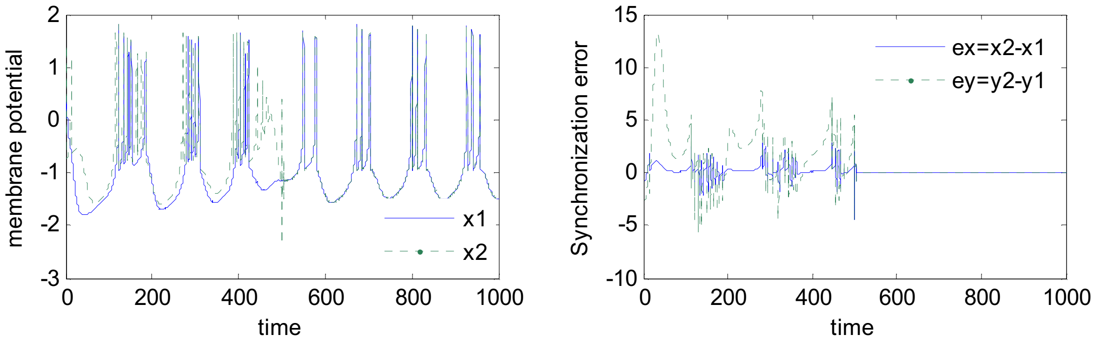


Figure 5: The responses of two coupled HR neurons before and after the application of the control law in (32): (a) the $x_2 x_1$ phase portrait, (b) the synchronization errors $e_x = x_2 x_1$ and $e_y = y_2 y_1$

Remark 3.2. When the applied currents on two neurons are different, say I_1 for the master neuron (22) and I_2 for the slave neuron (23), the control law is extended as follows.

$$u = -h_1(x_1,x_2,e_x) - [h_2(x_1,x_2)+1]e_y - (rb-1)e_z - (I_2 - I_1). \quad (36)$$

Figure 6a shows the time responses of two neurons' membrane potentials when $I_1 = 2.2$ and $I_2 = 3.1$. Before applying the control law, the master neuron exhibits a regular bursting behaviour (solid line) while the slave neuron is chaotic bursting (dashed line). At time t = 500, we apply the control law in (36), the response of the slave neuron switches immediately from the chaotic bursting behaviour to the regular busting behaviour. The synchronization errors $e_x = x_2 - x_1$ and $e_y = y_2 - y_1$ are shown in Figure 6b.



(a) (b)

Figure 6: The responses of two coupled HR neurons before and after the application of the control law in (36) when $I_1 = 2.2$ and $I_2 = 3.1$: (a) the membrane potentials x_1 and x_2, (b) the synchronization errors $e_x = x_2 x_1$ and $e_y = y_2 y_1$

4. CONCLUSIONS

In this paper, the synchronization problem of two coupled chaotic HR neurons is investigated. From the achievement of the self-synchronization of two chaotic neurons, a sufficient condition of the coupling coefficient is derived. The results show that the self-synchronization occurs with the coupling coefficient larger than a critical value satisfying (13). Also, in case where the condition for self-synchronization not satisfied, a Lyapunov-based nonlinear control law that guarantees the synchronization of two coupled chaotic neurons is proposed. Finally, numerical simulations are performed, confirming the effectiveness of the proposed method.

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