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FINITE-DIMENSIONAL CHU SPACE, FUZZY SPACE AND THE GAME INVARIANCE THEOREM

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Abstract. By constructing the notion "$(n+1)$-fuzzy functor", it is shown that the $(n+1)$-fuzzy category introduced in [3] is an equivalent system. Moreover, the game invariance theorem is proved in this note.

Tóm tắt. Chúng tôi đưa ra một lớp các hàm tử hiệp biến, được gọi là "$(n+1)$-*hàm tử fuzzy*", từ phạm trù các n-tập hợp vào phạm trù các $(n+1)$-không gian fuzzy; chỉ ra rằng $(n+1)$-phạm trù fuzzy là một hệ thống tương đương và chứng minh rằng phạm trù các $(n+1)$-không gian fuzzy và phạm trù các $(n+1)$-không gian Chu hoàn toàn đầy đủ là đẳng cấu với nhau. Cuối cùng, khi đưa ra các khái niệm về chuẩn, trung bình và độ lệch tiêu chuẩn, chúng tôi chỉ ra rằng các đại lượng này là bất biến trò chơi.

1. INTRODUCTION

This work is motivated by recent attempt to model information flow in distributed system of Bariwise and Seligman in 1977 as well as the work of V. R. Pratt in computer science in which a general algebraic scheme, known as Chu space, is systematically used. In this paper we continue to study the finite-dimensional Chu space introduced in [3]. This paper is organized as follows. In section we recall the notion of finite-dimensional Chu space in general settings, and define some numerical data which used in section 4. In section 3 we introduce a new class of covariant functors, called the "$(n+1)$-fuzzy functors", from the n-set category into the category of $(n+1)$-fuzzy spaces. We show that the $(n+1)$-fuzzy category is an equivalent system and prove that the two categories of $(n+1)$-fuzzy spaces and of fully complete $(n+1)$-Chu spaces are isomorphic. In section 4 we define some statistical data as norm, mean, standard deviation of a game space. These data are proved to be game invariance.

2. FINITE-DIMENSIONAL CHU SPACES

By a $(n+m)$-*Chu space* we mean the set $\widetilde{C} = (X_1 \times X_2 \times \ldots \times X_n; f; A_1 \times A_2 \times \ldots \times A_m)$, where X_i, A_j $(i = 1, \ldots, n; j = 1, \ldots, m)$ are arbitrary sets and $f : X_1 \times \ldots \times X_n \times A_1 \times \ldots \times A_m \to [0, 1]$ is a map, called the *probability function* of $\widetilde{C}$.

If $\widetilde{C} = (X_1 \times X_2 \times \ldots \times X_n; f; A_1 \times A_2 \times \ldots \times A_m)$ and $\widetilde{D} = (Y_1 \times Y_2 \times \ldots \times Y_n; g; B_1 \times B_2 \times \ldots \times B_m)$ are $(n+m)$-Chu spaces, then a $(n+m)$-*Chu morphism* $\Phi : \widetilde{C} \to \widetilde{D}$ is a $(n+m)$-tuple of maps $\Phi = (\varphi_1, \varphi_2, \ldots, \varphi_n; \psi_1, \psi_2, \ldots, \psi_m)$, with $\varphi_i : X_i \to Y_i$ for $i = 1, \ldots, n$ and $\psi_j : B_j \to A_j$ for $j = 1, \ldots, m$ such that the diagram below commutes:

$$\begin{array}{ccc} \prod_{i=1}^n X_i \times \prod_{j=1}^m B_j & \xrightarrow{(\prod_{i=1}^n \varphi_i, 1_{\prod_{j=1}^m B_j})} & \prod_{i=1}^n Y_i \times \prod_{j=1}^m B_j \\ {\scriptstyle (1_{\prod_{i=1}^n X_i}, \prod_{j=1}^m \psi_j)} \downarrow & & \downarrow {\scriptstyle g} \\ \prod_{i=1}^n X_i \times \prod_{j=1}^m A_j & \xrightarrow[f]{} & [0,1] \end{array} \tag{1}$$

where $1_{\prod_{i=1}^n X_i}$, $1_{\prod_{j=1}^m B_j}$ denote identity maps. That is

$$f \circ (1_{\prod_{i=1}^n X_i}, \prod_{j=1}^m \psi_j) = g \circ (\prod_{i=1}^n \varphi_i, 1_{\prod_{j=1}^m B_j}),$$

or equivalently,

$$f(\prod_{i=1}^n x_i \times \prod_{j=1}^m \psi_j(b_j)) = g(\prod_{i=1}^n \varphi_i(x_i) \times \prod_{j=1}^m b_j) \text{ for } \prod_{i=1}^n x_i \in \prod_{i=1}^n X_i \text{ and } \prod_{j=1}^m b_j \in \prod_{j=1}^m B_j. \quad (2)$$

If $\Phi = (\varphi_1, \dots, \varphi_n; \psi_1, \dots, \psi_m) : \widetilde{C} = (X_1 \times \dots \times X_n; f; A_1 \times \dots \times A_m) \to \widetilde{D} = (Y_1 \times \dots \times Y_n; g; B_1 \times \dots \times B_m)$ is a $(n+m)$ - Chu morphism, then the $(n+m)$ - Chu space $(\prod_{i=1}^n X_i; f \times_\Phi g; \prod_{j=1}^m B_j)$, where

$$(f \times_\Phi g) = f \circ (1_{\prod_{i=1}^n X_i}, \prod_{j=1}^m \psi_j) = g \circ (\prod_{i=1}^n \varphi_i, 1_{\prod_{j=1}^m B_j})$$

is called the *cross product of* $\widetilde{C}$ *and* $\widetilde{D}$ *over* Φ, denoted by $\widetilde{C} \times_\Phi \widetilde{D}$.

For $\prod_{i=1}^n x_i \in \prod_{i=1}^n X_i$ we define the following notation:

1. The number $\| \prod_{i=1}^n x_i \|^* = \sup \{ f(\prod_{i=1}^n x_i \times \prod_{j=1}^m a_j) : \prod_{j=1}^m a_j \in \prod_{j=1}^m A_j \}$ is called the *upper value* of $\prod_{i=1}^n x_i$.
2. The number $\| \prod_{i=1}^n x_i \|_* = \inf \{ f(\prod_{i=1}^n x_i \times \prod_{j=1}^m a_j) : \prod_{j=1}^m a_j \in \prod_{j=1}^m A_j \}$ is called the *lower value* of $\prod_{i=1}^n x_i$.
3. The number $\| \prod_{i=1}^n x_i \| = \frac{1}{2}(\| \prod_{i=1}^n x_i \|^* + \| \prod_{i=1}^n x_i \|_*)$ is called the *value* of $\prod_{i=1}^n x_i$.
4. The number $d(\prod_{i=1}^n x_i) = \| \prod_{i=1}^n x_i \|^* - \| \prod_{i=1}^n x_i \|_*$ is called the *deviation* of $\prod_{i=1}^n x_i$.

For $(n+m)$ - Chu spaces $\widetilde{C} = (X_1 \times \dots \times X_n; f; A_1 \times \dots \times A_m)$ and $\widetilde{D} = (Y_1 \times \dots \times Y_n; g; B_1 \times \dots \times B_m)$ let $\mathcal{M}(\widetilde{C}, \widetilde{D})$ denote the set of all $(n+m)$ - Chu morphisms from $\widetilde{C}$ into $\widetilde{D}$. If $\mathcal{M}(\widetilde{C}, \widetilde{D}) \neq \emptyset$, then we say that $\widetilde{C}$ is *dominated* by $\widetilde{D}$ and denote $\widetilde{C} \preceq \widetilde{D}$. We say that $\widetilde{C}$ and $\widetilde{D}$ are *equivalent*, denoted by $\widetilde{C} \approx \widetilde{D}$, if $\widetilde{C} \preceq \widetilde{D}$ and $\widetilde{D} \preceq \widetilde{C}$; $\widetilde{C}$ and $\widetilde{D}$ are *connected* if either $\widetilde{C} \preceq \widetilde{D}$ or $\widetilde{D} \preceq \widetilde{C}$. A class of (n+m)-Chu spaces $\mathcal{G}$ is called a *connected system* if any two members of $\mathcal{G}$ are connected. If $\widetilde{C} \approx \widetilde{D}$ for every $\widetilde{C}, \widetilde{D} \in \mathcal{G}$, then we say that $\mathcal{G}$ is an *equivalent system*. A connected system is called a *closed system* if $\mathcal{G}$ is closed under cross products. That is, $\widetilde{C} \times_\Phi \widetilde{D} \in \mathcal{G}$ for any $\widetilde{C}, \widetilde{D} \in \mathcal{G}$ and $\Phi \in \mathcal{M}(\widetilde{C}, \widetilde{D})$. A *complete system* is a closed equivalent system.

Let $\widetilde{C} = (X_1 \times \dots \times X_n; f; A_1 \times \dots \times A_m)$ and $\widetilde{D} = (Y_1 \times \dots \times Y_n; g; B_1 \times \dots \times B_m)$ be $(n+m)$ - Chu spaces, we say that $\widetilde{C}$ and $\widetilde{D}$ are *isomorphic*, denoted by $\widetilde{C} \cong \widetilde{D}$, if $\widetilde{C}$ and $\widetilde{D}$ are isomorphic objects in the category $\mathcal{C}$ of $(n+m)$ - Chu spaces. It is easy to see that a $(n+m)$ - Chu morphism $\Phi = (\varphi_1, \dots, \varphi_n; \psi_1, \dots, \psi_m) : (X_1 \times \dots \times X_n; f; A_1 \times \dots \times A_m) \to (Y_1 \times \dots \times Y_n; g; B_1 \times \dots \times B_m)$ is an isomorphism if and only if $\varphi_i : X_i \to Y_i$ for $i = 1, \dots, n$ and $\psi_j : B_j \to A_j$ for $j = 1, \dots, m$ are one-to-one and onto.

If $\Phi = (\varphi_1, \dots, \varphi_n; \psi_1, \dots, \psi_m)$ is a $(n+m)$ - monomorphism, then we say that $\widetilde{C} = (X_1 \times \dots \times X_n; f; A_1 \times \dots \times A_m)$ is a *subspace* of $\widetilde{D} = (Y_1 \times \dots \times Y_n; g; B_1 \times \dots \times B_m)$, denoted by $\widetilde{C} \subseteq \widetilde{D}$. It is easy to see that a $(n+m)$ - Chu morphism $\Phi = (\varphi_1, \dots, \varphi_n; \psi_1, \dots, \psi_m) : (X_1 \times \dots \times X_n; f; A_1 \times \dots \times A_m) \to (Y_1 \times \dots \times Y_n; g; B_1 \times \dots \times B_m)$ is a mornomorphism iff $\varphi_i : X_i \to Y_i$ for $i = 1, \dots, n$ are one-to-one and $\psi_j : B_j \to A_j$ for $j = 1, \dots, m$ are onto.

3. FUZZY SPACE AND FUZZY FUNCTOR

Recall that by a *fuzzy subset* of a set $X = \prod_{i=1}^n X_i$, we mean a fuction $f : X \to [0, 1]$, see [3]. Observe that if A is a subset of X, then the characteristic function χ_A of A is a fuzzy subset of X. So by identifying A with χ_A we can say that any subset of X is a fuzzy subset of X. A fuzzy subset of X is also simply called a *fuzzy set*.

Let $\mathcal{S}$ denote the category of sets. For a given set $X = \prod_{i=1}^n X_i$, let $X^* = [0, 1]^X$ denote collection of all fuzzy sets of X.

For any map $\alpha : X = X_1 \times X_2 \times ... \times X_n \to Y = Y_1 \times Y_2 \times ... \times Y_n$ we define the conjugate $\alpha^* : Y^* \to X^*$ of α by the formula

$$\alpha^*(a)(x) = a(\alpha(x))$$

for every $x \in X$ and $a \in Y^*$.

It is easy to see that

$$(\beta\alpha)^* = \alpha^*\beta^* \text{ for every } \alpha : X \to Y \text{ and } \beta : Y \to Z.$$

For any set $A \subset X^*$ we define $f_A : X_1 \times X_2 \times ... \times X_n \times A \to [0,1]$ by

$$f_A(x_1, \dots, x_n, a) = a(x_1, \dots, x_n) \text{ for } (x_1, \dots, x_n, a) \in X_1 \times X_2 \times ... \times X_n \times A.$$

Clearly that $\widetilde{C} = (X_1 \times X_2 \times ... \times X_n; f_A; A)$ is a $(n+1)$ - Chu space. This space is called a $(n+1)$ - *pre-fuzzy space* on $X = X_1 \times X_2 \times ... \times X_n$. In the case $A = (X_1 \times X_2 \times ... \times X_n)^*$, the $(n+1)$ - Chu space $F(X) = (X_1 \times X_2 \times ... \times X_n; f_{X^*}; X^*)$ is uniquely determined by $X = X_1 \times X_2 \times ... \times X_n$, and is called $(n+1)$ - *fuzzy space associated with* X, or shortly a $(n+1)$ - *fuzzy space.*

The category of $(n+1)$ - -pre-fuzzy spaces with $(n+1)$ - Chu morphisms is called the *(n+1) - pre-fuzzy category*, denoted by $\mathcal{F}_P$. The $(n+1)$ - *fuzzy category*, denoted by $\mathcal{F}$, is the subcategory of $\mathcal{F}_P$ consisting of fuzzy spaces.

Observe that a $(n+1)$ - Chu morphism $\Phi : \widetilde{C} = (X_1 \times X_2 \times ... \times X_n; f_A; A) \to \widetilde{D} = (Y_1 \times Y_2 \times ... \times Y_n; f_B; B)$ in the $(n+1)$ - pre-fuzzy category is a collection of maps $\Phi = (\varphi_1, \varphi_2, \dots, \varphi_n; \psi)$, where

$$\prod_{i=1}^n \varphi_i : \prod_{i=1}^n X_i \to \prod_{i=1}^n Y_i \quad \text{with } (\prod_{i=1}^n \varphi_i)(\prod_{i=1}^n x_i) = \prod_{i=1}^n \varphi_i(x_i) \in \prod_{i=1}^n Y_i,$$

and $\psi : B \to A$ satisfy the condition

$$\psi(b)(\prod_{i=1}^n x_i) = b(\prod_{i=1}^n \varphi_i(x_i)) \text{ for } (x_1, \dots, x_n, b) \in X \times B.$$

It is easy to see that, in general $(n+1)$ - Chu spaces are not connected. Forturnately it is not the case in the $(n+1)$ - fuzzy category. In fact, we have the following theorem.

Theorem 1. *The $(n+1)$ - fuzzy category $\mathcal{F}$ is an equivalent system.*

Proof. Let $X = X_1 \times X_2 \times ... \times X_n$, $Y = Y_1 \times Y_2 \times ... \times Y_n$, we need to show that $\mathcal{M}(F(X), F(Y)) \neq \emptyset$ for any $(n+1)$ - fuzzy spaces $F(X) = (X_1 \times X_2 \times ... \times X_n; f_{X^*}; X^*)$ and $F(Y) = (Y_1 \times Y_2 \times ... \times Y_n; f_{Y^*}; Y^*)$.

Let $\alpha : X \to Y$ be any map (in the set category). Define $\alpha^* : Y^* \to X^*$ by $\alpha^*(y^*)(x_1, \dots, x_n) = y^*(\alpha(x_1, \dots, x_n))$ for $(x_1, \dots, x_n) \in X_1 \times X_2 \times ... \times X_n$ and $y^* \in Y^*$.

We have

$$\begin{aligned} \alpha^*(y^*)(x_1, \dots, x_n) &= f_{X^*}(x_1, \dots, x_n, \alpha^*(y^*)) \\ &= y^*(\alpha(x_1, \dots, x_n)) \\ &= f_{Y^*}(\alpha(x_1, \dots, x_n), y^*). \end{aligned}$$

Therefore the diagram bellow commutes

$$\begin{array}{ccc} \prod_{i=1}^n X_i \times Y^* & \xrightarrow{(\alpha, 1_{Y^*})} & \prod_{i=1}^n Y_i \times Y^* \\ {\scriptstyle (1_{\prod_{i=1}^n X_i}, \alpha^*)} \downarrow & & \downarrow {\scriptstyle f_{Y^*}} \\ \prod_{i=1}^n X_i \times X^* & \xrightarrow[f_{X^*}]{} & [0,1]. \end{array}$$

Thus, $\Phi = (\alpha, \alpha^*) \in \mathcal{M}(F(X), F(Y))$ and the theorem is proved.

By *n - set* we mean the cartesian product $X = X_1 \times ... \times X_n$. We will show that $F(X) = (X_1 \times ... \times X_n; f_{X^*}; X^*)$ is a covariant functor from the n-set category $\mathcal{S}$ into the $(n+1)$ - fuzzy category $\mathcal{F}$ and then F will be called a $(n+1)$ - *fuzzy functor.*

In fact, let $\alpha : \prod_{i=1}^n X_i \to \prod_{i=1}^n Y_i$ be a map. Define $F(\alpha) : F(X) \to F(Y)$ by $F(\alpha) = (\alpha, \alpha^*)$, where $\alpha^* : Y^* \to X^*$ is the conjugate of α.

We observe that

$$F(\beta\alpha) = (\beta\alpha, (\beta\alpha)^*) = (\beta\alpha, \alpha^*\beta^*) = F(\beta)F(\alpha)$$

for any $\alpha : \prod_{i=1}^n X_i \to \prod_{i=1}^n Y_i$ and $\beta : \prod_{i=1}^n Y_i \to \prod_{i=1}^n Z_i$. Therefore F preserves the composition.

Theorem 2. *The two categories $\mathcal{F}$ and C_F are isomophic.*

Proof. The functor F defined in the proof of Theorem 2 in [3] is an isomorphism between the $(n+1)$-fuzzy category $\mathcal{F}$ and the category C_F of fully complete $(n+1)$-Chu spaces.

From Theorem 1 and Theorem 2 we get:

Corollary 1. *The category C_F of all fully complete $(n+1)$-Chu spaces is an equivalent system.*

Remark 1. Since any subset of a set X is a fuzzy set, we can consider the family $A = 2^X \subset X^*$ consisting of all subsets of $X = X_1 \times ... \times X_n$. The resulting $(n+1)$-pre-fuzzy space $D(X) = (\prod_{i=1}^n X_i; f_{2^X}; 2^X)$ will be called the $(n+1)$-*Crisp space associated with* X, and the category $\mathcal{D}$ of all crisp spaces is called the *crisp category*.

We will show that

Proposition 1. *Every $(n+1)$-Crisp space is biextensional.*

Proof. By Proposition 7 in [3], every $(n+1)$-pre-fuzzy space is separated, therefore we need to claim that it is extensional.

Assume

$$0 = \|\prod_{i=1}^n x_i - \prod_{i=1}^n y_i\| = \sup\{|f(\prod_{i=1}^n x_i, a) - f(\prod_{i=1}^n y_i, a)| : a \in 2^X\},$$

then $a(\prod_{i=1}^n x_i) = a(\prod_{i=1}^n y_i)$ for every $a \in 2^X$. From that it follows $\prod_{i=1}^n x_i = \prod_{i=1}^n y_i$, since if it is not the case, setting $a = \chi_{\{\prod_{i=1}^n x_i\}} \in 2^X$, we get $a(\prod_{i=1}^n x_i) = 1$, but $a(\prod_{i=1}^n y_i) = 0$.

The crisp category $\mathcal{D}$ is a subcategory of $\mathcal{F}$. We observe that

Proposition 2. *The map D defined in Remark 1 is a covariant functor from the n-set category S into the $(n+1)$-crisp category $\mathcal{D}$.*

Proof. Let $\alpha : \prod_{i=1}^n X_i \to \prod_{i=1}^n Y_i$ be a map. Then the morphism

$$D(\alpha) : D(X) = (\prod_{i=1}^n X_i; f_{2^X}; 2^X) \to D(Y) = (\prod_{i=1}^n Y_i; f_{2^Y}; 2^Y)$$

is defined by $D(\alpha) = (\alpha, \alpha^{-1})$, where $\alpha^{-1}(D) \in 2^X$ for every $D \in 2^Y$.

We will show that the following diagram commutes

$$\begin{array}{ccc} \prod_{i=1}^n X_i \times 2^Y & \xrightarrow{(\alpha, 1_{2^Y})} & \prod_{i=1}^n Y_i \times 2^Y \\ {\scriptstyle (1_{\prod_{i=1}^n X_i}, \alpha^{-1})}\downarrow & & \downarrow{\scriptstyle f_{2^Y}} \\ \prod_{i=1}^n X_i \times 2^X & \xrightarrow[f_{2^X}]{} & [0,1]. \end{array}$$

In fact, by definition of f_{2^X} and f_{2^Y}, we need to claim that

$$\alpha^{-1}(b)(\prod_{i=1}^n x_i) = b(\alpha(\prod_{i=1}^n x_i)) \text{ for every } b \in 2^Y.$$

Since $\alpha^{-1}(b)$ and b are two characteristic functions of the set $\alpha^{-1}(b)$ in the space 2^X and 2^Y, respectively, they admit only two values 0 or 1. If $\alpha^{-1}(b)(\prod_{i=1}^n x_i) = 1$, then $\prod_{i=1}^n x_i \in \alpha^{-1}(b)$ which implies $\alpha(\prod_{i=1}^n x_i) \in b$, hence $b(\alpha x) = 1$. If $\alpha^{-1}(b)(\prod_{i=1}^n x_i) = 0$, then $\prod_{i=1}^n x_i \notin \alpha^{-1}(b)$ which implies $\alpha(\prod_{i=1}^n x_i) \notin b$, hence $b(\alpha(\prod_{i=1}^n x_i)) = 0$.

Thus, in both cases we have

$$\alpha^{-1}(b)(\prod_{i=1}^n x_i) = b(\alpha(\prod_{i=1}^n x_i)) \text{ for } \prod_{i=1}^n x_i \in \prod_{i=1}^n X_i.$$

Therefore the proposition is proved.

4. GAME SPACE AND THE GAME INVARIANCE THEOREM

Given a set $A = \prod_{j=1}^m A_j$, by a *game space over* A $= \prod_{j=1}^m A_j$, we mean a $(n+m)$ - Chu space $\widetilde{G} = (\prod_{i=1}^n X_i; f; \prod_{j=1}^m A_j)$, where:

1. $\prod_{i=1}^n X_i$ is a cartesian product of finite sets, called the *team game*. If $\prod_{i=1}^n x_i \in \prod_{i=1}^n X_i$, then $\prod_{i=1}^n x_i$ is called the *players* of the game space $\widetilde{G}$.

2. $\prod_{j=1}^m A_j$ is a cartesian product of any sets, called the *field game*. If $\prod_{j=1}^m a_j \in \prod_{j=1}^m A_j$, then $\prod_{j=1}^m a_j$ is called a *position* in the field game $\prod_{j=1}^m A_j$.

3. $f(\prod_{i=1}^n x_i, \prod_{j=1}^m a_j)$ is called the *winning probability* of the players $\prod_{i=1}^n x_i$ while they are in the position $\prod_{j=1}^m a_j$ in the field game.

Observe that if $\widetilde{G} = (\prod_{i=1}^n X_i; f; \prod_{j=1}^m A_j)$ is a game space, then the upper value $\| \prod_{i=1}^n x_i\|^*$ measures the 11skill" of $\prod_{i=1}^n x_i$ in the best situation and the lower value $\| \prod_{i=1}^n x_i\|_*$ measures the "skill" of the set $\prod_{i=1}^n x_i$ in the worst situation.

Dually, for a state $\prod_{j=1}^m a_j \in \prod_{j=1}^m A_j$ the upper value $\| \prod_{j=1}^m a_j\|^*$ describes the *quality* of the position $\prod_{j=1}^m a_j$ in hands of the best players and the lower value $\| \prod_{j=1}^m a_j\|_*$ describes the *quality* of the position $\prod_{j=1}^m a_j$ if the worst players are staying there.

Since the set $\prod_{i=1}^n X_i$ of a game space $\widetilde{G} = (\prod_{i=1}^n X_i; f; \prod_{j=1}^m A_j)$ is finite, we can define the following statistical data for a game space:

1. The number $\|\widetilde{G}\| = \sqrt{\sum_{\prod_{i=1}^n x_i \in \prod_{i=1}^n X_i} \| \prod_{i=1}^n x_i\|^2}$ is called the *norm* of $\widetilde{G}$.

2. The number $D(\widetilde{G}) = \sqrt{\sum_{\prod_{i=1}^n x_i \in \prod_{i=1}^n X_i} [d(\prod_{i=1}^n x_i)]^2}$ is called the *standard deviation* of $\widetilde{G}$.

3. The number $M(\widetilde{G}) = \frac{1}{|\prod_{i=1}^n X_i|}\sum_{\prod_{i=1}^n x_i \in \prod_{i=1}^n X_i} \| \prod_{i=1}^n x_i\|$, where $|\prod_{i=1}^n X_i|$ denotes the cardinality of $\prod_{i=1}^n X_i$, is called the *mean* of $\widetilde{G}$.

Now given a set $\prod_{j=1}^m A_j$, we define the game category over the field $\prod_{j=1}^m A_j$, denoted $\mathcal{G}_A$ as follows:

1. The objects of $\mathcal{G}_A$ are game spaces over $\prod_{j=1}^m A_j$.

2. If $\widetilde{S} = (\prod_{i=1}^n X_i; f; \prod_{j=1}^m A_j)$ and $\widetilde{T} = (\prod_{i=1}^n Y_i; g; \prod_{j=1}^m A_j)$ are two game spaces over $\prod_{j=1}^m A_j$, then a morphism $\Phi = (\varphi_1, \dots, \varphi_n; 1_{\prod_{j=1}^m A_j}) : \widetilde{S} \to \widetilde{T}$, where $\varphi_i : X_i \to Y_i$, for i=1,...,n are maps satisfying the condition:

$$f(\prod_{i=1}^n x_i \times \prod_{j=1}^m a_j) \le g(\prod_{i=1}^n \varphi_i(x_i) \times \prod_{j=1}^m a_j)$$

for $\prod_{i=1}^n x_i \in \prod_{i=1}^n X_i$ and $\prod_{j=1}^m a_j \in \prod_{j=1}^m A_j$.

Consequently morphisms in the game category $\mathcal{G}_A$ are $(n+m)$ - Chu upper-morphisms.

The existence of a $(n+m)$ - morphism $\Phi : \widetilde{S} \to \widetilde{T}$ in the game category over the field $\prod_{j=1}^m A_j$ implies that for any set of players $\prod_{i=1}^n x_i$ of the team $\prod_{i=1}^n X_i$, there exists a set of players $\prod_{i=1}^n \varphi_i(x_i)$ of the team $\prod_{i=1}^n Y_i$ such that at any situation $\prod_{j=1}^m a_j$ in the game field $\prod_{j=1}^m A_j$, the set of players $\prod_{i=1}^n \varphi_i(x_i)$ have better chance to win than the set of players $\prod_{i=1}^n x_i$ at the same situations $\prod_{j=1}^m a_j$. It follows that the team $\prod_{i=1}^n Y_i$ have some advantages over the team $\prod_{i=1}^n X_i$ in the field $\prod_{j=1}^m A_j$.

We have

Lemma 1. *If* $\widetilde{S} = (\prod_{i=1}^n X_i); f; \prod_{j=1}^m A_j)$ *is a subset of* $\widetilde{G} = (\prod_{i=1}^n Y_i); g; \prod_{j=1}^m A_j)$, *then* $\|\widetilde{S}\| \le \|\widetilde{G}\|$.

Proof. Since the game space $\widetilde{S} = (\prod_{i=1}^n X_i); f; \prod_{j=1}^m A_j)$ is a subset of the the game space $\widetilde{G} = (\prod_{i=1}^n Y_i); g; \prod_{j=1}^m A_j)$, there is a monorphism $\Phi = (\varphi_1, \dots, \varphi_n; \psi_1, \dots, \psi_m) : \widetilde{S} \to \widetilde{G}$ with $\varphi_i : X_i \to Y_i$ for $i = 1, \dots, n$ are one-to-one and $\psi_j : A_j \to A_j$ are identical maps for $j = 1, \dots, m$, so that

$$f(\prod_{i=1}^n x_i \times \prod_{j=1}^m a_j) \le g(\prod_{i=1}^n \varphi_i(x_i) \times \prod_{j=1}^m a_j)$$

for $\prod_{i=1}^n x_i \in \prod_{i=1}^n X_i$ and $\prod_{j=1}^m a_j \in \prod_{j=1}^m A_j$. We have

$$\begin{aligned} \|\prod_{i=1}^n x_i\|^* &= \sup \{f(\prod_{i=1}^n x_i \times \prod_{j=1}^m a_j) : \prod_{j=1}^m a_j \in \prod_{j=1}^m A_j\} \\ &\le \sup \{g(\prod_{i=1}^n \varphi_i(x_i) \times \prod_{j=1}^m a_j) : \prod_{j=1}^m a_j \in \prod_{j=1}^m A_j\} \\ &= \sup \{g(\prod_{i=1}^n y_i \times \prod_{j=1}^m a_j) : \prod_{j=1}^m a_j \in \prod_{j=1}^m A_j\} \\ &= \|\prod_{i=1}^n y_i\|^*, \end{aligned}$$

and

$$\begin{aligned} \|\prod_{i=1}^n x_i\|_* &= \inf \{f(\prod_{i=1}^n x_i \times \prod_{j=1}^m a_j) : \prod_{j=1}^m a_j \in \prod_{j=1}^m A_j\} \\ &\le \inf\{g(\prod_{i=1}^n \varphi_i(x_i) \times \prod_{j=1}^m a_i) : \prod_{j=1}^m a_j \in \prod_{j=1}^m A_j\} \\ &= \inf\{g(\prod_{i=1}^n y_i \times \prod_{j=1}^m a_i) : \prod_{j=1}^m a_j \in \prod_{j=1}^m A_j\} \\ &= \|\prod_{i=1}^n y_i\|_*. \end{aligned}$$

So

$$\|\prod_{i=1}^n x_i\| \le \|\prod_{i=1}^n y_i\|.$$

On the other hand, since φ_i are one-to-one for $i = 1, \dots, n$, $|\prod_{i=1}^n X_i| \le |\prod_{i=1}^n Y_i|$.

Therefore

$$\sqrt{\sum_{\prod_{i=1}^n x_i \in \prod_{i=1}^n X_i} \|\prod_{i=1}^n x_i\|^2} \le \sqrt{\sum_{\prod_{i=1}^n y_i \in \prod_{i=1}^n Y_i} \|\prod_{i=1}^n y_i\|^2}.$$

Consequently

$$\|\widetilde{S}\| \le \|\widetilde{G}\|.$$

Remark 2. With the same assumption in the Lemma 1, we will show that $M(\widetilde{S}) \le M(\widetilde{G})$ is in general not true.

In fact, suppose that for a given set $\prod_{i=1}^n X_i$, let $\prod_{i=1}^n Y_i = \prod_{i=1}^n X_i \cup \{\prod_{i=1}^n x_i^0\}$, where $\prod_{i=1}^n x_i^0 \notin \prod_{i=1}^n X_i$. We put

$$g(\prod_{i=1}^n y_i \times \prod_{j=1}^m a_j) = f(\prod_{i=1}^n x_i \times \prod_{j=1}^m a_j) \text{ if } \prod_{i=1}^n y_i = \prod_{i=1}^n x_i,$$

and

$$g(\prod_{i=1}^n x_i^0 \times \prod_{j=1}^m a_j) = 0 \text{ for every } \prod_{j=1}^m a_j \in \prod_{j=1}^m A_j.$$

Then $\|\prod_{i=1}^n x_i^0\| = 0$ and $\widetilde{S} = (\prod_{i=1}^n X_i; f; \prod_{j=1}^m A_j)$ is a subset of the $\widetilde{G} = (\prod_{i=1}^n Y_i; g; \prod_{j=1}^m A_j)$.

Let $\Phi = (\varphi_1, \ldots, \varphi_n, 1_{\prod_{j=1}^m A_j}) : \widetilde{S} \to \widetilde{G}$, be a morphism from $\widetilde{S}$ into $\widetilde{G}$. Then

$$f(\prod_{i=1}^n x_i \times \prod_{j=1}^m a_j) = g(\prod_{i=1}^n \varphi_i(x_i) \times \prod_{j=1}^m a_j) \text{ for every } \prod_{i=1}^n x_i \in \prod_{i=1}^n X_i.$$

We have

$$\|\prod_{i=1}^n x_i\| = \|\prod_{i=1}^n \varphi_i(x_i)\| = \|\prod_{i=1}^n y_i\| \text{ and } |\prod_{i=1}^n X_i| < |\prod_{i=1}^n Y_i|.$$

Hence

$$\begin{aligned} M(\widetilde{S}) &= \frac{1}{|\prod_{i=1}^n X_i|} \sum_{\prod_{i=1}^n x_i \in \prod_{i=1}^n X_i} \|\prod_{i=1}^n x_i\| \\ &= \frac{1}{|\prod_{i=1}^n X_i|} \Big(\sum_{\prod_{i=1}^n x_i \in \prod_{i=1}^n X_i} \|\prod_{i=1}^n x_i\| + \|\prod_{i=1}^n x_i^0\| \Big) \\ &= \frac{1}{|\prod_{i=1}^n X_i|} \sum_{\prod_{i=1}^n y_i \in \prod_{i=1}^n Y_i} \|\prod_{i=1}^n y_i\| \\ &> \frac{1}{|\prod_{i=1}^n Y_i|} \sum_{\prod_{i=1}^n y_i \in \prod_{i=1}^n Y_i} \|\prod_{i=1}^n y_i\| \\ &= M(\widetilde{G}). \end{aligned}$$

It shows that, in this case, $\widetilde{S}$ is a subset of $\widetilde{G}$ but $M(\widetilde{S}) > M(\widetilde{G})$.

Theorem 3 (The game invariance theorem). *The numbers $\|\widetilde{G}\|, M(\widetilde{G})$ and $D(\widetilde{G})$ are invariance in the game category over the field A. That is, if $\widetilde{S}$ and $\widetilde{G}$ are isomorphic, then $\|\widetilde{S}\| = \|\widetilde{G}\|, M(\widetilde{S}) = M(\widetilde{G})$ and $D(\widetilde{S}) = D(\widetilde{G})$.*

Proof. From Lemma 1 it follows $\|\widetilde{S}\| = \|\widetilde{G}\|$. For every $\prod_{i=1}^n x_i \in \prod_{i=1}^n X_i$, since $\widetilde{S}$ and $\widetilde{G}$ are isomorphic, there exists unique $\prod_{i=1}^n y_i = \prod_{i=1}^n \varphi_i(x_i) \in \prod_{i=1}^n Y_i$, such that $f(\prod_{i=1}^n x_i \times \prod_{j=1}^m a_j) = g(\prod_{i=1}^n \varphi_i(x_i) \times \prod_{j=1}^m a_j) = g(\prod_{i=1}^n y_i \times \prod_{j=1}^m a_j)$.

We have

$$\|\prod_{i=1}^{n} x_i\|^* = \|\prod_{i=1}^{n} \varphi_i(x_i)\|^* = \|\prod_{i=1}^{n} y_i\|^* \text{ and } \|\prod_{i=1}^{n} x_i\|_* = \|\prod_{i=1}^{n} \varphi_i(x_i)\|_* = \|\prod_{i=1}^{n} y_i\|_*.$$

It implies that $\| \prod_{i=1}^n x_i \| = \| \prod_{i=1}^n \varphi_i(x_i) \| = \| \prod_{i=1}^n y_i \|$.

Thus

$$\begin{aligned} M(\widetilde{S}) &= \frac{1}{|\prod_{i=1}^n X_i|} \sum_{\prod_{i=1}^n x_i \in \prod_{i=1}^n X_i} \|\prod_{i=1}^{n} x_i\| \\ &= \frac{1}{|\prod_{i=1}^n X_i|} \Big(\sum_{\prod_{i=1}^n x_i \in \prod_{i=1}^n X_i} \|\prod_{i=1}^{n} \varphi_i(x_i)\| \Big) \\ &= \frac{1}{\prod_{i=1}^n Y_i|} \Big(\sum_{\prod_{i=1}^n y_i \in \prod_{i=1}^n Y_i} \|\prod_{i=1}^{n} y_i\| \Big) \\ &= M(\widetilde{G}). \end{aligned}$$

The similar argument proves the equality $D(\widetilde{S}) = D(\widetilde{G})$.
The theorem is proved.

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REFERENCES

[1] Barry Mitchell, *Theory of Categories*, New York and London, 1965.

[2] Nguyen Nhuy and Pham Quang Trinh, Chu spaces, Fuzzy sets and Game Invariances, accepted for publication in *Viet. J. Math.* (2000).

[3] Nguyen Nhuy, Pham Quang Trinh, and Vu Thi Hong Thanh, Finite dimensional Chu space, *Journal of Computer Science and Cybernetics* **15** (4) (1999) 7-18.

[4] H. T. Nguyen and E. Walker, A First Course in Fuzzy Logic, Boca Raton, FL: CRC, 1997 (2nd ed., 1999).

[5] V. R. Pratt, Type as processes, via Chu spaces, *Electronic Notes in Theoretical Computer Science* **7** (1997).

[6] V. R. Pratt, Chu spaces as a sematic bridge between linear logic and mathematics, *Electronic Notes in Theoretical Computer Science* **12** (1998).

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