

DIFFERENCE SCHEMES FOR GENERALIZED SOLUTIONS OF SOME ELLIPTIC DIFFERENTIAL EQUATIONS, II

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Abstract. The approximate methods for the problems of differential equations with non-regular data are studied by some authors. For example, in [1–3, 6, 7] are considered the cases of data belonging to the Sobolev spaces $W_p^l(G)$. In this paper, which is a continuation of [4], we consider the difference schemes for solutions of some elliptic problems in the case where the region of definition for variable has arbitrary form. In the last section the result is generalized to a class of problems with data defined by the continuous linear functionals in $W_2^{(-l)}(G)$.

1. DIFFERENCE SCHEME FOR THE DIRICHLET PROBLEM OF POISSON EQUATION

Consider the following Dirichlet problem:

$$\begin{aligned} \Delta u &= -f(x), \quad x \in G, \\ u(x) &= 0, \quad x \in \partial G. \end{aligned} \tag{1}$$

To simplify the exposition, assume that G is a convex region in R^2 with $\partial G \in C^2$. We shall keep some notations in [4], [7].

Let R_h be a rectangle grid covered the x -plane and defined by

$$R_h \equiv \{x = (x_1, x_2) : x_i = x^{(j_i)} = j_i h_i, j_i = 0, \pm 1, \pm 2, \dots, i = 1, 2\},$$

where the straight lines x_i are the parallels to the coordinate lines, h_i are positive mesh sizes in the x_i -directions, $i = 1, 2$, respectively. Denote by $\omega = R_h \cap G$ the set of all gridpoints in G , and by $\gamma = R_h \cap \partial G$ the set of boundary gridpoints, by γ_i^+ and γ_j^- the set of right and left boundary gridpoints in the x_i - directions respectively. Let ω_γ be the subset of interior netpoints that the lie in the neighbourhood of ∂G , $\omega_0 \equiv \omega \setminus \omega_\gamma$, $\bar{\omega} \equiv \omega \cup \gamma$.

Let us introduce a supplementary grid of the parallels x_β to the lines x_i :

$$x_\beta \equiv x_i^{(j_i+0.5)} = 0.5(x_i^{(j_i)} + x_i^{(j_i+1)}).$$

Let every gridpoint $x \in \omega$ be corresponding to the subregion $\tilde{e}(x) \in G$ bounded by the straight lines $x_\beta = x_i^{(j_i+0.5)}$, $i = 1, 2$. If $x \in \omega_\gamma$, $\tilde{e}(x)$ is limited by not only the x_β but also an arc of the curve ∂G . The boundary segments x_β of $\tilde{e}(x)$ perpendicular to the coordinate lines Ox_i are denoted by $l_i^{(\pm 0.5)}$, $i = 1, 2$, respectively.

Denote by $x^{(\pm 1_i)}$, $i = 1, 2$, the neighbourhood netpoints of the netpoint $x \in \bar{\omega}$ in the x_i -direction, $h_i^{(\pm 1_i)} \equiv |x_i^{(\pm 1_i)} - x_i|$, x_i and $x_i^{(\pm 1_i)}$ being the coordinates of the netpoints x and $x^{(\pm 1_i)} \in \bar{\omega}$ respectively. We see that there are the differences of steplengths $h^{(\pm 0.5)}$ and h_i only in the neighbourhoods of ∂G .

The points of intersection of the straight lines $x_i = x_i^{(j_i)}$ with $x_\beta = x_i^{(j_i \pm 0.5)}$ are denoted by $x^{(\pm 0.5_i)}$ that are called the stream gridpoints in the x_i -direction. Denote by ω'_i the set of these points, $\omega' \equiv \omega'_1 \cup \omega'_2$.

Let every gridpoint $x^{(\pm 0.5i)}$ correspond to a following area, $i = 1, 2$,

$$e_i(x^{(\pm 0.5i)}) = \{\zeta = (\zeta_1, \zeta_2) : x_i < \zeta_i < x_i + h_i, |x_\beta - \zeta_\beta| < 0.5h_\beta, \beta = 3 - i\}.$$

Let $x \in \omega_\gamma$ and in the area $\tilde{e}(x)$ the segment $\overline{\Delta}l$ correspond to the arc $\Delta\Gamma$ of ∂G ($\Gamma \equiv \partial G$). Denote by $e(x)$ the area bounded by the segments $l_i^{(\pm 0.5)}$ and $\overline{\Delta}l$. Note that, by assumptions for G , with $x \in \omega_\gamma$ the different value between the areas $\tilde{e}(x)$ and $e(x)$ is equal to $O(h)$, where $|h|^2 = h_1^2 + h_2^2$.

1.1. Construction of difference schemes

The generalized solution of the problem (1) is considered in the spaces $W_2^m(G)$, $m = 2, 3$. As in [4] the generalized solution (denoted by the *GS*) $u(x)$ satisfies the following equalities:

$$Pu \equiv \iint_G \Delta u(x)v(x)dx = - \iint_G f(x)v(x)dx, \quad \forall v(x) \in L_2(G); \quad u(x) = 0, \quad x \in \partial G. \quad (2)$$

Let $x \in \omega_\gamma$. For deriving finite-difference methods, we may take the solution of (2) in the neighbourhood area $e(x)$ of the gridpoint x by the form:

$$\begin{aligned} P^e u &= \frac{1}{h_1 h_2} \iint_{e(x)} \alpha(\zeta_1, \zeta_2) \Delta u(\zeta) d\zeta = \frac{1}{h_1 h_2} \iint_{e(x)} \left[\sum_{i=1}^2 \frac{\partial u}{\partial \zeta_i} \left(\alpha \frac{\partial u}{\partial \zeta_i} \right) - \sum_{i=1}^2 \frac{\partial \alpha}{\partial \zeta_i} \frac{\partial u}{\partial \zeta_i} \right] d\zeta \\ &= - \frac{1}{h_1 h_2} \iint_{e(x)} \alpha(\zeta) f(\zeta) d\zeta \equiv -Rf, \end{aligned} \quad (3)$$

where $v(x) = (h_1 h_2)^{-1} \alpha(x)$, $\alpha(x) = \begin{cases} \frac{1}{4\pi h_1^{m-1} h_2^{m-1}} \exp \left\{ -\frac{x_1^2 + x_2^2}{4h_1^m h_2^m} \right\}, & x \in e, \\ 0, & x \in \overline{G} \setminus e. \end{cases}$

From (3), applying the Green-Ostrogradski formula one has

$$\begin{aligned} P^e u &= \frac{1}{h_1 h_2} \left\{ \sum_{i=1}^2 l_i^{(0.5)} \overline{\omega}_i^{(+0.5i)} - \sum_{i=1}^2 l_i^{(0.5)} \overline{\omega}_i^{(-0.5i)} + \Delta l \beta(x) \overline{\omega}^{(0)} \right\} \\ &\quad - \frac{1}{h_1 h_2} \iint_{e(x)} \sum_{i=1}^2 \frac{\partial \alpha}{\partial \zeta_i} \frac{\partial u}{\partial \zeta_i} d\zeta = -Rf, \quad x \in \omega_\gamma; \quad u(x) = 0, \quad x \in \gamma, \end{aligned} \quad (4)$$

where

$$\overline{\omega}_i^{(\pm 0.5i)} = \frac{1}{l_i^{(\pm 0.5)}} \int_{l_i^{(\pm 0.5)}} \alpha \frac{\partial u}{\partial x_i} dl, \quad \overline{\omega}^{(0)} = \frac{1}{\Delta l} \int_{\overline{\Delta}l} \alpha \frac{\partial u}{\partial n} dl, \quad (5)$$

the net function $\beta(x)$ is equal to 1 as $x \in \omega_\gamma$ and is zero as $x \in \omega_0$, the lengths of segments $l_i^{(\pm 0.5)}$ are denoted also by $l_i^{(\pm 0.5)}$, $\overline{\omega}^{(0)}$ is calculated by a contour integral of the first kind. The notation “ \sum_t ” signifies that this sum has no the i -th summand corresponding to the $l_i^{(\pm 0.5)} = \emptyset$ respectively, \vec{n} being the outer normal to ∂e .

Note that if the netpoint $x \in \omega_0$, one has the form of $P^e u$ analogous to (4) in which the sum \sum has no the sign “ \cdot ” and $l_{3-i}^{(\pm 0.5)}$, $i = 1, 2$, are replaced by h_i respectively.

Now, to construct the difference schemes one may do in the same way as in Section 2.1 for the net problem (8), [4]. Thus, by (4) we obtain the following difference approximations analogous to (9) and (12) in [4] respectively:

$$\begin{aligned} K\overline{y} &\equiv - (a_1 \overline{y}_{x_1})_{\widehat{x}_1} - (a_2 \overline{y}_{x_2})_{\widehat{x}_2} + \frac{1}{h_1 h_2} \iint_{e(x)} \sum_{i=1}^2 \alpha_{\overline{x}_i}(x) \overline{y}_{\overline{x}_i}(x) \overline{y}_{\overline{x}_i}(x) d\zeta = \varphi(x) \equiv Rf(x), \\ x &\in \omega; \quad \overline{y}(x) = 0, \quad x \in \gamma, \end{aligned} \quad (6)$$

$$\text{and} \quad L\widehat{y} = -\widehat{y}_{x_1 \overline{x}_1} - \widehat{y}_{x_2 \overline{x}_2} = \varphi(x), \quad x \in \omega; \quad \widehat{y}(x) = 0, \quad x \in \gamma, \quad (7)$$

where

$$y_{x_i}^{(+0.5_i)} = \frac{y^{+1_i} - y}{h_i^{(+0.5)}}, \quad y_{x_i}^{-0.5_i} = \frac{y - y^{(-1_i)}}{h_i^{(-0.5)}}, \quad \widehat{y}_{x_i} = \frac{y^{(+0.5_i)} - y^{(-0.5_i)}}{h_i},$$

$$a_i^{(\pm 0.5_i)} = \frac{1}{l_i^{(\pm 0.5_i)}} \int_{l_i^{(\pm 0.5_i)}} \alpha(\zeta) dl \text{ for } l_i^{(\pm 0.5_i)} \neq 0, \quad a_i^{(\pm 0.5_i)} = \frac{1}{\Delta l} \int_{\Delta l} \alpha(\zeta) dl \text{ for } l_i^{(\pm 0.5_i)} = 0.$$

Note that the integrals should be taken along the segments $l_i^{(\pm 0.5)}$ and $\overline{\Delta l}$ lying inside the region \overline{G} . For $x \in \omega_0$ one has the formula similar to (6).

1.2. Estimation of the convergence rate

We shall estimate the method error and the approximate error of the scheme (7) and (6).

1.2.1. Consider first the difference scheme (7). The left-hand side of the difference equation (7) coincides with a standard fivepoints approximation for the one of the differential equation (1) in the case of the variable region G of any form. Consider now the convergence of the approximate solution \widehat{y} to the GS u of form (4). Denote the method error by $z \equiv \widehat{y} - u$. By (7) one has

$$Lz = \Psi(x), \quad x \in \omega; \quad z(x) = 0, \quad x \in \gamma, \quad (8)$$

where $\Psi(x)$ is the approximate error of the scheme (7): $\Psi(x) = \varphi(x) - LU$. Then, using the expression (4) of $\varphi = Rf$ we get

$$Lz = \Psi = \sum_{i=1}^2 \eta_{i \pm i} + \eta_0, \quad (9)$$

where

$$\eta_i^{(\pm 0.5_i)} = u_{x_i}^{(\pm 0.5_i)} - \overline{w}_i^{(\pm 0.5_i)} \quad \text{if } l_i^{(\pm 0.5_i)} = h_{3-1}, \quad (10)$$

$$\eta_i^{(\pm 0.5_i)} = u_{x_i}^{(\pm 0.5_i)} - \overline{w}_i^{(\pm 0.5_i)} + \left(1 - \frac{l_i^{(\pm 0.5_i)}}{h_{3-i}}\right) (\overline{w}_i^{(\pm 0.5_i)} - \tilde{w}_i) \quad \text{if } 0 < l_i^{(\pm 0.5_i)} < h_{3-i}, \quad (11)$$

$$\eta_i^{(\pm 0.5_i)} = u_{x_i}^{(\pm 0.5_i)} - \tilde{w}_i \quad \text{if } l_i^{(\pm 0.5_i)} = 0, \quad (12)$$

$$\tilde{w}_i = \frac{1}{\Delta l} \int_{\Delta l} \alpha \frac{\partial u}{\partial x_i} dl, \quad \eta_0 = \frac{1}{h_1 h_2} \iint_{e(x)} \sum_{i=1}^2 \frac{\partial \alpha}{\partial \zeta_i} \frac{\partial u}{\partial \zeta_i} d\zeta. \quad (13)$$

Now, to obtain a priori estimation, let us scalar multiply both sides of (9) by $z(x)$ and, then, arguing by the same way as in [4, Sec. 2.2], we get

$$\|z\|_{1,\omega} \leq M (\|\eta_1\|_{0,\omega'} + \|\eta_2\|_{0,\omega'} + \|\eta_0\|_{0,\omega'}), \quad (14)$$

where M is a constant independent of h and $z(x)$,

$$\|v\|_{1,\omega}^2 = \|v\|_{0,\omega}^2 + \|\nabla v\|_{0,\omega'}, \quad \|v\|_{0,\omega} \equiv \sqrt{(v, v)}$$

$$\|\nabla v\|_{0,\omega'}^2 \equiv (\nabla v, \nabla v)', \quad \nabla v(x^{(\pm 0.5_i)}) \equiv v_{x_i}^{(\pm 0.5_i)} \quad \text{for } x^{(\pm 0.5_i)} \in \omega'_i,$$

(u, v) is the scalar product on the set of net functions $\overset{\circ}{H}_h$: $(u, v) \equiv \sum_{x \in \omega} h_1 h_2 u(x) v(x)$, $(u, v)'$ is the scalar product on the set of functions defined on the net ω' of stream gridpoints H'_h :

$$(u, v)' \equiv \sum_{i=1}^2 \sum_{x^{(\pm 0.5_i)} \in \omega'_i} h_i^{(\pm 0.5_i)} h_{3-i} u(x^{(\pm 0.5_i)}) v(x^{(\pm 0.5_i)}).$$

The estimation of summands in the right-hand side of (14) is analogous to that of (18) in Section 2.2 [4], and one has

$$\|z\|_{1,\omega} \leq M|h|^{m-1}\|u\|_{m,\overset{\circ}{G}} + M|h|\|u\|_{2,\widehat{G}}, \quad (15)$$

where $m = 2, 3$; $\overset{\circ}{G} = \bigcup_{x' \in \omega'_0} e_i(x')$, $\widehat{G} = \bigcup_{x' \in \omega'_\gamma} \widehat{e}_i(x')$, $x' = x^{(\pm 0.5)}$, $i = 1, 2$, ω_0 being the subset of gridpoints x' that $e_i(x') \subset G$ (Fig. a), $\omega'_\gamma \equiv \omega' \setminus \omega'_0$; If $x' \in \omega'_\gamma$, then $e_i(x') \equiv \widehat{e}_i \cup \widetilde{e}_i$ with $\widehat{e}_i \subset G$ and $\widetilde{e}_i \not\subset G$ (Fig. b).

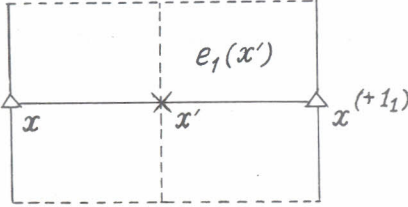


Fig. a

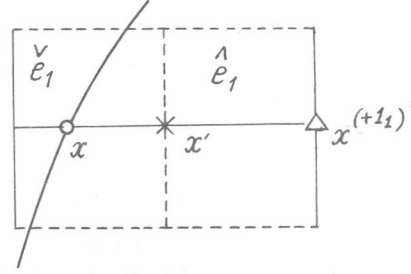


Fig. b

The set \widehat{G} can be bounded by a boundary strip G_ε with its width $\varepsilon = M|h|$. Then, if $u \in W_2^3(G)$ one has the following estimation [7]

$$\|u\|_{2,\widehat{G}} \leq M|h|^{1/2}\|u\|_{3,G}. \quad (16)$$

Finally, it follows from (15) and (16)

$$\|\widehat{y} - u\|_{1,\omega} \leq M|h|^{m/2}\|u\|_{m,G}, \quad (17)$$

where $m = 2, 3$; the constant M is independent of h and $u(x)$.

1.2.2. Consider now the difference scheme (6). By the same way as we did for the scheme (9) in the Section 2.2 [4], and for the scheme (7) above, with employing (17) one obtains the following result

Theorem 1. Let $\alpha(x)f(x) \in L_2(G)$. Then the solution \widetilde{y} of the scheme (6) converges to the GS (4) $u(x)$ of the problem (1) in the grid norm $W_2^1(\omega)$ with the rate $O(|h|^{m/2})$ that is, there is a number M such that

$$\|\widetilde{y} - u\|_{1,\omega} \leq M|h|^{m/2}\|u\|_{m,G}, \quad (18)$$

where the constant M is independent of h and $u(x)$, $m = 2, 3$.

2. ELLIPTIC DIFFERENTIAL EQUATION OF THE SECOND ORDER WITH VARIABLE COEFFICIENTS

Consider the elliptic problem

$$P_u \equiv \sum_{i=1}^2 \frac{\partial}{\partial x_i} \left(k_i(x) \frac{\partial u}{\partial x_i} \right) = -f(x), \quad x \in G; \quad u(x) = 0, \quad x \in \partial G, \quad (19)$$

where G is defined as in the problem (1), $k_i(x) \in C(\overline{G})$, $i = 1, 2$,

$$0 < C_1 \leq k_i(x) \leq C_2, \quad x \in \overline{G}, \quad (20)$$

here C_i being constants.

2.1. Construction of difference scheme

Consider the GS of the problem (19) $u(x)$ in the space $W_2^m(G) \cap \overset{\circ}{W}_2^1(G)$ satisfying the equality:

$$\iint_G \sum_{i=1}^2 \left(k_i(x) \frac{\partial u}{\partial x_i} \right) v(x) = - \iint_G f(x) v(x) dx, \quad \forall v(x) \in \overset{\circ}{W}_2^m(G) \quad (21)$$

From the last equation, arguing as in Sections 1.1 and 3.1, [4], one has the following net problem for the *GS* of the problem (19):

$$P^e u = \frac{1}{h_1 h_2} \left\{ \sum_{i=1}^2 l_i^{(+0.5)} \bar{\omega}_i^{(+0.5)} - \sum_{i=1}^2 l_i^{(-0.5)} \bar{\omega}_i^{(-0.5)} + \Delta l \beta(x) \bar{\omega}^{(0)} \right\} - \frac{1}{h_1 h_2} \iint_{e(x)} \sum_{i=1}^2 k_i \frac{\partial \alpha}{\partial \zeta_i} \frac{\partial u}{\partial \zeta_i} d\zeta = Rf, \quad u(x) = 0, \quad x \in \gamma, \quad (22)$$

where $Rf \equiv \varphi$ and $\alpha(x)$ have the form (3), $\beta(x)$ is defined as in (4),

$$\bar{\omega}_i^{(\pm 0.5)} = \frac{1}{l_i^{(\pm 0.5)}} \int_{l_i^{(\pm 0.5)}} k_i \alpha \frac{\partial u}{\partial x_i} dl, \quad \bar{\omega}^{(0)} = \frac{1}{\Delta l} \int_{\Delta l} k_i \alpha \frac{\partial u}{\partial l} dl.$$

By (22), in a manner analogous to the proof of the forms (6) and (7) one obtains the following difference schemes of the net problem (22):

$$\tilde{K}y \equiv - \sum_{i=1}^2 (b_i \bar{y}_{x_i})_{\hat{x}_i} + \frac{1}{h_1 h_2} \iint_{e(x)} \sum_{i=1}^2 k_i(x) \alpha_{\bar{x}_i} \bar{y}_{\bar{x}_i}(x) d\zeta = \varphi(x) \bar{y}(x) = 0, \quad x \in \gamma, \quad (23)$$

$$\tilde{L}\hat{y} \equiv - \sum_{i=1}^2 (d_i \hat{y}_{x_i})_{\hat{x}_i} = \varphi(x); \quad \hat{y}(x) = 0, \quad x \in \gamma, \quad (24)$$

where

$$\begin{aligned} b_i^{(\pm 0.5)} &= \frac{1}{l_i^{(\pm 0.5)}} \int_{l_i^{(\pm 0.5)}} k_i(\zeta) \alpha(\zeta) dl \quad \text{for } l_i^{(\pm 0.5)} \neq 0, \\ b_i^{(\pm 0.5)} &= \frac{1}{\Delta l} \int_{\Delta l} k_i(\zeta) \alpha(\zeta) dl \quad \text{for } l_i^{(\pm 0.5)} = 0, \\ d_i^{(\pm 0.5)} &= \frac{1}{l_i^{(\pm 0.5)}} \int_{l_i^{(\pm 0.5)}} k_i(\zeta) dl \quad \text{for } l_i^{(\pm 0.5)} \neq 0, \\ d_i^{(\pm 0.5)} &= \frac{1}{\Delta l} \int_{\Delta l} k_i(\zeta) dl \quad \text{for } l_i^{(\pm 0.5)} = 0. \end{aligned}$$

2.2. Estimate of convergence rate

By (22)–(24), arguing as in the proof of the Theorem 1, we have following.

Theorem 2. Let $k_i(x) \in W_\infty^{m-1}(G)$, $i = 1, 2$, satisfying the condition (20), $m = 2, 3$; $\alpha(x)f(x) \in L_2(G)$. Then the solution y of the scheme (23) or (24) ($y = \bar{y}$ or \hat{y}) converges to the *GS* (22) u of the problem (19) in the net norm $W_2^1(\omega)$ with the rate $O(|h|^{m/2})$, that is there is a number M such that

$$\|y - u\|_{1,\omega} \leq M |h|^{m/2} \|u\|_{m,G}, \quad (25)$$

where the constant M is independent of h and $u(x)$.

Remark

a. For simplicity of presentation, the homogeneous boundary condition was considered. The Theorems 1 and 2 are also valid in the case where $u(x) = g(x)$, $x \in \partial G$.

b. Some generalizations given in Section 2.3, [4] are also true for the problems (1) and (19):

- The Theorems 1 and 2 are also valid, if in the formulas (2) and (21) of the *GS* $u(x)$, $v(x)$ is any function in the space $\mathcal{D}(G)$ of Schwartz basic functions [8].

- It is known (see [5], [9], etc.) that the right hand side of differential equations in the environment problems may be a functional (e.g., f is the Dirac delta functions δ). The estimates (17), (18) and (25) are obtained with the assumption $f \in L_2(G)$, now we show that the results may be generalized to the equations with right-hand side $f \in W_2^{(-l)}(G)$ - the space of continuous linear functionals on the space $\overset{\circ}{W}_2^l(G)$, l is a nonnegative integer. Indeed, by this assumption, $f(x) \in \mathcal{D}'(G)$ - the space of Schwartz distributions [8]. Then, by the theorem on local structure of the distributions (see [9, chap.1, n.2]) there exist a function $g(x) \in L_\infty(e)$ and an integer $k \geq 0$ such that

$$f(x) = D_1^k \dots D_n^k g(x), \quad (26)$$

where $x \in e$, the set e is compact in $G \subset R^n$.

Let $v(x) \in \mathcal{D}(e)$, from (26) and (21) one has

$$\iint_e \sum_{i=1}^2 \frac{\partial}{\partial x_i} \left(k_i(x) \frac{\partial u}{\partial x_i} \right) v(x) dx = - \iint_e g(x) \tilde{v}(x) dx, \quad (27)$$

where

$$\tilde{v}(x) = D_1^k D_2^k v(x) \quad (n = 2).$$

We have $\tilde{v}(x) \in \mathcal{D}(e)$, $g(x) \in L_2(e)$. Therefore, the equation (27) has the form (21), (22). Then one may repeat the procedure used above and obtains the following

Theorem 3. *Let the coefficients $k_i(x)$ of the problem (19) belong to the space $W_\infty^{m-1}(G)$, satisfying the condition (20), $m = 2, 3$, and let the right-hand side $f(x) \in W_2^{(-l)}(G)$. Then the solution y of the scheme (23) or (24) converges to the GS (22) u of the problem (19) in the grid norm $W_2^1(\omega)$ with the rate $O(h^{m/2})$, that is, there is a number M such that*

$$\|y - u\|_{1,\omega} \leq M |h|^{m/2} \|u\|_{m,G},$$

where the constant M is independent of h and $u(x)$.

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