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SOME OPERATIONS ON TYPE-2 INTUITIONISTIC FUZZY SETS

BUI CONG CUONG[1], TONG HOANG ANH, BUI DUONG HAI

[1]*Institute of Information Technology, Vietnam Academy of Science and Technology*

Tóm tắt. Trong những thập niên gần đây, một số mở rộng của khái niệm tập mờ được đề xuất. Tập mờ loại hai và tập mờ trực cảm là hai khái niệm mới đã thu hút được nhiều sự quan tâm của các nhà nghiên cứu vì sự phong phú của các ứng dụng. Bài báo giới thiệu một khái niệm mới - tập mờ trực cảm loại hai và chứng minh một số tính chất trên đó.

Abstract. In the last decades, there have been some extensions of fuzzy sets and their applications. Recently, type-2 fuzzy set and intuitionistic fuzzy set are two of them, drawing a great deal of scientist's attention because of their widespread range of applications. In this paper, we introduce a new concept - type-2 intuitionistic fuzzy set and propose some properties of their operations.

1. INTRODUCTION

Type-2 fuzzy sets - that is, fuzzy sets with fuzzy sets as truth values - seem destined to play an increasingly important role in applications. They were introduced by Zadeh [13], extending the notion of ordinary fuzzy sets. The Mendel's book [5] on Uncertain Rule-based Fuzzy Logic Systems and other researches [6, 7, 8, 11] are discussions of both theoretical and practical aspects of type-2 fuzzy sets.

In [1] Atanassov K. introduced the concept of intuitionistic fuzzy set characterized by a membership function and a non-membership function, which is a generalization of fuzzy set. In [1] Atanassov K. also defined some operators of IFSs. Recently, intuitionistic fuzzy set (IFS) theory have been applied to many different fields, such as decision making, medical diagnosis, pattern recognition.

In this paper, we introduce concept of type-2 intuitionistic fuzzy sets. We define some basic operations and derive some their properties. The paper is organized as follow: Section 2 gives a briefly review some basic definitions of fuzzy sets, type-2 fuzzy sets and intuitionistic fuzzy sets. Section 3 is devoted to the new main definitions and some their properties. Section 4 is a first discussion of a subclass of the type-2 IFS.

2. BASIC DEFINITION

2.1. Definition of fuzzy sets

Let T, S be nonempty sets. The $Map(S,T)$ be the set of all function from S into T. A

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fuzzy set A of a set S is a maping $A : S \to [0,1]$. The set S has no operations on it. So operations on the set $Map(S,[0,1])$ of all fuzzy subsets of S come from operations on $[0,1]$. Common operations on $[0,1]$ of interest in fuzzy theory are $\vee, \wedge$, and $'$ given by

$$x \wedge y = \min\{x,y\}, \qquad x \vee y = \max\{x,y\}, \qquad x' = 1 - x$$

The constant 0 and 1 are generally considered as part of the algebraic structure. So the algebra basic to fuzzy set theory is $[0,1], \vee, \wedge,', 0, 1$.

The corresponding operations on the set $Map(S,[0,1])$ of all fuzzy subsets of S are given pointwise by the following formulas

$$(A \wedge B)(s) = A(s) \wedge B(s), \qquad (A \vee B)(s) = A(s) \vee B(s), \qquad A'(s) = (A(s))'$$

and the two nullary operations are given by $1(s) = 1$ and $0(s) = 0$ for all $s \in S$.
We use the same symbols for the pointwise operations on the elements of $Map(S,[0,1])$. There are many properties hold in the algebra $I = ([0,1], \vee, \wedge,' .0.1)$ (see [9, 11]).

Thus, I is a bounded distributive lattice with an involution $'$ that satisfies De Morgan's laws and the Kleene inequality. Thus is, I is a Kleene algebra. Thus $Map(S,[0,1])$ is also a Kleene algebra. Basic knowledge of fuzzy sets has been presented in [3, 9].

2.2. Definition of type-2 fuzzy sets

Let S be a universe of discourse, the a type-2 fuzzy set (T2 FS) is defined as following.

Definition 2.2.1. *[5] A type-2 fuzzy set, denoted by A, is characterized by a type-2 membership function $\mu_A(x,u)$, where $x \in S$ and $u \in J_x \subseteq [0.1]$, i.e. $A = \{((x,u), \mu_A(x,u)) | \forall x \in X, \forall u \in J_x \subseteq [0.1]\}$, in which $0 \le \mu_A(x,u) \le 1$. A can be express as* $A = \int_{x \in X} \int_{u \in J_x} \mu_A(x,u)/(x,u)$, $J_x \subseteq [0,1]$

2.3. Basic definition and some properties of IFS

Let Y be a universe of discourse, then a fuzzy set $A = \{< y, \mu_A(y) > | y \in Y\}$ defined by Zadeh [13] is characterized by a membership function $\mu_A : Y \to [0.1]$, where $\mu_A(y)$ denotes the degree of membership of element y to the set A.

Definition 2.3.1. *[1] An intuitionistic fuzzy set (IFS) $A = \{< y, \mu_A(y), \nu_A(y) > | y \in Y\}$ is characterized by a membership function $\mu_A : Y \to [0.1]$, and a non-membership function $\nu_A : Y \to [0,1]$ with the condition $0 \le \mu_A(y) + \nu_A(y) \le 1$ for all $y \in Y$, where the numbers $\mu_A(y)$ and $\nu_A(y)$ represent the degree of membership and the degree of non-membership of the element y to the set A, respectively.*

Definition 2.3.2. *[1, 4] If A and B are two IFS of the set Y, then*
$A \subset B$ *iff* $\forall y \in Y$, $\mu_A(y) \le \mu_B(y)$ *and* $\nu_A \ge \nu_B(y)$, $A \supset B$ *iff* $B \subset A$,
$A = B$ *iff* $\forall y \in Y$, $\mu_A(y) = \mu_B(y)$ *and* $\nu_A = \nu_B(y)$,
$A \cap B = \{< y, \min(\mu_A(y), \mu_B(y)), \max(\nu_A(y), \nu_B(y)) > | y \in Y\}$,
$A \cup B = \{< y, \max(\mu_A(y), \mu_B(y)), \min(\nu_A(y), \nu_B(y)) > | y \in Y\}$,

Definition 2.3.3. *[1, 4] If A_1 and A_2 are two intuitionistic fuzzy sets, then*
$\overline{A_1} = \{< y, \nu_{A_1}(y), \mu_{A_1}(y) > |y \in Y\}$
$A_1 + A_2 = \{< y, \mu_{A_1}(y) + \mu_{A_2}(y) - \mu_{A_1}(y).\mu_{A_2}(y), \nu_{A_1}(y).\nu_{A_2}(y) > |y \in Y\}$
$A_1.A_2 = \{< y, \mu_{A_1}(y).\mu_{A_2}(y), \nu_{A_1}(y) + \nu_{A_2}(y) - \nu_{A_1}(y).\nu_{A_2}(y) > |y \in Y\}$
$\lambda A_1 = \{< y, 1 - (1 - \mu_{A_1}(y))^\lambda, (\nu_{A_1}(y))^\lambda > |y \in Y\}$
${A_1}^\lambda = \{< y, (\mu_{A_1}(y))^\lambda, 1 - (1 - \nu_{A_1}(y))^\lambda > |y \in Y\}$

3. DEFINITION OF OPERATIONS ON TYPE-2 IFS

Now we introduce the notion of a type-2 intuitionistic fuzzy set.

3.1. Definition

Definition 3.1.1. *Let S be a nonempty set. A is a type-2 intuitionistic fuzzy set (T2IFS) of S. A is defined by:*
$A : S \to Map(D, [0, 1]) \times Map(D, [0, 1])$, *where* $D = \{(u, v) \in [0, 1] \times [0, 1] : u + v \leq 1\}$.

For convenience in description, the binary operations between $f(u, v)$ and $g(u, v)$ in $Map(D, [0, 1])$ are written in the forms $(f \wedge g) = (f \wedge g)(u, v) = f(u, v) \wedge g(u, v)$, $(f \vee g) = (f \vee g)(u, v) = f(u, v) \vee g(u, v)$ and $(f, g) = (f(u, v), g(u, v))$.
Now we give the definitions of main operations in T2IFS theory

3.1.1. The operation AND

Definition 3.1.2. *Let (f_1, g_1) and (f_2, g_2) be in $Map(D, [0, 1]) \times Map(D, [0, 1])$. We define the **intersection** operation denoted $\sqcap$ and it is defined by: $(f_1, g_1) \sqcap (f_2, g_2) = (f, g)$ where for any $(u, v) \in D$*

$$f(u, v) = \bigvee_{u_1 \wedge u_2 = u, v_1 \vee v_2 = v} f_1(u_1, v_1) \wedge f_2(u_2, v_2)$$
$$g(u, v) = \bigvee_{u_1 \wedge u_2 = u, v_1 \vee v_2 = v} g_1(u_1, v_1) \wedge g_2(u_2, v_2)$$

3.1.2. The operation OR

Definition 3.1.3. *Let (f_1, g_1) and (f_2, g_2) be in $Map(D, [0, 1]) \times Map(D, [0, 1])$. We define the **union** operation denoted $\sqcup$ and it is defined by: $(f_1, g_1) \sqcup (f_2, g_2) = (f, g)$ where for any $(u, v) \in D$*

$$f(u, v) = \bigvee_{u_1 \vee u_2 = u, v_1 \wedge v_2 = v} f_1(u_1, v_1) \wedge f_2(u_2, v_2),$$
$$g(u, v) = \bigvee_{u_1 \vee u_2 = u, v_1 \wedge v_2 = v} g_1(u_1, v_1) \wedge g_2(u_2, v_2)$$

3.1.3. The operation NEGATION

$$(f(u, v), g(u, v))^* = (f(v, u), g(v, u))$$

The followings are definitions of identities. They are $\mathbf{1} = (\mathbf{1}_1, \mathbf{1}_0)$ and $\mathbf{0} = (\mathbf{1}_0, \mathbf{1}_1)$.

$$\mathbf{1}_1(u,v) = \begin{cases} 1 & \text{if } u=1, v=0 \\ 0 & \text{if } u \neq 1, v \neq 0 \end{cases} \quad \mathbf{1}_0(u,v) = \begin{cases} 1 & \text{if } u=0, v=1 \\ 0 & \text{if } u \neq 0, v \neq 1 \end{cases}$$

Thus, we defined the algebra $\mathbf{M} = (Map(D,[0,1]) \times Map(D,[0,1]), \sqcap, \sqcup, ^*, \mathbf{1}, \mathbf{0})$ for T2IFS with operations $\sqcap$,$\sqcup$, *. Our aim in this paper is to examine some properties on the algebra of T2IFS such as idempotent, involution, commutative laws, associative laws or distributive laws.

3.2. Some properties of these operations.

In this section, we are going to demonstrate some properties of the operations on T2 IFS. We start with the below theorem which clarifies the first properties such as idempotent, commutative, absorption laws with identities, involution, and De Morgan's laws.

Theorem 3.2.1. *For every $(f,g),(f_1,g_1),(f_2,g_2) \in \boldsymbol{M}$, we have*
1.$(f,g) \sqcap (f,g) = (f,g)$ *and* $(f,g) \sqcup (f,g) = (f,g)$
2.$(f_1,g_1) \sqcap (f_2,g_2) = (f_2,g_2) \sqcap (f_1,g_1)$ *and* $(f_1,g_1) \sqcup (f_2,g_2) = (f_2,g_2) \sqcup (f_1,g_1)$
3.$(\boldsymbol{1}_1, \boldsymbol{1}_0) \sqcap (f,g) = (f,g)$ *and* $(\boldsymbol{1}_0, \boldsymbol{1}_1) \sqcup (f,g) = (f,g)$
4.$(f,g)^{**} = (f,g)$
5.$\{(f_1,g_1) \sqcap (f_2,g_2)\}^* = (f_1,g_1)^* \sqcup (f_2,g_2)^*$ *and*
$\{(f_1,g_1) \sqcup (f_2,g_2)\}^* = (f_1,g_1)^* \sqcap (f_2,g_2)^*$

Proof We omit properties 1 ,2 and 4.
The proof of property 3. First, we handle the absorption law of identity $(\mathbf{1}_1, \mathbf{1}_0)$. Let (f,g) be in $\mathbf{M}$.Suppose that $(\mathbf{1}_1, \mathbf{1}_0) \sqcap (f,g) = (f', g')$, we have

$$f'(u,v) = \bigvee_{u_1 \wedge u_2 = u, v_1 \vee v_2 = v} \mathbf{1}_1(u_1,v_1) \wedge f(u_2,v_2)$$

To find $f'(u,v)$ we look for all values of $\mathbf{1}_1(u_1,v_1) \wedge f(u_2,v_2)$, where $u_1 \wedge u_2 = u, v_1 \vee v_2 = v$ and then find their *sup*. In the first case, if $(u_1,v_1) = (1,0)$ then $\mathbf{1}_1(u_1,v_1) = 1$ and (u_2,v_2) must be (u,v). We have $\mathbf{1}_1(u_1,v_1) \wedge f(u_2,v_2) = 1 \wedge f(u,v) = f(u,v)$.
In the other case, if $(u_1,v_1) \neq (1,0)$, then $\mathbf{1}_1(u_1,v_1) = 0$, we have $\mathbf{1}_1(u_1,v_1) \wedge f(u_2,v_2) = 0 \wedge f(u_2,v_2) = 0$.
Hence, $f'(u,v)$ is the *sup* of two results for the two cases or $f'(u,v) = 0 \vee f(u,v) = f(u,v)$.
In similar way, we prove that $g'(u,v) = g(u,v)$.
With the same arguments, the absorption law of the other identity can be proven.
For property 5,we prove only the first formula. The second formula automatically has the analogous proof.
Let $(f_1,g_1),(f_2,g_2) \in \mathbf{M}$. Suppose that $(f', g') = (f_1,g_1) \sqcap (f_2,g_2)$, we have

$$f'(u,v) = \bigvee_{u_1 \wedge u_2 = u, v_1 \vee v_2 = v} f_1(u_1,v_1) \wedge f_2(u_2,v_2),$$
$$g'(u,v) = \bigvee_{u_1 \wedge u_2 = u, v_1 \vee v_2 = v} g_1(u_1,v_1) \wedge g_2(u_2,v_2).$$

then $(f'(u,v), g'(u,v))^* = (f'(v,u), g'(v,u))$, where

$$f'(v,u) = \bigvee_{u_1 \wedge u_2 = u, v_1 \vee v_2 = v} f_1(v_1,u_1) \wedge f_2(v_2,u_2),$$
$$g'(v,u) = \bigvee_{u_1 \wedge u_2 = u, v_1 \vee v_2 = v} g_1(v_1,u_1) \wedge g_2(v_2,u_2).$$

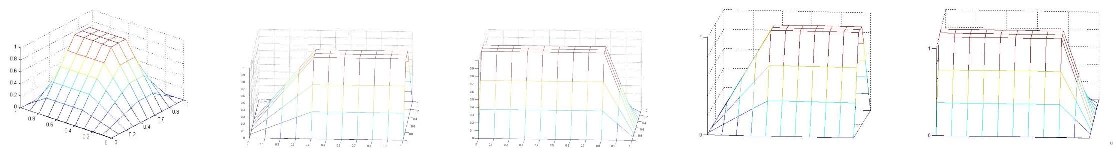
Otherwise,
$(f_1, g_1)^* \sqcup (f_2, g_2)^* = (f_1(u, v), g_1(u, v))^* \sqcup (f_2(u, v), g_2(u, v))^*$
$= (f_1(v, u), g_1(v, u)) \sqcup (f_2(v, u), g_2(v, u) = (f^{''}(v, u), g^{''}(v, u)),$
where

$$f^{''}(v, u) = \bigvee_{v_1 \vee v_2 = v, u_1 \wedge u_2 = u} f_1(v_1, u_1) \wedge f_2(v_2, u_2),$$
$$g^{''}(v, u) = \bigvee_{v_1 \vee v_2 = v, u_1 \wedge u_2 = u} g_1(v_1, u_1) \wedge g_2(v_2, u_2).$$

Comparing $(f^{'}, g^{'})$ and $(f^{''}, g^{''})$, we can imply $(f^{'}, g^{'}) = (f^{''}, g^{''})$. Thus, we have $\{(f_1, g_1) \sqcap (f_2, g_2)\}^* = (f_1, g_1)^* \sqcup (f_2, g_2)^*$ is proved.

Definition 3.2.2. *For $f \in Map(D, [0, 1])$. Let f^L, f^R, f_L and f_R be elements of $Map(D, [0, 1])$ defined by $f^L(u, v) = \vee_{u' \leq u} f(u^{'}, v)$, $f^R(u, v) = \vee_{u' \geq u} f(u^{'}, v)$, $f_L(u, v) = \vee_{v' \leq v} f(u, v^{'})$, $f_R(u, v) = \vee_{v' \geq v} f(u, v^{'})$.*

The below figures visualize our definitions.



(a) f (b) f^L (c) f^R (d) f_L (e) f_R

Figure 3.1: Geometrical interpretation of f, f^L, f^R, f_L, and f_R

Theorem 3.2.3. *The following properties hold for all $(f_1, g_1)(f_2, g_2) \in \boldsymbol{M}$:*
1.$(f_1, g_1) \sqcap (f_2, g_2) = (f, g)$ provided

$$f = ({f_1}_L \wedge {f_2}^R) \vee ({f_1}^R \wedge {f_2}_L) \vee ({f_1}_L^R \wedge f_2) \vee (f_1 \wedge {f_2}_L^R),$$
$$g = ({g_1}_L \wedge {g_2}^R) \vee ({g_1}^R \wedge {g_2}_L) \vee ({g_1}_L^R \wedge g_2) \vee (g_1 \wedge {g_2}_L^R).$$

2.$(f_1, g_1) \sqcup (f_2, g_2) = (f, g)$ provided

$$f = ({f_1}^L \wedge {f_2}_R) \vee ({f_1}_R \wedge {f_2}^L) \vee ({f_1}_R^L \wedge f_2) \vee (f_1 \wedge {f_2}_R^L),$$
$$g = ({g_1}^L \wedge {g_2}_R) \vee ({g_1}_R \wedge {g_2}^L) \vee ({g_1}_R^L \wedge g_2) \vee (g_1 \wedge {g_2}_R^L).$$

Proof
Let $(f_1, g_1), (f_2, g_2) \in \mathbf{M}$. We have $(f_1, g_1) \sqcap (f_2, g_2) = (f, g)$ such that

$$f(u, v) = \bigvee_{u_1 \wedge u_2 = u, v_1 \vee v_2 = v} (f_1(u_1, v_1) \wedge f_2(u_2, v_2)),$$
$$g(u, v) = \bigvee_{u_1 \wedge u_2 = u, v_1 \vee v_2 = v} (g_1(u_1, v_1) \wedge g_2(u_2, v_2))$$

First,

$$
\begin{aligned}
f(u,v) &= \bigvee_{u_1\wedge u_2=u, v_1\vee v_2=v} (f_1(u_1,v_1)\wedge f_2(u_2,v_2)) \\
&= \bigvee_{u_1\wedge u_2=u} [\bigvee_{v_1\vee v_2=v} (f_1(u_1,v_1)\wedge f_2(u_2,v_2))] \\
&= \bigvee_{u_1\wedge u_2=u} \{[\bigvee_{v_1\vee v=v} (f_1(u_1,v_1)\wedge f_2(u_2,v))]\vee[\bigvee_{v\vee v_2=v} (f_1(u_1)\wedge f_2(u_2,v_2))]\} \\
&= \bigvee_{u_1\wedge u_2=u} \{[(\bigvee_{v_1\le v} f_1(u_1,v_1))\wedge f_2(u_2,v)]\vee[f_1(u_1,v)\wedge(\bigvee_{v2\le v} f_2(u_2,v_2))]\} \\
&= \bigvee_{u_1\wedge u_2=u} ((f_{1L}(u_1,v)\wedge f_2(u_2,v))\vee(f_1(u_1,v)\wedge f_{2L}(u_2,v))) \\
&= \bigvee_{u_1\wedge u_2=u} (f_{1L}(u_1,v)\wedge f_2(u_2,v))\vee \bigvee_{u_1\wedge u_2=u} (f_1(u_1,v)\wedge f_{2L}(u_2,v)) \\
&= \bigvee_{u\wedge u_2=u} (f_{1L}(u,v)\wedge f_2(u_2,v))\vee \bigvee_{u_1\wedge u=u} (f_{1L}(u_1,v)\wedge f_2(u,v)) \\
&\vee \bigvee_{u\wedge u_2=u} (f_1(u,v)\wedge f_{2L}(u_2,v))\vee \bigvee_{u_1\wedge u=u} (f_1(u_1,v)\wedge f_{2L}(u,v)) \\
&= (f_{1L}(u,v)\wedge \bigvee_{u_2\ge u} f_2(u_2,v))\vee(\bigvee_{u_1\ge u} f_{1L}(u_1,v)\wedge f_2(u,v)) \\
&\vee(f_1(u,v)\wedge \bigvee_{u_2\ge u} f_{2L}(u_2,v))\vee(\bigvee_{u_1\ge u} f_1(u_1,v)\wedge f_{2L}(u,v)) \\
&= (f_{1L}(u,v)\wedge f_2{}^R(u,v))\vee(f_{1L}^R(u_1,v)\wedge f_2(u,v))\vee(f_1(u,v)\wedge f_{2L}^R(u,v)) \\
&\vee(f_1{}^R(u,v)\wedge f_{2L}(u,v)) \\
&= (f_{1L}\wedge f_2{}^R)\vee(f_1{}^R\wedge f_{2L})\vee(f_{1L}^R\wedge f_2)\vee(f_1\wedge f_{2L}^R)
\end{aligned}
$$

Analogously, we can easily prove the fomulas stated for g. It is no doubt that the operation $\sqcup$ between (f_1,g_1) and (f_2,g_2) has the similar proof.

Note.To make convenient for further descriptions and proofs, let denote operations between (f_1,g_1) and (f_2,g_2) as the followings:
1.$(f_1,g_1)\sqcup(f_2,g_2)=(f_1\Cup f_2, g_1\Cup g_2)$, where
$f_1\Cup f_2=(f_1{}^L\wedge f_{2R})\vee(f_{1R}\wedge f_2{}^L)\vee(f_{1R}^L\wedge f_2)\vee(f_1\wedge f_{2R}^L)$
$g_1\Cup g_2=(g_1{}^L\wedge g_{2R})\vee(g_{1R}\wedge g_2{}^L)\vee(g_{1R}^L\wedge g_2)\vee(g_1\wedge g_{2R}^L)$
$(f_1,g_1)\sqcap(f_2,g_2)=(f_1\Cap f_2, g_1\Cap g_2)$where
$f_1\Cap f_2=(f_{1L}\wedge f_2{}^R)\vee(f_1{}^R\wedge f_{2L})\vee(f_{1L}^R\wedge f_2)\vee(f_1\wedge f_{2L}^R)$
$g_1\Cap g_2=(g_{1L}\wedge g_2{}^R)\vee(g_1{}^R\wedge g_{2L})\vee(g_{1L}^R\wedge g_2)\vee(g_1\wedge g_{2L}^R)$
2.$f\Cap g=(f_L\bigtriangledown g)\vee(f\bigtriangledown g_L)$ and $f\Cup g=(f_R\bigtriangleup g)\vee(f\bigtriangleup g_R)$
such that
$f\bigtriangledown g=(f\wedge g^R)\vee(f^R\wedge g)$ and $f\bigtriangleup g=(f\wedge g^L)\vee(f^L\wedge g)$

Corollary 3.2.4. *Let $f,g\in Map(D,[0,1])$. We have*
1.$(f\Cup g)^L=(f_R^L\wedge g^L)\vee(f^L\wedge g_R^L)$ and $(f\Cup g)_R=(f_R^L\wedge g_R)\vee(f_R\wedge g_R^L)$.
2. $(f\Cap g)_R=(f_{RL}\bigtriangledown g_R)\vee(f_R\bigtriangledown g_{RL})$and $(f\Cup g)_L=(f_{RL}\bigtriangleup g_L)\vee(f_L\bigtriangleup g_{RL})$.
3.$(f\Cup g)_R^L=f_R^L\wedge g_R^L$.

Proof
1.We have

$$\begin{aligned}
(f \Cup g)^L(u,v) &= \bigvee_{u' \le u} (f \Cup g)(u',v) = \bigvee_{u'\le u} \Big(\bigvee_{u_1 \vee u_2 = u', v_1 \wedge v_2 = v} (f(u_1,v_1) \wedge g(u_2,v_2)) \Big) \\
&= \bigvee_{u_1 \vee u_2 \le u} \Big(\bigvee_{v_1 \wedge v_2 = v} (f(u_1,v_1) \wedge g(u_2,v_2)) \Big) \\
&= \bigvee_{u_1 \vee u_2 \le u} ((f_R(u_1,v) \wedge g(u_2,v)) \vee (f(u_1,v) \wedge g_R(u_2,v))) \\
&= \bigvee_{u_1 \vee u_2 \le u} (f_R(u_1,v) \wedge g(u_2,v)) \vee \bigvee_{u_1 \vee u_2 \le u} (f(u_1,v) \wedge g_R(u_2,v)) \\
&= \bigvee_{u_1 \le u, u_2 \le u} (f_R(u_1,v) \wedge g(u_2,v)) \vee \bigvee_{u_1 \le u, u_2 \le u} (f(u_1,v) \wedge g_R(u_2,v)) \\
&= \Big(\bigvee_{u_1 \le u} f_R(u_1,v) \wedge \bigvee_{u_2 \le u} g(u_2,v) \Big) \vee \Big(\bigvee_{u_1 \le u} f(u_1,v) \wedge \bigvee_{u_2 \le u} g_R(u_2,v) \Big) \\
&= (f_R^L(u,v) \wedge g^L(u,v)) \vee (f^L(u,v) \wedge g_R^L(u,v)). \ \square
\end{aligned}$$

There is no difficulty to prove the other formula by using the same arguments used in the above proof.

2.We have

$$\begin{aligned}
(f \Cap g)_R(u,v) &= \bigvee_{v' \ge v} (f \Cap g)(u,v') = \bigvee_{v' \ge v} \bigvee_{u_1 \wedge u_2 = u, v_1 \vee v_2 = v'} f(u_1,v_1) \wedge g(u_2,v_2) \\
&= \bigvee_{u_1 \wedge u_2 = u, v_1 \vee v_2 \ge v} f(u_1,v_1) \wedge g(u_2,v_2) \\
&= \bigvee_{v_1 \vee v_2 \ge v} \bigvee_{u_1 \wedge u_2 = u} f(u_1,v_1) \wedge g(u_2,v_2) \\
&= \bigvee_{v_1 \vee v_2 \ge v} \Big[\Big(\bigvee_{u \wedge u_2 = u} f(u,v_1) \wedge g(u_2,v_2) \Big) \vee \Big(\bigvee_{u_1 \wedge u = u} f(u_1,v_1) \wedge g(u,v_2) \Big) \Big] \\
&= \bigvee_{v_1 \vee v_2 \ge v} \Big[\Big(f(u,v_1) \wedge \bigvee_{u_2 \ge u} g(u_2,v_2) \Big) \vee \Big(\Big(\bigvee_{u_1 \ge u} f(u_1,v_1) \Big) \wedge g(u,v_2) \Big) \Big] \\
&= \bigvee_{v_1 \vee v_2 \ge v} [(f(u,v_1) \wedge g^R(u,v_2)) \vee (f^R(u,v_1) \wedge g(u,v_2))] \\
&= \bigvee_{v_1 \ge v, v_2} [(f(u,v_1) \wedge g^R(u,v_2)) \vee (f^R(u,v_1) \wedge g(u,v_2))] \vee \\
&\quad \bigvee_{v_2 \ge v, v_1} [(f(u,v_1) \wedge g^R(u,v_2)) \vee (f^R(u,v_1) \wedge g(u,v_2))] \\
&= [(f_R(u,v) \wedge g_{RL}^R(u,v)) \vee (f_R^R(u,v) \wedge g_{RL}(u,v))] \vee \\
&\quad [(f_{RL}(u,v) \wedge g_R^R(u,v)) \vee (f_{RL}^R(u,v) \wedge g_R(u,v))] \\
&= (f_R \wedge g_{RL}^R) \vee (f_R^R \wedge g_{RL}) \vee (f_{RL} \wedge g_R^R) \vee (f_{RL}^R \wedge g_R) \\
&= (f_R \bigtriangledown g_{RL}) \vee (f_{RL} \bigtriangledown g_R).
\end{aligned}$$

Thus, $(f \Cap g)_R = (f_{RL} \bigtriangledown g_R) \vee (f_R \bigtriangledown g_{RL})$.

With the same arguments, it becomes easy to prove the following formula:

$$(f \Cup g)_L = (f_{RL} \bigtriangleup g_L) \vee (f_L \bigtriangleup g_{RL}).$$

3. We have

$$\begin{aligned}
(f \Cup g)_L^R &= \bigvee_{u' \le u, v' \ge v} (f \Cup g)(u',v') = \bigvee_{u' \le u, v' \ge v} \Big(\bigvee_{u_1 \vee u_2 = u', v_1 \wedge v_2 = v'} (f(u_1,v_1) \wedge g(u_2,v_2)) \Big) \\
&= \bigvee_{u_1 \vee u_2 \le u, v_1 \wedge v_2 \ge v} (f(u_1,v_1) \wedge g(u_2,v_2)) = \bigvee_{u_1 \vee u_2 \le u} \Big(\bigvee_{v_1 \wedge v_2 \ge v} (f(u_1,v_1) \wedge g(u_2,v_2)) \Big) \\
&= \bigvee_{u_1 \vee u_2 \le u} \Big(\bigvee_{v_1 \ge v} f(u_1,v_1) \wedge \bigvee_{v_2 \ge v} g(u_2,v_2) = \bigvee_{u_1 \vee u_2 \le u} (f_R(u_1,v) \wedge g_R(u_2,v) \\
&= \Big(\bigvee_{u_1 \le u} (f_R(u_1,v)) \wedge \Big(\bigvee_{u_2 \le u} g_R(u_2,v) \Big) = f_R^L(u,v) \wedge g_R^L(u,v) \\
&= f_R^L \wedge g_R^L
\end{aligned}$$

Theorem 3.2.5. *The following associative laws hold for* $\sqcup$ *and* $\sqcap$.

$$(f_1,f_2) \sqcup [(g_1,g_2) \sqcup (h_1,h_2)] = [(f_1,f_2) \sqcup (g_1,g_2)] \sqcup (h_1,h_2)$$

$$(f_1, f_2) \sqcap [(g_1, g_2) \sqcap (h_1, h_2)] = [(f_1, f_2) \sqcap (g_1, g_2)] \sqcap (h_1, h_2)$$

Proof
We are going to prove the following formula. For $f, g, h \in Map(D, [0,1])$

$$f \Cup (g \Cup h) = (f \Cup g) \Cup h \qquad (3.1)$$

From the right side of (1), we have

$$\begin{aligned}
(f \Cup g) \Cup h \quad &= [(f \Cup g)^L \wedge h_R] \vee [(f \Cup g)_R \wedge h^L] \vee [(f \Cup g)_R^L \wedge h] \vee [(f \Cup g) \wedge h_R^L] \\
&= \{[(f_R^L \wedge g^L) \vee (f^L \wedge g_R^L)] \wedge h_R\} \vee \{[(f_R^L \wedge g_R) \vee (f_R \wedge g_R^L)] \wedge h^L\} \\
&\vee [(f_R^L \wedge g_R^L) \wedge h] \vee \{[(f^L \wedge g_R) \vee (f_R \wedge g^L) \vee (f_R^L \wedge g) \vee (f \wedge g_R^L)] \wedge h_R^L\} \\
&= [f_R^L \wedge g^L \wedge h_R] \vee [f^L \wedge g_R^L \wedge h_R] \vee [f_R^L \wedge g_R \wedge h^L] \vee [f_R \wedge g_R^L \wedge h^L] \\
&\vee [f_R^L \wedge g_R^L \wedge h] \vee [f^L \wedge g_R \wedge h_R^L] \vee [f_R \wedge g^L \wedge h_R^L] \vee [f_R^L \wedge g \wedge h_R^L] \\
&\vee [f \wedge g_R^L \wedge h_R^L]
\end{aligned}$$

From the left side of (1), we have

$$\begin{aligned}
f \Cup (g \Cup h) \quad &= [f^L \wedge (g \Cup h)_R] \vee [f_R \wedge (g \Cup h)^L] \vee [f_R^L \wedge (g \Cup h)] \vee [f \wedge (g \Cup h)_R^L] \\
&= \{f^L \wedge [(g_R^L \wedge h_R) \vee (g_R \wedge h_R^L)]\} \vee \{f_R \wedge [(g_R^L \wedge h^L) \vee (g^L \wedge h_R^L)]\} \\
&\vee \{f_R^L \wedge [(g^L \wedge h_R) \vee (g_R \wedge h^L) \vee (g_R^L \wedge h) \vee (g \wedge h_R^L)]\} \vee [f \wedge (g_R^L \wedge h_R^L)] \\
&= [f^L \wedge g_R^L \wedge h_R] \vee [f^L \wedge g_R \wedge h_R^L] \vee [f_R \wedge g_R^L \wedge h^L] \vee [f_R \wedge g^L \wedge h_R^L] \\
&\vee [f_R^L \wedge g^L \wedge h_R] \vee [f_R^L \wedge g_R \wedge h^L] \vee [f_R^L \wedge g_R^L \wedge h] \vee [f_R^L \wedge g \wedge h_R^L] \\
&\vee [f \wedge g_R^L \wedge h_R^L]
\end{aligned}$$

Hence, $f \Cup (g \Cup h) = (f \Cup g) \Cup h$ is proved. Because of this property, we imply

$$\begin{aligned}
(f_1, f_2) \sqcup [(g_1, g_2) \sqcup (h_1, h_2)] \quad &= (f_1 \Cup (g_1 \Cup h_1), f_2 \Cup (g_2 \Cup h_2)) \\
&= ((f_1 \Cup g_1) \Cup h_1, (f_2 \Cup g_2) \Cup h_2) \\
&= [(f_1, f_2) \sqcup (g_1, g_2)] \sqcup (h_1, h_2)
\end{aligned}$$

To prove the absorption law of $\sqcap$, we have the similar proof.

Corollary 3.2.6. *The following distributive laws hold for operations $\bigtriangledown$, $\bigtriangleup$ and $\vee$:*

$$f \bigtriangledown (g \vee h) = (f \bigtriangledown g) \vee (f \bigtriangledown h)$$

$$f \bigtriangleup (g \vee h) = (f \bigtriangleup g) \vee (f \bigtriangleup h)$$

This is one of the results which has been proved for type-2 fuzzy sets in [11].
The following results will show that the operations of type-2 intuitionistic fuzzy sets do not have distributive laws.

Theorem 3.2.7. *Let f, g, h be in $\boldsymbol{M}$ We have*

1. $f \Cup (g \Cap h) \quad = [f_R \bigtriangleup (g_L \bigtriangledown h)] \vee [f_R \bigtriangleup (g \bigtriangledown h_L)] \vee [f \bigtriangleup (g_{RL} \bigtriangledown h_R)] \vee$
$[f \bigtriangleup (g_R \bigtriangledown h_{RL})]$.

2.
$$\begin{aligned}
(f \Cup g) \Cap (f \Cup h) \quad &= [(f_{RL} \bigtriangleup g_L) \bigtriangledown (f_R \bigtriangleup h)] \vee [(f_{RL} \bigtriangleup g_L) \bigtriangledown (f \bigtriangleup h_R)] \\
&\vee [(f_L \bigtriangleup g_{RL}) \bigtriangledown (f_R \bigtriangleup h)] \vee [(f_L \bigtriangleup g_{RL}) \bigtriangledown (f \bigtriangleup h_R)] \\
&\vee [(f_R \bigtriangleup g) \bigtriangledown (f_{RL} \bigtriangleup h_L)] \vee [(f_R \bigtriangleup g) \bigtriangledown (f_L \bigtriangleup h_{RL})] \\
&\vee [(f \bigtriangleup g_R) \bigtriangledown (f_{RL} \bigtriangleup h_L)] \vee [(f \bigtriangleup g_R) \bigtriangledown (f_L \bigtriangleup h_{RL})].
\end{aligned}$$

Proof

1.From the left side, we have.

$$\begin{aligned} f \Cup (g \Cap h) &= [f_R \bigtriangledown (g \Cap h)] \vee [f \bigtriangledown (g \Cap h)_R] \\ &= \{f_R \bigtriangledown [(g_L \bigtriangleup h) \vee (g \bigtriangleup h_L)]\} \vee \{f \bigtriangledown [(g_R \bigtriangleup h_{RL}) \vee (g_{RL} \bigtriangleup h_R)]\} \\ &= [f_R \bigtriangledown (g_L \bigtriangleup h)] \vee [f_R \bigtriangledown (g \bigtriangleup h_L)] \vee [f \bigtriangledown (g_R \bigtriangleup h_{RL})] \\ &\quad \vee [f \bigtriangledown (g_{RL} \bigtriangleup h_R)] \end{aligned}$$

2.We have $(f \Cup g) \Cap (f \Cup h) = [(f \Cup g)_L \bigtriangledown (f \Cup h)] \vee [(f \Cup g) \bigtriangledown (f \Cup h)_L]$
the first expression, we have

$$\begin{aligned} (f \Cup g)_L \bigtriangledown (f \Cup h) &= [(f_{RL} \bigtriangleup g_L) \vee (f_L \bigtriangleup g_{RL})] \bigtriangledown [(f_R \bigtriangleup h) \vee (f \bigtriangleup h_R)] \\ &= [(f_{RL} \bigtriangleup g_L) \bigtriangledown (f_R \bigtriangleup h)] \vee [(f_{RL} \bigtriangleup g_L) \bigtriangledown (f \bigtriangleup h_R)] \\ &\quad \vee [(f_L \bigtriangleup g_{RL}) \bigtriangledown (f_R \bigtriangleup h)] \vee [(f_L \bigtriangleup g_{RL}) \bigtriangledown (f \bigtriangleup h_R)] \end{aligned}$$

Deal with the second expression, we have

$$\begin{aligned} (f \Cup g) \bigtriangledown (f \Cup h)_L &= \\ &= [(f_R \bigtriangleup g) \bigtriangledown (f_{RL} \bigtriangleup h_L)] \vee [(f_R \bigtriangleup g) \bigtriangledown (f_L \bigtriangleup h_{RL})] \\ &\quad \vee [(f \bigtriangleup g_R) \bigtriangledown (f_{RL} \bigtriangleup h_L)] \vee [(f \bigtriangleup g_R) \bigtriangledown (f_L \bigtriangleup h_{RL})] \end{aligned}$$

Hence,

$$\begin{aligned} (f \Cup g) \Cap (f \Cup h) = \; & [(f_{RL} \bigtriangleup g_L) \bigtriangledown (f_R \bigtriangleup h)] \vee [(f_{RL} \bigtriangleup g_L) \bigtriangledown (f \bigtriangleup h_R)] \\ & \vee [(f_L \bigtriangleup g_{RL}) \bigtriangledown (f_R \bigtriangleup h)] \vee [(f_L \bigtriangleup g_{RL}) \bigtriangledown (f \bigtriangleup h_R)] \\ & \vee [(f_R \bigtriangleup g) \bigtriangledown (f_{RL} \bigtriangleup h_L)] \vee [(f_R \bigtriangleup g) \bigtriangledown (f_L \bigtriangleup h_{RL})] \\ & \vee [(f_R \bigtriangleup g) \bigtriangledown (f_{RL} \bigtriangleup h_L)] \vee [(f_R \bigtriangleup g) \bigtriangledown (f_L \bigtriangleup h_{RL})] \\ & \vee [(f_R \bigtriangleup g) \bigtriangledown (f_{RL} \bigtriangleup h_L)] \vee [(f_R \bigtriangleup g) \bigtriangledown (f_L \bigtriangleup h_{RL})] \\ & \vee [(f \bigtriangleup g_R) \bigtriangledown (f_{RL} \bigtriangleup h_L)] \vee [(f \bigtriangleup g_R) \bigtriangledown (f_L \bigtriangleup h_{RL})] \end{aligned}$$

In short, this theorem shows that the distributive laws do not hold.

4. CONCLUSION

In this paper, we have introduced a new concept of type-2 intuitionistic fuzzy set and their operations. It is hopefully more general and applicable than the ordinary intuitionistic fuzzy set. There would be an overwhelmingly large amount of applications on many different fields that derive from it.

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