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A NOTE ON REGULARIZATION BY LINEAR OPERATORS

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Abstract. The aim of this note is to give an improvement in our results of convergence rates of the regularized solutions for ill-posed operator equations involving monotone operators and in their convergence rates in combination with finite-dimensional approximations of reflexive Banach spaces .

Tóm tắt. Bài này trình bày một cải tiến tốt hơn cho tốc độ hội tụ của nghiệm hiệu chỉnh của bài toán không chính quy với toán tử đơn điệu và sự hội tụ đó cùng với việc xấp xỉ hữu hạn chiều không gian Banach.

1. INTRODUCTION

Let X be a real reflexive Banach space and X^* be dual space of X. For the sake of simplicity norms of X and X^* will be denoted by one symbol $\|.\|$. We write $\langle x^*, x\rangle$ instead of $x^*(x)$ for $x^* \in X^*$ and $x \in X$. Let A be a monotone, continuous and bounded operator with domain of definition $D(A) = X$ and range $R(A) \subseteq X^*$.

We are interested in solving the ill-posed problem

$$A(x) = f, \quad f \in R(A). \tag{1.1}$$

By ill-posedness we mean that the solutions of (1.1) do not depend continuously on the data (A, f). To solve it we have to use stable methods. One of them was shown in [1]: Let B is a linear operator such that

$$\langle Bx, x\rangle \geq m_B\|x\|^2, \ \forall x \in D(B), \ m_B > 0, \ S_0 \subset D(B), \ \overline{D(B)} = X,$$

where S_0 denotes the set of solutions of (1.1), then the regularized equation

$$A_h(x) + \alpha Bx = f_\delta, \tag{1.2}$$

where (A_h, f_δ) are the approximations of (A, f) with the following properties

$$\|A_h(x) - A(x)\| \leq hg(\|x\|), \ \forall x \in X,$$
$$\|f_\delta - f\| \leq \delta, \ h, \delta \to 0,$$

$g(t)$ is a real and nondecreasing function with $g(0) = 0$, $g(t) \to +\infty$, as $t \to +\infty$, and A_h are also monotone, for every $\alpha > 0$, has a unique solution $x_{h\delta}^\alpha$; if h/α, $\delta/\alpha \to 0$, as $\alpha \to 0$, the sequence $\{x_{h\delta}^\alpha\}$ converges to $x_1 \in S_0$,

$$\langle Bx_1, x - x_1\rangle \geq 0, \ \forall x \in S_0;$$

and the solution $x_{h\delta}^\alpha$ of (1.2) can be approximated by solution of the finite-dimensional problem

$$A_h^n(x) + \alpha B^n x = f_\delta^n \tag{1.3}$$

with $A_h^n = P_n^* A_h P_n$, $B^n = P_n^* B P_n$, $f_\delta^n = P_n^* f_\delta$, under the conditions that

$$X_n \subset D(B), \ X_n \subset X_{n+1}, \ B^n x = P_n^* BP_n x \to Bx, \ \forall x \in D(B).$$

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The convergence rates of the sequences $\{x^\alpha_{h\delta}\}$ and $\{x^{\alpha n}_{h\delta}\}$, where $x^{\alpha n}_{h\delta}$ denotes the solution of (1.3), are given by (see [1]).

Theorem 1.1. *Assume that the following conditions hold:*

(i) *A is Fréchet differentiable in some neighbourhood $\mathcal{U}(S_0)$ of S_0.*

(ii) *There exists a constant $L > 0$ such that*

$$\|A'(x) - A'(y)\| \le L\|x - y\|, \ \forall\, x \in S_0, \ y \in \mathcal{U}(S_0).$$

(iii) *There exists an element $v \in D(B)$ such that*

$$A'(x_1)^* v = Bx_1.$$

(iv) $L\|v\| < 2m_B$.

Then, if α is chosen as $\alpha \sim (h+\delta)^\mu$, $0 < \mu < 1$, we obtain

$$\|x^\alpha_{h\delta} - x_1\| = O((h+\delta)^\theta), \ \theta = \min\{1-\mu, \ \mu/2\}.$$

Remark: $\theta_{\max} = 1/3$, when $\mu = 2/3$.

Set

$$\beta_n = \|P_n^* B P_n x_1 - Bx_1\|,$$
$$\gamma_n = \|(I - P_n)x_1\|.$$

Theorem 1.2. *Let the following conditions hold:*

(i) *Conditions* (i) - (iv) *of Theorem 2.1 are fulfilled.*

(ii) *α is chosen as $\alpha \sim (h + \delta + \gamma_n)^\mu + \beta_n$.*

Then

$$\|x^{\alpha n}_{h\delta} - x_1\| = O\Big((h + \delta + \gamma_n)^\theta + \beta_n^{1/2}\Big),$$

where $\theta = \min\{1-\mu, \ \mu/2\}$.

In this note, by using the approach in [2] we can prove that the sequences $\{x^\alpha_{h\delta}\}$ and $\{x^{\alpha n}_{h\delta}\}$ converge with faster rates.

2. RESULTS

Theorem 2.1. *Suppose that the following conditions hold:*

(i) *A is twice-Fréchet differentiable with $\|A''\| \le M$, M is a positive constant.*

(ii) *There exists an element $v \in D(B)$, $Bv \ne 0$, such that*

$$A'(x_1)v = Bx_1.$$

(iii) $M\|v\| < 2m_B$.

Then, if α is chosen such that $\alpha \sim (h+\delta)^\mu$, $0 < \mu < 1$, we have

$$\|x^\alpha_{h\delta} - x_1\| = O((h+\delta)^\theta), \ \theta = \min\{1-\mu, \mu\}.$$

Proof. From (1.1) and (1.2) it follows that

$$A(x^\alpha_{h\delta}) - A(x_1) + \alpha B(x^\alpha_{h\delta} - x_1) = f_\delta - f_0 + A(x^\alpha_{h\delta}) - A_h(x^\alpha_{h\delta}) - \alpha Bx_1.$$

Set

$$P^\alpha_{h\delta} = \int_0^1 A'(x_1 + t(x^\alpha_{h\delta} - x_1))\,dt + \alpha B.$$

It is easy to see that $P^\alpha_{h\delta}$ has the inversion $P^{\alpha(-1)}_{h\delta}$ with $\|P^{\alpha(-1)}_{h\delta}\| \le 1/(m_B\alpha)$. And, we have

$$\|x_{h\delta}^{\alpha} - x_1\| \leq (\delta + hg(\|x_{h\delta}^{\alpha}\|))/(m_B\alpha) + \alpha\|P_{h\delta}^{\alpha(-1)} Bx_1\|.$$

On the other hand,

$$\begin{aligned}
\alpha\|P_{h\delta}^{\alpha(-1)} Bx_1\| &= \alpha\Big[\|P_{h\delta}^{\alpha(-1)}(P_{h\delta}^{\alpha} + A'(x_1) - P_{h\delta}^{\alpha})v\|\Big] \\
&\leq \alpha\|v\| + \alpha\|P_{h\delta}^{\alpha(-1)}(P_{h\delta}^{\alpha} - A'(x_1)v\| \\
&\leq \alpha(\|v\| + \frac{\|Bv\|}{m_B}) + \|(\int_0^1 A'(x_1 + t(x_{h\delta}^{\alpha} - x_1))dt - A'(x_1))v/m_B\| \\
&\leq \alpha(\|v\| + \frac{\|Bv\|}{m_B}) + M\|v\|\|x_{h\delta}^{\alpha} - x_1\|/(2m_B).
\end{aligned}$$

Therefore,

$$\Big(1 - \frac{M\|v\|}{2m_B}\Big)\|x_{h\delta}^{\alpha} - x_1\| \leq (\delta + hg(\|x_{h\delta}^{\alpha}\|))/(m_B\alpha) + \alpha\Big(\|v\| + \frac{\|Bv\|}{m_B}\Big).$$

Consequently,

$$\|x_{h\delta}^{\alpha} - x_1\| \leq O\big((h+\delta)^{\theta}\big).$$

Hence,

$$\|x_{h\delta}^{\alpha} - x_1\| = O((h+\delta)^{\theta}) \ \ \text{(see [2])}.$$

Remark: With $\mu = 1/2$ the parameter θ achieves the maximal value $1/2$.

Set

$$\tilde{\beta}_n = \max\{\beta_n, \|B^n v - Bv\|\}.$$

Theorem 2.2. *Suppose that conditions* (i) - (iii) *of Therem 2.1 hold and* α *is chosen such that* $\alpha \sim (h + \delta + \gamma_n)^{\mu} + \tilde{\beta}_n$, $0 < \mu < 1$. *Then we have*

$$\|x_{h\delta}^{\alpha n} - x_1\| = O\big((h+\delta+\gamma_n)^{\theta} + \tilde{\beta}_n\big), \ \theta = \min\{1-\mu, \mu\}.$$

Proof. First, we estimate the value $\|x_{h\delta}^{\alpha n} - x_1^n\|$, where $x_1^n = P_n x_1$. From (1.1) and (1.3) it implies that

$$\begin{aligned}
A^n(x_{h\delta}^{\alpha n}) - A^n(x_1^n) + \alpha B^n(x_{h\delta}^{\alpha n} - x_1^n) = f_\delta^n - f^n - \alpha B^n x_1^n + P_n^*(A(x_1) - A(x_1^n)) \\
+ A^n(x_{h\delta}^{\alpha n}) - A_h^n(x_{h\delta}^{\alpha n})
\end{aligned}$$

where $A^n = P_n^* A P_n$, and $f^n = P_n^* f$.
Set

$$P_{h\delta}^{\alpha n} = \int_0^1 P_n^* A'(x_1^n + t(x_{h\delta}^{\alpha n} - x_1^n))dt + \alpha B^n.$$

Clearly, the operator $P_{h\delta}^{\alpha n}$ is linear, bounded and monotone from X_n onto X_n^* with $\|P_{h\delta}^{\alpha n(-1)}\| \leq 1/(m_B\alpha)$. Since

$$\|P_{h\delta}^{\alpha n(-1)} P_n^*(f_\delta - f)\| \leq \delta/(m_B\alpha),$$

$$\|P_{h\delta}^{\alpha n(-1)} P_n^*(A_h(x_{h\delta}^{\alpha n}) - A(x_{h\delta}^{\alpha n}))\| \leq \frac{h}{m_B\alpha} g(\|x_{h\delta}^{\alpha n}\|),$$

$$\begin{aligned}
&\|P_{h\delta)}^{\alpha n(-1)} P_n^*(A(x_1) - A(x_1^n))\| \leq \\
&\leq \|P_{h\delta}^{\alpha n(-1)}[A'(x_1)(I - P_n)x_1 + A''(x_1 + \tau(P_n - I)x_1)(I - P_n)x_1(I - P_n)x_1]\| \\
&\leq \gamma_n\|A'(x_1)\|/(m_B\alpha) + \gamma_n^2 M/(2m_B\alpha) \\
&\leq O(\gamma_n/\alpha),
\end{aligned}$$

$$\alpha \| P_{h\delta}^{\alpha n(-1)} B^n x_1^n \| \le \alpha \| P_{h\delta}^{\alpha n(-1)} P_n^* (B^n x_1 - B x_1) + P_{h\delta}^{\alpha n(-1)} P_n^* B x_1 \|$$
$$\le \beta_n / m_B + \alpha \| P_{h\delta}^{\alpha n(-1)} B x_1 \|,$$
$$\alpha \| P_{h\delta}^{\alpha n(-1)} P_n^* B x_1 \| = \alpha \| P_{h\delta}^{\alpha n(-1)} P_n^* (P_{h\delta}^{\alpha n} + A'(x_1) - P_{h\delta}^{\alpha n}) v \|,$$

$$\alpha \|v\| + \frac{M\|v\|\gamma_n}{m_B} + \alpha \| P_{h\delta}^{\alpha n(-1)} P_n^* (P_{h\delta}^{\alpha n} - A'(x_1^n)) v \| \le$$
$$\le \alpha\|v\| + \frac{M\|v\|\gamma_n}{m_B} + \frac{\alpha\tilde{\beta}_n + \alpha\|Bv\|}{m_B} + \alpha \| P_{h\delta}^{\alpha n(-1)} \Big(\int_0^1 A'(x_1^n + t(x_{h\delta}^{\alpha n} - x_1^n)) dt - A'(x_1^n) \Big) v \|$$
$$\le \alpha\|v\| + \frac{M\|v\|\gamma_n}{m_B} + \frac{\alpha(\tilde{\beta}_n + \|Bv\|)}{m_B} + \frac{M\|v\|}{2m_B} \|x_{h\delta}^{\alpha n} - x_1^n\|.$$

Therefore,

$$\|x_{h\delta}^{\alpha n} - x_1^n\| \le O\big((h + \delta + \gamma_n)/\alpha + \alpha + \tilde{\beta}_n + \gamma_n\big).$$

Hence (see [2]),

$$\|x_{h\delta}^{\alpha n} - x_1\| = O\big((h + \delta + \gamma_n)^\theta + \tilde{\beta}_n\big).$$

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