Tạp chí Tin học và Điều khiển học, T. 17, S. 1 (2001), 10-16

DIFFERENCE SCHEMES OF GENERALIZED SOLUTIONS FOR A CLASS OF ELLIPTIC NON-LINEAR DIFFERENTIAL EQUATIONS

HOANG DINH DUNG

Abstract. It is known (see [1], [2], etc.) that in many applied problems the data are nonregular. The approximate methods for the problems of nonlinear differential equations with data belonging the Sobolev spaces $W_p^l(G)$ are presented in [3-5]. In this paper the finite-difference schemes of generalized solutions for a class of elliptic nonlinear differential equations are considered. The theorem for the convergence of approximate solution to generalized one and error norm estimations is proved in the class of equations with the right-hand side defined by a continuous linear functional in $W_2^{(-l)}(G)$.

Tóm tắt. Nhiều bài toán thực tiễn được dẫn về giải các bài toán đối với phương trình vi phân riêng với dữ kiện không trơn (xem [10], [2]). Phương pháp xấp xỉ giải một số bài toán đối với các phương trình vi phân phi tuyến với vế phải thuộc các lớp hàm khả tích khác nhau (các không gian Sobolev $W_p^l(G)$) được nghiên cứu trong các công trình [3-5]. Bài này xét lược đồ sai phân, nghiên cứu sự hội tụ và đánh giá sai số của nghiệm bài toán đối với một lớp phương trình vi phân phi tuyến loại ellip với vế phải không trơn độ cao kiểu các phiếm hàm tuyến tính liên tục (các không gian $W_2^{(-l)}(G)$).

1. INTRODUCTION

Let G be a rectangle with the boundary ∂G. Consider the following problem

$$\Delta u + T\Big(x, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\Big) = -f(x), x \in \partial,\ \ u(x) = 0,\ x \in \partial G, \tag{1}$$

where the given $f(x) \in W_2^{-l}(G)$ - the space of continuous linear functionals on the space $\overset{\circ}{W}{}_2^l(G), l$ being a nonegative integer, the function $T(x,a)$, $a = (a_0, a_1, a_2)$, satisfies the conditions:

$$\begin{aligned} &\big[T(x,a) - T(x,b)\big](a_0 - b_0) \geq C_1 \sum_{i=0}^{2} (a_i - b_i)^2, \\ &|T(x,a) - T(x,b)| \leq C_2 \Big[\sum_{i=0}^{2} (a_i - b_i)^2\Big]^{1/2}, \end{aligned} \tag{2}$$

where C_j, $j = 1, 2$, are the positive constants (see [3, chap. 3, sec. 4]).

We shall use the same notations as in [6]. Consider the generalized solution $u(x)$ of the problem (1) in the space $\overset{\circ}{W}{}_2^1(G)$ satisfying the following equality:

$$P(u,v) = \iint\limits_G \Big[\Delta u + T\big(x, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\big)\Big] v(x)dx = -\iint\limits_G f(x)v(x)dx, \tag{3}$$

where $v(x)$ is a function in the space $\mathcal{D}(G)$ of Schwartz basic functions [7].

One has $v(x) \in \overset{\circ}{W}{}_2^1(G)$. Then, by [3] (chap. 3, sec. 4), if the function $T\big(x, u, \dfrac{\partial u}{\partial x_1} \dfrac{\partial u}{\partial x_2}\big)$ satisfies the conditions (2), $f(x) \in L_2(G)$, there exists uniquely a solution of integral equation (3) $u(x) \in \overset{\circ}{W}{}_2^1(G) \cap W_2^2(G)$.

* This work is partially supported by the National Basics Research Program in Natural Sciences

2. CONSTRUCTION OF DIFFERENCE SCHEMES

We first consider the case where $f(x) \in L_2(G)$ and let G be the unit square $G = \{x = (x_1, x_2) : 0 < x_n < 1,\ n = 1, 2\}$.

Let us introduce in the region G a grid $\overline{\omega}$ with interior and boundary gridpoints denoted by ω and γ respectively [6].

To construct the difference schemes one may take the test functions $v(x)$ in the form:

$$v(x) = \begin{cases} \frac{1}{4\pi h_1^k h_2^k} \exp\left\{ -\frac{|x|^2}{4 h_1^k h_2^k} \right\}, & x \in e, \\ 0, & x \in \overline{G} \setminus e, \end{cases} \tag{4}$$

where $e = e(x) \equiv \{\varsigma = (\varsigma_1, \varsigma_2) : |\varsigma_n - x_n| < 0,5h_n,\ n = 1, 2\}, h_n$ being the steplengths, k being a natural number.

Let every gridpoint $x \in \omega$ be corresponding to a mesh $e(x)$. The generalized solution (denoted by the GS) $u(x)$ of the problem (1) in e satisfies the following integral equation:

$$P^e(u, \alpha) = \frac{1}{h_1 h_2} \int\limits_{x_1 - 0,5h_1}^{x_1 + 0,5h_1} \int\limits_{x_2 - 0,5h_2}^{x_2 + 0,5h_2} \left[\Delta u(\varsigma) + T\left(\varsigma, u, u(\varsigma), \frac{\partial u}{\partial \varsigma_1}, \frac{\partial u}{\partial \varsigma_2}\right) \right] \alpha(\varsigma) d\varsigma$$
$$= -Rf, \quad x \in \omega, \tag{5}$$

where $\alpha(\varsigma) \equiv h_1 h_2 v(\varsigma)$,

$$Rf \equiv \frac{1}{h_1 h_2} \iint\limits_{e} f(\varsigma) \alpha(\varsigma) d\varsigma. \tag{6}$$

One may rewrite the equation (5) as follows

$$P^e(u, \alpha) = \sum_{i=1}^{2} \frac{1}{h_i} S_{3-i} \left[\left(\alpha \frac{\partial u}{\partial x_i}\right)^{(+0,5_i)} - \left(\alpha \frac{\partial u}{\partial x_i}\right)^{(-0,5_i)} \right] - S_1 S_2 \sum_{i=1}^{2} \frac{\partial \alpha}{\partial \varsigma_i} \frac{\partial u}{\partial \varsigma_i}$$
$$+ S_1 S_2 \left[\alpha(\varsigma) T\left(\varsigma, u(\varsigma), \frac{\partial u}{\partial \varsigma_1}, \frac{\partial u}{\partial \varsigma_2}\right) \right] = -Rf, \quad x \in \omega, \tag{7}$$

where

$$S_i u(x) = \frac{1}{h_i} \int\limits_{x_1 - 0,5h_i}^{x_i + 0,5h_i} u(x_1, \dots, \varsigma_i, \dots, x_n) d\varsigma_i, \quad u^{(\pm 0,5_i)}(x) = u(x_1, \dots, x_i \pm 0,5h_i, \dots, x_n).$$

Now, to obtain the difference schemes of the operator (7) $P^e(u, \alpha)$ one may approximate the mean integral operators S_i by the quadrature formula of average rectangles and the partial derivatives by difference quotients as in [6] (see 2.1). Hence, one get the following difference approximations corresponding to (7), (3):

$$K(\check{y}) \equiv {}^1P_1^e(\check{y}, \alpha) = \sum_{i=1}^{2} \left(\alpha_i \check{y}_{\overline{x}_i}\right)_{x_i} - S_1 S_2 \sum_{i=1}^{2} \alpha_{\overline{x}_i}(x) \check{y}_{\overline{x}_i} + S_1 S_2 \alpha(\varsigma) T\left(\varsigma, \check{y}(x), \check{y}_{\overline{x}_1}, \check{y}_{\overline{x}_2}\right) = -\varphi, \quad x \in \omega,$$
$$\check{y}(x) = 0, \quad x \in \gamma, \tag{8}$$

and (cf. [3, chap. 3, sec. 4])

$$L(\widehat{y}) \equiv {}^2P_h^e(\widehat{y}, \alpha) = \sum_{i=1}^{2} \widehat{y}_{\overline{x}_i x_i} + S_1 S_2 \alpha(\varsigma) T\left(\varsigma, \widehat{y}(x), \widehat{y}_{\overline{x}_1}, \widehat{y}_{\overline{x}_2}\right) = -\varphi, \quad x \in \omega,$$
$$\widehat{y}(x) = 0, \quad x \in \gamma, \tag{9}$$

where

$$u_{x_i} = \frac{1}{h_i}\big[u^{(+1_i)} - u\big],\ u_{\bar{x}_i} = \frac{1}{h_i}\big[u - u^{(-1_i)}\big],$$
$$u^{(\pm 1_i)} \equiv u^{(\pm 1_i)}(x) = u(x_1, ..., x_i \pm h_i, ..., x_n),\ i \geq 1,$$
$$\alpha_i = \alpha^{(-0,5_i)}(x),\ \varphi = Rf. \tag{10}$$

Note that by [3] (see chap. 3, sec. 4) there exists uniquely a solution of the operator equation ${}^2P_h^e(y, \alpha) = -\varphi$ and, then, of the equation ${}^1P_h^e(y, \alpha)$.

3. ESTIMATION OF THE CONVERGENCE RATE

Estimate now the method error and the approximate one of the scheme (8) and (9).

3.1. Consider the difference scheme (9) with φ defined by (10), (7). Denote the method error by $z = \widehat{y} - u$, where $\widehat{y}$ being the solution of the problem (9). It follows from (9) that.

$$Lz = -\psi(x), x \in \omega;\ z(x) = 0,\ x \in \gamma, \tag{11}$$

where $\psi(x)$ is the approximation error of the scheme (9):

$$\Psi(x) = \varphi + Lu.$$

From (10), (7) and by formulas (10), (11) in [6, sec. 2], for the sufficiently small mesh sizes h_1 and h_2, one has

$$\varphi = -\sum_{i=1}^{2}\Big[S_{3-i}\big(\alpha\frac{\partial u}{\partial x_i}\big)^{-0,5_i)}\Big]_{x_i} + S_1S_2\Big(\sum_{i=1}^{2}\frac{\partial \alpha}{\partial \varsigma_i}\frac{\partial u}{\partial \varsigma_i}\Big)$$
$$- S_1S_2T\Big(\varsigma, u(\varsigma), \frac{\partial u}{\partial \varsigma_1}, \frac{\partial u}{\partial \varsigma_2}\Big),\quad x \in \omega. \tag{12}$$

Thus,

$$\Psi = \sum_{i=1}^{2}\Big[u_{\bar{x}_i} - S_{3-i}\big(\alpha\frac{\partial u}{\partial x_i}\big)^{-0,5_i)}\Big]_{x_i} + S_1S_2\sum_{i=1}^{2}\frac{\partial \alpha}{\partial \varsigma_1}\frac{\partial u}{\partial \varsigma_i}$$
$$- S_1S_2\Big[T\big(\varsigma, u(\varsigma), \frac{\partial u}{\partial \varsigma_1}, \frac{\partial u}{\partial \varsigma_2}\big) - T\big(\varsigma, u(x), u_{\bar{x}_1}(x),\ u_{\bar{x}_2}(x)\big)\Big].$$

By (9) one has

$$L_0\widehat{y} \equiv \sum_{i=1}^{2}\widehat{y}_{\bar{x}_i x_i} = -S_1S_2\Big[T\big(\varsigma, \widehat{y}(x), \widehat{y}_{\bar{x}_1}, \widehat{y}_{\bar{x}_2}\big)\Big] - \varphi \equiv \varphi_0,\ x \in \omega. \tag{13}$$

Then,

$$L_0x = L_0\widehat{y} - L_0u \equiv -\Psi_0(x),\ x \in \omega;\ \ z(x) = 0,\ x \in \gamma. \tag{14}$$

From (12) - (14) it follows that

$$\Psi_0 = L_0u - \varphi_0 = \sum_{i=1}^{2}u_{\bar{x}_i x_i} - \sum_{i=1}^{2}\Big[S_{3-i}\big(\alpha\frac{\partial u}{\partial x_i}\big)^{(-0,5_i)}\Big]_{x_i} + S_1S_2\sum_{i=1}^{2}\frac{\partial \alpha}{\partial \varsigma_i}\frac{\partial u}{\partial \varsigma_i}$$
$$+ S_1S_2T\big(\varsigma, \widehat{y}(x), \widehat{y}_{\bar{x}_1}, \widehat{y}_{\bar{x}_2}\big) - S_1S_2T\Big(\varsigma, u(\varsigma), \frac{\partial u}{\partial \varsigma_1}, \frac{\partial u}{\partial \varsigma_2}\Big).$$

Hence,

$$-L_0z = -\sum_{i=1}^{2}z_{\bar{x}_i x_i} = \sum_{i=1}^{2}(\eta_i)_{x_i} + \lambda_0 + \beta_0,\ x \in \omega, \tag{15}$$

where

$$\eta_i = u_{x_i} - S_{3-i}\big(\alpha\frac{\partial u}{\partial x_i}\big)^{(-0,5_i)},$$

$$\lambda_0 = S_1 S_2 \sum_{i=1}^{2} \frac{\partial \alpha}{\partial \varsigma_i}\frac{\partial u}{\partial \varsigma_i}, \tag{16}$$

$$\beta_0 = -S_1 S_2\Big[T\big(\varsigma, u(\varsigma), \frac{\partial u}{\partial \varsigma_1}, \frac{\partial u}{\partial \varsigma_2}\big) - T\big(\varsigma, u(x), u_{\bar{x}_1}, u_{\bar{x}_2}\big)\Big]$$

$$- S_1 S_2\Big[T\big(\varsigma, u(\varsigma), u_{\bar{x}_1}, u_{\bar{x}_2}\big) - T\big(\varsigma, \widehat{y}(x), \widehat{y}_{\bar{x}_1}, \widehat{y}_{\bar{x}_2}\big)\Big] \equiv \beta_0^1 + \beta_0^2.$$

Now, to obtain a priori estimation, let us scalar multiply both sides of (15) by $z(x)$:

$$-\sum_{i=1}^{2}\big(z_{\bar{x}_i x_i}, z\big) = \sum_{i=1}^{2}\big(\eta_{i_{x_i}}, z\big) + (\lambda_0, z) + (\beta_0, z),$$

where (a, b) is the scalar product on the set of net functions:

$$(a, b) = \sum_{x\in\omega} a(x) b(x) h_1 h_2.$$

Since $z(x) = 0$ for $x \in \gamma$, one has

$$\sum_{i=1}^{2} \|z_{\bar{x}_i}\|_i^2 \equiv \|\nabla z\|_{0,\omega}^2 \le \sum_{i=1}^{2} \|\eta_i\|_i \, \|z_{\bar{x}_i}\|_i + \big(\|\lambda_0\| + \|\beta_0\|\big)\|z\|, \tag{17}$$

where

$$\|z_{\bar{x}_i}\|_i^2 \equiv \big(z_{\bar{x}_i}, z_{\bar{x}_i}\big]_i,$$

$$(a, z]_1 = \sum_{y_1=1}^{N_1}\sum_{j_2=1}^{N_2-1} a(j_1 h_1, j_2 h_2) z(j_1 h_1, j_2 h_2) h_1 h_2,$$

$$(a, z]_2 = \sum_{y_1=1}^{N_1-1}\sum_{j_2=1}^{N_2} a(j_1 h_1, j_2 h_2) z(j_1 h_1, j_2 h_2) h_1 h_2,$$

$$\|a\|^2 = \sum_{y_1=1}^{N_1}\sum_{j_2=1}^{N_2} a^2(j_1 h_1, j_2 h_2) h_1 h_2.$$

Then

$$\|z\|_{1,\omega} \le C\big(\|\eta_1\|_1 + \|\eta_2\|_2 + \|\lambda_0\| + \|\beta_0\|\big), \tag{18}$$

where the constant C is independent of h ($|h|^2 = h_1^2 + h_2^2$) and $z(x)$,

$$\|z\|_{1,\omega}^2 \equiv \|z\|_{0,\omega}^2 + \|\nabla z\|^2, \|z\|_{0,\omega} \equiv \|z\|.$$

Now, we first consider the functional $\eta_1(x)$ defined by (16):

$$\eta_1(x) = u_{x_1} - \frac{1}{h_2}\int_{x_2-0,5h_2}^{x_2+0,5h_2} \alpha(x_1 - 0,5h_1, \varsigma_2)\frac{\partial u}{\partial \varsigma_1}(x_1 - 0,5h_1, \varsigma_2)d\varsigma_2.$$

This expression coincides with the one of $\eta_1(x)$ (19) in [6]. Hence, by (23) in [6] we have

$$|\eta_1(x)| \le M|h|(h_1h_2)^{-\frac{1}{2}}\|u\|_{2,e^1},$$

where e^1 is the following mesh of the grid ω:

$$e^i = e^i(x) \equiv \{\varsigma = (\varsigma_1, \varsigma_2) \,:\, x_i - h_i < \varsigma_i < x_i, |\varsigma_{3-i} - x_{3-i}| < 0,5h_{3-i}\},$$

$$\|u\|_{m,e^1} \equiv \|u\|_{W_2^m(e^1)} = \Big(\sum_{|\alpha|\le m}\int_{e^1} |D^\alpha u|^2 dx\Big)^{1/2}.$$

The functional $\eta_2(x)$ is estimated similarly. Then,

$$\|\eta_i\|_i \le C|h|\big(\sum_x \|u\|^2_{2,e^i}\big)^{1/2} \le C|h|\ \|u\|_{2,G}. \tag{19}$$

The expression of λ_0 coincides with the one of η_0 (15) in [6]. Then, by (26) in [6] we have

$$\|\lambda_0\| \le C|h|\ |H(h)|\ \|u\|_{1,G}, \tag{20}$$

where $H(h) \to 0$ as $h_1, h_2 \to 0$.

Consider now β_0^1 in (18). The difference of the form β_0^1 is estimated in [3] (see chap. 3, sec. 4), one has

$$\|\beta_0^1\| \le C|h|\ \|u\|_{2,G}.$$

From the last inequality and (16) it follows that

$$\|\beta_0\| \le C|h|\ \|u\|_{2,G}. \tag{21}$$

Finally, combining (18) - (21) we get

$$\|z\|_{1,\omega} = \|\widehat{y} - u\|_{1,\omega} \le C|h|\ \|u\|_{2,G}. \tag{22}$$

3.2. Consider the following difference scheme

$$My = \frac{1}{2}(K + L)y = -\varphi,\ x \in \omega;\ \ y(x) = 0,\ x \in \gamma, \tag{23}$$

where $y = \dfrac{1}{2}(\breve{y} + \widehat{y})$, $\breve{y}$ and $\widehat{y}$ are defined (8) and (9) respectively. Then,

$$\begin{aligned} My &= \frac{1}{2}\sum_{i=1}^{2}\big[(1+\alpha_i)y_{\overline{x}_i}\big]_{x_i} - \frac{1}{2}S_1S_2\sum_{i=1}^{2}\alpha_{\overline{x}_i}y_{\overline{x}_i} + \\ &\quad + \frac{1}{2}S_1S_2\big[\alpha(\varsigma)T(\varsigma, y(x), y_{\overline{x}_1}, y_{\overline{x}_2}) + T(\varsigma, y(x), y_{\overline{x}_1}, y_{\overline{x}_2})\big] \\ &= -\varphi,\ x \in \omega, \\ y(x) &= 0,\ x \in \gamma. \end{aligned}$$

Thus,

$$\begin{aligned} M_0y &\equiv \sum_{i=1}^{2}\big[(1+\alpha_i)y_{\overline{x}_i}\big]_{x_i} \\ &= S_1S_2\sum_{i=1}^{2}\alpha_{\overline{x}_i}y_{\overline{x}_i} - S_1S_2\big[\alpha T(\varsigma, y(x), y_{\overline{x}_1}, y_{\overline{x}_2}) + T(\varsigma, y(x), y_{\overline{x}_1}, y_{\overline{x}_2})\big] - 2\varphi \\ &\equiv \varphi_0,\ x \in \omega, \\ y(x) &= 0,\ x \in \gamma. \end{aligned}$$

From the last equality, (7) and (12) one has

$$M_0z = \varphi_0 - M_0u = -\Psi(x),\ x \in \omega;\quad z(x) = 0,\ x \in \gamma \tag{24}$$

where $z = y - u$ is the method error,

$$
\begin{aligned}
\Psi(x) &= -\sum_{i=1}^{2}\left[(1+\alpha_i)z_{\bar{x}_i}\right]_{x_i} = \sum_{i=1}^{2}[\eta_i+\mu_i]_{x_i} + \lambda_0 + \beta_0 + q_0,\\
\eta_i &= u_{x_i} - S_{3-i}\Big(\alpha\frac{\partial u}{\partial x_i}\Big)^{(-0,5_i)}, \mu_i = \alpha_i = \alpha_i u_{x_i} - S_{3-i}\Big(\alpha\frac{\partial u}{\partial x_i}\Big)^{(-0,5_i)},\\
\lambda_0 &= S_1S_2\sum_{i=1}^{2}\Big[2\frac{\partial\alpha}{\partial\varsigma_i}\frac{\partial u}{\partial\varsigma_i} - \alpha_{\bar{x}_i}y_{\bar{x}_i}\Big],\\
\beta_0 &= -\,S_1S_2\Big[T\big(\varsigma, u(\varsigma), \frac{\partial u}{\partial\varsigma_1}, \frac{\partial u}{\partial\varsigma_2}\big) - T(\varsigma, y, y_{\bar{x}_1}, y_{\bar{x}_2})\Big],\\
q_0 &= -\,S_1S_2\alpha(\varsigma)\Big[T\big(\varsigma, u(\varsigma), \frac{\partial u}{\partial\varsigma_1}, \frac{\partial u}{\partial\varsigma_2}\big) - T(\varsigma, y, y_{\bar{x}_1}, y_{\bar{x}_2})\Big].
\end{aligned}
\tag{25}
$$

By (24), (25), in the same way as in 3.1 one has

$$\|z\|_{1,\omega} \le C\Big(\sum_{i=1}^{2}(\|\eta_i\|_i + \|\mu_i\|_i) + \|\lambda_0\| + \|\beta_0\| + \|q_0\|\Big). \tag{26}$$

In (26) η_i has the form (16), then one has the estimation (19) for η_i.

The expression of μ_i coincides with the one of γ_i (31) in [6], then by (39) in [6] one has

$$\|\mu_i\| \le C|h|^{m-1}|H_i(h)|\,\|u\|_{m,G},\quad m=2,\,3, \tag{27}$$

where $H_i(h)$, $i=1,\,2$ tend to zero as $h\to 0$.

λ_0 has the form (33) of $\tilde{\gamma}_0$ in [6], then by (44) in [6],

$$\|\lambda_0\| \le C|h|\,|H(h)|\,\|u\|_{2,G}. \tag{28}$$

β_0 has the form (16), then by (21) one has

$$\|\beta_0\| \le C|h|\,\|u\|_{2,G}. \tag{29}$$

Consider the last summand q_0 in (26). The form of q_0 is analogous to β_0 and one may easily verify that

$$\|q_0\| \le C|h|\,|H(h)|\,\|u\|_{2,G}. \tag{30}$$

Now, combining (19), (2) - (30) yields

$$\|\check{y} + \hat{y} - 2u\|_{1,\omega} \le C|h|^{m-1}\|u\|_{m,G},\quad m=2,\,3. \tag{31}$$

Finally, by (22) and (31) we get the estimation of method error for the difference scheme (8):

$$\|\check{y} - u\|_{1,\omega} \le C|h|^{m-1}\|u\|_{m,G},\quad m=2,\,3. \tag{32}$$

Remark. In a manner analogous to the proof of the inequalities (22) and (32), one may verify that these inequalities are also valid if in the formula of the GS $u(x)$ (5), (7), $v(x)$ $\big(=\dfrac{\alpha(x)}{h_1h_2}\big)$ is a Schwartz basic function.

3.3. The estimates (22) and (32) are obtained with the assumption $f\in L_2(G)$, now we show that the results may be generalized to the equations with right-hand side $f\in W_2^{(-l)}(G)$, $W_2^{(-l)}(G)$ being the space of continuous linear functionals on the space $\overset{\circ}{W}{}_2^l(G)$, l is a nonegative integer. For example, f is the Dirac delta function δ.

Indeed, by our assumption, $f(x)\in\mathcal{D}'(G)$, $\mathcal{D}'(G)$ being the space of Schwartz distributions. Therefore, by the theorem on local structure of the distributions (see [7, chap. 3, sec. 6]) there exists a function $g(x)\in L_\infty(e)$ and an integer $k\ge 0$ such that

$$f(x) = D_1^k ... D_n^k g(x), \tag{33}$$

where $x \in e$, the set e is compact in $G \in R^n$, $D_i \equiv \partial / \partial x_i$.

Let $v(x) \in \mathcal{D}(e)$, By (5) and (33) one has

$$\iint_e \Big[\Delta u(x) + T(x, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2})\Big] v(x) dx = - \iint_e g(x) \tilde{v}(x) dx, \tag{34}$$

where

$$\tilde{v}(x) = D_1^k D_2^k v(x) (n = 2).$$

We see that $\tilde{v}(x)$ is also a test function: $\tilde{v}(x) \in \mathcal{D}(e) \subset \overset{\circ}{W}{}_2^l(e)$ and $g(x) \in L_2(e)$. Thus, the equation (34) has the form (5). Hence, one may repeat the procedure used above for the difference schemes (8), (9) and obtaines the following.

Theorem. *Let in the problem* (1) *the function* $T(\,.\,)$ *satisfy the conditions* (2) *and the right-hand side* $f \in W_2^{(-l)}(G)$. *Then the solution* y *of the difference scheme* (8) *or* (9) ($y = \check{y}$ *or* $\hat{y}$) *converges to the GS* (5) $u(x)$ *of the problem* (1) *in the grid norm* $W_2^1(\omega)$ *with the rate* $O(|h|)$, *that is, one has the following error estimation*

$$\|y - u\|_{1,\omega} \leq C|h| \, \|u\|_{2,G},$$

where the constant C *is independent of* h *and* $u(x)$.

REFERENCES

[1] G. I. Marchuk, *Mathematical Modelling in the Environment Problems*, Nauka, Moscow, 1982 (Russian).

[2] V. S. Vladimirov, *Generalized Functions in Mathematical Physics*, Mir, Moscow, 1979.

[3] A. A. Samarskii, R. D. Lazarov, V. I. Makarov, *Difference Schemes for Generalized Solutions of Differential Equations*, Vus. Univ., Moscow, 1987.

[4] C. Padra, A posterior error estimators for nonconforming approximation of some quasi-Newtonian flows, *SIAM J. Numer. Anal.* **34** (4) (1997) 1600-1615.

[5] C. N. Davson, M. F. Wheeler, C. S. Woodward, A two-grid finite difference scheme for non-linear parabolic equations, *SIAM J. Numer. Anal.* **35** (2) (1998) 435-452.

[6] Hoang Dinh Dung, Difference schemes for generalized solutions of some elliptic differential equations, I, *Journal of Computer Science and Cybernetics* **15** (1) (1999) 49-61.

[7] L. Schwartz, *Théorie des Distributions*, Hermann, Paris, 1978.

Received March 20, 2000
Revised January 5, 2001

Institute of Mathematics, NCST of Vietnam