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# ON THE RELATIONAL DEPENDENCY COALITIONAL GAMES

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**Abstract.** Cooperating game theory is becoming increasingly popular in AI, data science, and game theory applications in sharing and circular economy. Social media shows us the impact of many influencers on millions (even hundreds of millions) of followers, which raised the need to have a new model of coalition game, in which the influence or dependence of players on others are not equal, some have more than others. In this paper, we introduce a relational dependency game with new properties of minimal winning coalitions and maximal losing coalitions and their in-depth relationship with different approaches from simple games. In this new model, unlike simple games, all winning coalitions have the same payoff but losing coalitions have different payoffs, which coincides with Leo Tolstoy's philosophy: all happy families are alike, but each unhappy family is unhappy in its way. The algorithm to find a minimal winning coalition among maximal losing coalitions is addressed in this paper. In this new model, unlike a simple game, we present the relational dependency coalition game model in which players depend on or do not depend on one another when they share a common interest in achieving a specific goal or outcome. The players must find a minimal winning coalition on which all players of the game depend on achieving the highest payoff. Closure operations and choice functions arise naturally in this game when there is a one-to-one correspondence between the winning coalition/losing coalition and the closure operation/choice function. And the game becomes more complex when relational independence lives with dependency among players. How to have a structural representation of relational independence along with dependency and how to describe a minimal winning collation on a simple hypergraph is also addressed in the paper.

**Keywords.** Relational dependency coalitional games, minimal winning coalition, maximal losing coalition, closure transversal, choice transversal, anti-transversal, economics.

## 1. INTRODUCTION

With increased competition over natural resources and environmental amenities, decision-makers face strategic decisions in various management and use aspects. Since founded in 1944

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by John von Neumann and Morgenstein [1], coalitional games had ups and downs in research interests. John Nash in 1953 argued that solutions to cooperative games should always be verified by showing that they are also solutions to formally equivalent noncooperative games [2]. One way of interpreting this was as demonstrating the ultimate redundancy of cooperative game theory. Since then, cooperative game theory or coalitional games has been understudied for long. However, not all things are for sale or competition, just as universities don't auction off admission places to the highest bidder in general, and transplant organs are not for sale legally. The coalitional game models become so important in our society, and they demonstrably have improved efficiency and saved lives.

Coalition formation, which has been widely studied and researched in game theory and economics in [3], now attracts much more attention in AI and data science as a means of forming coalitions of autonomous selfish players that need to work together to perform certain jobs in [4] and [5]. Game theory with coalitional games becomes a very powerful means to study some novel concepts like the sharing economy as a new wave of integration of economy and technology in research. Game theory continues to be one of the 20th-century inventions that are driving social revolutions in the 21st, as Larry Samuelson in 2016 [6] predicts a coming surge of renewed interest in the deeper mathematics of cooperative games. Following that trend of deeper mathematics of coalitional games, our introduction of the Relational Dependency Coalitional Game (RDCG) model is a marriage between a relational database model with coalitional games built based on relational dependencies among players. No work before us merges coalitional games with the idea of functional dependencies in the relational database model proposed by E. F. Codd. The major benefit of putting relational dependencies into a coalition game is its inheritance of a massive knowledge of relational models with powerful combinatorics tools like keys/anti-keys as well as hypergraph tools like transversal/anti-transversal, and graph theory of strongly connected/Hamiltonian graph, which in turn could help us more powerful quantifying tools to find new properties under both certain and uncertain conditions as well as in the world of complete and incomplete information. Also, by using relational dependencies we could easily apply Pawlak's rough set on decision table [7] that each player faces when forming coalitions under uncertainty. By integrating coalition games with a relational database model, we introduce relational dependencies into cooperative games, which have yielded results that could help us understand deeper the value of relational dependencies, founded by E. F. Codd, applied in coalition games.

Cooperative games, with grand coalition, coalition structure containing disjoint coalitions, and overlapping coalitions. The relational dependency (RD) coalition model could be used as an underlying model with the capability to describe all three kinds of coalition above. Besides that, to reduce the algorithm complexity, using data along with upper and lower boundary are two major ways. Introducing relational dependencies into coalition games is a direct way of reducing algorithm complexity, and by investigating properties of minimal winning and maximal losing coalitions, we could add more upper and lower boundary properties to support further. In this study, we keep deterministic assumptions in the world of complete information:

- First, it assumes that the payoff to each coalition is given by a fixed and deterministic value.

- Secondly, it assumes that these values are common knowledge among all agents.

However, in future papers, we will replace those two assumptions by: Each player has private information that other players don't know, which leads to uncertainty of the payoff to each coalition, which in turn leads to payoff uncertainty for each player depends on the honesty of other players. The type of each person is the network of relations or the number of relational dependencies of other players on that person. Thus, agent type is a key factor determining the quality of a potential coalition. The higher the number of relational dependencies the player has, the more contribution he could make to the coalition, which in turn decreases the number of players needed to form a minimal winning coalition. In a incomplete information world, that type of the private information is only known to that player and is unknown to all others. This could lead to many new results in studying coalitional games in uncertainty.

In this paper, we focus our studies on the concept and properties of minimal winning coalitions based on the concept and properties of minimal keys. Similarly, we study the concept and properties of maximal losing coalitions is based on the concept and properties of maximal losing coalitions. Based on our massive knowledge of the relational database model, we inherit the algorithm for finding a minimal winning coalition. Also, linking with the hypergraphs, we introduced a new concept of anti-transversal and other properties besides describing a minimal winning coalition on a hypergraph.

## 2. PRELIMINARIES

This section presents some major concepts related to closure operations/choice functions and maximal losing coalition/minimal winning coalition in the relational dependency coalition game model, see [8–11] for details. First, we define our relational dependency game as follows:

Let $U = \{a_1, ..., a_n\}$ be a nonempty finite set of n players. A relational dependency (RD) is a statement of the form $A \to B$, where $A, B \subseteq U$, means B depends on A. For example, a group of doctors B depends on doctor $\{a_i\}$ due to his superior expertise and acting as a teacher to group B. And if $A \nrightarrow B$ we say B has no relational dependence on A.

Given a family $F$ of RDs over $U$, there exists a unique minimal f-family $F^+$ that contains $F$. It can be seen that $F^+$ contains all RDs which can be derived from $F$ by following axioms based on Armstrong axioms:

(F0) $X \to X \in F$: Group of doctors $X$ depends on themselves.

(F1) If $X \to Y \in F$ and $Y \to Z \in F$ then $X \to Z \in F$: If doctor group Y depends on group $X$ and doctor group $Z$ depends on group $Y$, then doctor group $Z$ depends on group $X$.

(F2) If $X \to Y \in F$ and $X \subseteq V$, $W \subseteq Y$ then $V \to W \in F$ : If doctor group $Y$ depends on group $X$ and doctor group $X \subseteq$ group $V$ and doctor group $W \subseteq$ group $Y$, then doctor group $W$ depends on group $V$.

(F3) If $X \to Y \in F$ and $V \to W$ then $X \cup V \to Y \cup W \in F$.

If doctor group $Y$ depends on group $X$ and doctor group $W$ depends on group $V$, then doctor group $Y \cup W$ depends on group $X \cup V$.

A family of RDs satisfying Armstrong's axioms is called an f-family over $U$.

**Definition 1.** Let $n \geq 2$ as the number of players in a game, $U = \{a_1, ...a_n\}$ be a nonempty finite set of n players. A coalition, s is defined as a subset of $U$, and the $P(U)$ is the set of all coalitions. We take an empty set as a coalition too and call it an empty coalition. The game with all players of $U$ is called a grand coalition.

**Examples.** If $n = 2$, which means there are two players then $P(U) = \{\emptyset, \{a_1\}, \{a_2\}, U\}$. If $n = 3$, then all subsets of $U$ are $2^{|U|} = 2^3 = 8$ we have the set of all coalitions as follows: $P(U) = \{\emptyset, \{a_1\}, \{a_2\}, \{a_3\}, \{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_1\}, U\}$. For $n$ players, we have $P(U) = 2^n$.

**Definition 2.** Coalition form and characteristics functions: $A$ coalition schema $S = \langle U, F, v\rangle$ (and a coalition game $r = \langle G, F, v\rangle$ with $G \subseteq U$) is defined as follows $F$ is a set of RDs over $U$, and a real-valued characteristic function $v$: $P(U) \to \mathbf{R}$ (and $v$:$P(G) \to \mathbf{R}$ for $r$) from the set of all possible coalitions of players of $U$ (and G respectively) to a set of payments that satisfies $v(\emptyset) = 0$. The quantitative $v(A)$ is a real number for each coalition $A \subset U$, which may be considered as the value (or worth, or power) of coalition $A$ when its members work together as a team.

**Definition 3.** Define $w_k$, a coalition of $k$ players in a game schema $S$ over $U$, as a winning coalition if all players of $U$ have relational dependencies on $k$, which implies $v(w_k) = v(U)$. That means all winning coalitions have the same payoffs as a grand coalition of all players. That definition of winning coalitions has a deeper connection with closure operation, explained further below in this section. A losing coalition $l_k$ is a coalition when all players of $U$ do not have relational dependencies on $k$.

The definitions above align with Leo Tolstoy's philosophy in his famous Anna Karenina: all happy families are the same, but each unhappy family is unhappy in its way.

Denote $A^+ = \{a | A \to \{a\} \in F^+\}$. $A^+$ is called the closure of $A$ over $U$. It is clear that $A \to B \in F^+$ if and only if $B \subseteq A^+$. Clearly, if coalition schema $S = \langle U, F, v\rangle$, then there is a game $r$ over $U$ such that $F_r = F^+$. Let $U$ be a nonempty finite set of players and $P(U)$ its power set, containing all subsets of $U$. A map $L : P(U) \to P(U)$ is called a closure over $U$ if it satisfies the following conditions:

(1) $A \subseteq L(A)$ (Extensiveness Property)

(2) $A \subseteq B$ implies $L(A) \subseteq L(B)$ (Monotonicity Property)

(3) $LL((A)) = L(A)$ (Closure Property)

Set $L(A) = \{a | A \to a \in F^+\}$, we can see that $L$ is a closure over $U$.

**Lemma 1.** *If F is a f-family and if* $L_F = \{a | a \in U and A \to \{a\} \in F\}$*, then* $L_F$ *is a closure. Inversely, if L is a closure, there exists only a f-family F over U such that* $L = L_F$ *and* $F = \{A \to B | A, B \subseteq U, B \subseteq L(A)\}$.

There is a 1-1 correspondence between closures and f-families of relational dependencies on $U$.

**Definition 4.** Let $M \subseteq P(U)$. $M$ is called a Sperner system over $U$ if $A, B \in M$, then $A$ is not a subset of $B$.

**Definition 5.** A map $C : P(U) \to P(U)$ is called a choice function, if every $A \in P(U)$ then $C(A) \subseteq A$.

Choice functions are widely studied in rational behaviors of individuals and groups in social studies as well as economics. We could understand the choice function as follows: $P(U)$ is interpreted as a set of all alternatives, $A$ as a set of alternatives given to the decision-maker to choose the best, and $C(A)$ as a choice of the best alternatives among $A$.

Choice functions have been also widely studied in connection with the theory of rational behaviors of individuals and groups, see [9–11]. There were introduced some properties to characterize the rational behavior of a decision-maker. The most important properties of choice functions are the following (in [9–11]). For all $X, Y \subseteq U$ we have:

- Heredity (H): if $X \subseteq Y$ then $C(Y) \cap X \subseteq C(X)$.
- Concordance (C): $C(X) \cap C(Y) \subseteq C(X \cap Y)$.
- Out casting (O): if $C(X) \subseteq Y \subseteq X$ then $C(X) = C(Y)$.
- Monotonicity (M): if $X \subseteq Y$ then $C(X) \subseteq C(Y)$.

Let $L$ be a closure operation. Based on [12–15], we have the following results.

**Definition 6.** Define two choice functions associated with L as follows: For $X \subseteq U$
$C(X) = U - L(U - X)$, denoted as Choice Function type-I (CF-I for short).
$C_L(X) = L(U - X) \cap X$, denoted as Choice Function type-II (CF-II for short).
Note that both choice functions related to closure operations above are uniquely determined by the closure $L$, in fact, $L(X) = X \cup C_L(U - X)$ and $L(X) = U - C(U - X)$.
For every $X \subseteq U$ the sets $C_L(X)$ and $C(X)$ form a partition of $X$, that means $C_L(X) \cup C(X) = X$ and $C_L(X) \cap C(X) = \emptyset$.

It is easy to see that CF-I satisfies the properties of (O) and (M) while CF-II satisfies the properties of (He) and (O).

Based on the definition above, we could easily have:

**Proposition 1.** *There is a 1-1 correspondence between CFs - I and closure operations on $U$.*

**Proposition 2.** *There is a 1-1 correspondence between CFs - II and closure operations on $U$.*

**Definition 7.** Let $L$ be a closure operation over $U$, and $A \subseteq U$ and $A$ is a winning coalition of $L$ if $L(A) = U$, that means $U$ of all players have a relational dependence on $A$. And $A$ is a minimal winning coalition of $L$ if $A$ is a winning coalition, but $L(B) \neq U$ for any proper subset $B$ of $A$.

Based on closure operations and choice functions, we define the concept of minimal winning coalition and maximal losing coalition as follows:

Denote $K$ is the set of all minimal winning coalitions. It is known that $K$ is the set of a minimal winning coalition of any closure operation if and only if $K$ is a nonempty Sperner system, that means $A, B \in K$: $A \notin B$.

**Proposition 3.** *A minimal winning coalition $W$ is a special case of closure operation $L$ over $U$: $W \subseteq L(W) = U$.*

**Definition 8.** Let $K$ be a Sperner system, we define the set of a maximal losing coalition of $K$, denoted $K^{-1}$ as follows

$$K^{-1} = \{A \subset U : (B \in K) \text{ implying } (B \nsubseteq A) \text{ and } (A \subset C) \text{ implying } (\exists B \in K)(B \subseteq C)\}.$$

It is easy to see that the maximal losing coalition $K^{-1}$ of $K$ are the subsets of $U$ not containing the elements of $K$ and which are maximal for this property and $K^{-1}$ is a Sperner system too. We have the following.

**Lemma 2.** *If $W$ is a minimal winning coalition of a closure operation over $U$, then maximal losing coalition $W^{-1}$ contains a choice function CF-II.*

*Proof.* Since for $W \subseteq U$, $W$ is a minimal winning coalition of $L$ if $L(W) = U$ and $L(W) = U - C(U - W) = U$, that means CF-I $C$ is the empty set in this case. In the other case, $W$ is a minimal winning coalition of $L$ if $L(W) = U$ and $L(W) = W \cup C_L(U - W) = U$. ■

We could see that the maximal losing coalition, denoted as $W^{-1}$ does not contain any element of $W$ so $W^{-1}$ contains choice function $C_L(U-W)$. Thus $K^{-1}$ family contains choice function CF-II. Thus, if $|W| = 1$ then $C_L(U - W)$ is a maximal losing coalition. It is easy to have the following results from Lemma 2.

**Proposition 4.** *There is a 1-1 correspondence between minimal winning coalitions and closure operations on $U$.*

**Proposition 5.** *There is a 1-1 correspondence between maximal losing coalitions and CFs - II on $U$ if $|W| = 1$.*

Based on the properties of choice functions, we then have the following properties of a maximal losing coalition $W^{-1}$ if $|W| = 1$.

**Proposition 6.** *If $|W| = 1$ then $W^{-1}$ satisfy following:*
- *Out casting (O): If $W^{-1}(X) \subseteq Y \subseteq X$ then $W^{-1}(X) = W^{-1}(Y)$.*
- *Heredity (He): If $X \subseteq Y$ then $W^{-1}(Y) \cap X \subseteq W^{-1}(X)$.*

*Proof.* This result comes directly from the nature of maximal losing coalition, $W^{-1}(X) = C_L(X) = L(U - X) \cap X$ and $L(X) = X \cup C_L(U - X) = U$, thus, if $|W| = 1$ then $W^{-1}(X) = C_L(X)$ is a CF-II. Thus, properties of O and He holds. ■

The maximal losing coalition presented in this paper originally from the idea of anti-key in relational databased proposed in [16], which plays an important role in understanding deeper the relationship between closure operations and choice functions and between functional dependencies and functional independencies, and is to understand the link between consistency and inconsistency in decision tables [7]. Now we have the following results presenting the relationship between maximal losing coalition and minimal winning coalition directly comes from [16].

**Proposition 7.** *For $K = \{W_1, ..., W_t\}$, and $K^{-1} = \{W_1^{-1}, ..., W_m^{-1}\}$ over $U$, then*

$$\bigcup_{i=1}^{t} W_i = \bigcup_{j=1}^{m} (U - W_j^{-1}) = U - \bigcap_{j=1}^{m} W_j^{-1}.$$

Based on all the results above, we have:

**Proposition 8.** *Winning coalition and losing coalition have the following properties over $U$:*

*a) Any coalition containing a minimal winning coalition is winning. Thus, the relational dependency game is monotonic.*

*b) The complement of a minimal winning coalition might contain other minimal winning coalitions in case there are many disjoint minimal winning coalitions, so the complement of a minimal winning coalition over $U$ might be not a losing coalition.*

*c) The complement of a losing coalition might be not a winning coalition if the only winning coalition is $U$.*

*d) The intersection of all winning coalitions is a veto set of some or one player, but that set might not be a winning coalition.*

Taylor and Zwicker, in 1999 [17], proposed a simple coalition by defining: a conventional coalitional game as considered simple if payoffs are either 1 or 0, which means coalitions are either "winning" or "losing" along with four axioms of monotonicity, properness, strongness, and non-weakness in [17–19]:

a) A simple game $W$ is monotonic if any coalition containing a winning coalition is also winning.

b) A simple game $W$ is proper if any winning coalition's complement (opposition) is losing.

c) A simple game $W$ is strong if the complement of any losing coalition is winning.

d) A veto player (vetoer) in a simple game is a player that belongs to all winning coalitions. Supposing there is a veto player, any coalition not containing a veto player is losing. A simple game $W$ is weak if it has a veto player.

So, based on the concept of simple games above, we can see that:

**Lemma 3.** *Predetermined relations of dependency among players make RD games find winning coalitions easier than simple games do.*

Without relational dependency among all players, like a relational dependency game, a simple game could not measure the algorithm complexity as quickly as a relational dependency coalition game. To illustrate that point, we go to an algorithm to find out how to find minimal winning coalition and maximal losing coalition in a relational dependency game:

Let $F$ be a closure operation over $U$. Set $Z(F) = \{A \subseteq U : F(A) = A\}$ and $T(F) = \{A \subset U : A \in Z(F) \text{ and } A \subset B \text{ implying } F(B) = U\}$. The following results come directly from [15, 16, 20].

**Lemma 4.** *Let $F$ be a closure operation over $U$, and $K_F$ is the set of minimal winning coalitions of $F$. Then $K_F^{-1} = T(F)$.*

Now, we construct Algorithm 1 to find a minimal winning coalition from a set of maximal losing coalitions.

**Lemma 5.** *If $H$ is a set of maximal losing collations, then $\{T(0), T(1), .....T(m)\}$ are winning coalitions, and $T(m)$ is the minimal winning coalition.*

The time complexity of this algorithm is $O(n^2|H|)$, where $|U| = n$ and $|H| =$ number of elements of $H$. It is easy to have those results based on the above algorithm:

**Algorithm 1** Finding a minimal winning coalition based on a set of maximal losing coalitions

**Input**: Let $r$ be a coalition game over $U$, and let $H$ be a Sperner system and $U \notin H$.
Define $B = \{b_1, ..., b_m\}$ and $B \in H$ and $a \in U - B$. And let $G = \{B_t \in H : a \notin B_t\}$
**Output**: $T(m)$ is a minimal winning coalition of $r$.
Define the recursive procedure (algorithm) as follows:
Step 1: $T(0) := B \cup a$ and for the $q = 0, 1, ..., m-1$ define
Step $(q+1)$ : $T(q+1) = \begin{cases} T(q) - \{b_{q+1}\} \text{ if } \forall B_i \in H - G : \{T(q) - b_{q+1}\} \notin B_i \\ T(q) \text{ otherwise} \end{cases}$

**Lemma 6.** *If $T(0) = \{b_1, ..., b_m\}$ is an arbitrary winning coalition, then the following recursive algorithm*

$$T(q+1) = \begin{cases} T(q) - \{b_{q+1}\} \; \textit{if} \;\; \forall B_i \in H - G : \{T(q) - b_{q+1}\} \notin B_i \\ T(q) \;\; \textit{otherwise.} \end{cases}$$

Then $T(m)$ is also a minimal winning coalition. Remark: It is best to choose $B$ because $|B|$ is minimal.

**Lemma 7.** *If there is a $B$ such that $\forall B_i \in H - \{B\} : B_i \cap B = \emptyset$ and $a \in \bigcup_{B_t \in H - \{B\}} B_t$, then $a \cup b$ is a winning coalition $\forall b \in B$.*

**Lemma 8.** *If $(U - \bigcup_{B_t \in H} B_t) \neq \emptyset$ then $a \in U - \bigcup_{B_t \in H} B_t$ is a winning coalition $\forall b \in B$.*

**Remark 1.** Let $H$ be a Sperner system and $U \notin H$ and $A \subset U$. We can give an algorithm analogous to the above one to decide whether $A$ is a winning coalition. If $A$ is a winning coalition, then this algorithm finds $A'$ such that $A' \subseteq A$ and $A'$ is a minimal winning coalition.

Next, we construct Algorithm 2 for finding the set of maximal losing coalition of a given Sperner system as follows:

**Algorithm 2** Finding the set of maximal losing coalitions from a given Sperner-system

**Input**: Let an arbitrary Sperner-system $V = \{K_1, ..., K_m\}$ over $U$ be given.
Step 1: $V_1 = \{U - \{a\} : a \in K_1\}$. It is obvious that: $V_1$ is losing coalition and $V_1 = \{K_1\}^{-1}$
Step $q+1$: Suppose we have constructed $V_q = \{K_1, ..., K_q\}^{-1}$ for $q < m$. We assume that $X_1, ..., X_t$ are the elements of $V_q$ containing $K_{q+1}$. So $V_q = F_q \cup \{X_1, ..., X_t\}$, where $F_q = \{B \in V_q : K_{q+1} \nsubseteq B\}$.
For all $i, (i = 1, ..., t_q)$ we construct a maximal losing coalition of $\{K_{q+1}\}$ on $X_i$, in the analogous way as $V_1$, which are the maximal subsets of $X$ not containing $K_{q+1}$. We denote them by $B_1^i, ..., B_{ri}^i (i = 1, .., t)$.
Let $V_{q+1} = F_q \cup \{B_p^i : B \in F_q \rightarrow B_p^i \not\subset B, 1 \leq i \leq t, 1 \leq p \leq r_i\}$.
**Output**: A set of maximal losing coalitions

Then we have following results come directly from [15].

**Lemma 9.** *For every $q$ $(1 \leq i \leq m)$ : $V_q = \{K_1, ..., K_q\}^{-1}$. Let $V_0 = U$, we have $V_q = F_q \cup \{X_1, ..., X_t\}$, where $1 \leq i \leq m-1$. Denote by $l_q$ the number of elements of $V_q$. When constructing $V_{q+1}$, the worst-case time is $O(n^2(l_q - t_q)t_q)$ if $t_q < l_q$ and $O(n^2 t_q)$ if $l_q = t_q$.*

*For the total time, we derive $O\left(n^2 \sum_{q=0}^{m-1} t_q \mu_q\right)$, where $\mu_q = \begin{cases} l_q - t_q \; if \; l_q > t_q \\ 1 \; otherwise. \end{cases}$*

## 3. RELATIONAL DEPENDENCY ON COALITIONAL GAME

A relational independency ($IN$) is a statement of the form $A \to B$, where $A, B \subset U$. The $IN$: $A \to B$ holds in a game of $r = \langle G, F, v \rangle$ over $U$ if $RD : A \to B$ does not hold. That means $B$ is independent from $A$. We also say that $r$ satisfies the $IN$ $A \to B$.

Let $r$ be a game over $U$, and $IN_r$ is the family of all $IN$ that holds for $r$. A family $RIN$ of $IN$'s is called complete if, for some game $r$, one has $RIN = IN_r$.

Let closure operation $L(A)$ over $r$ be defined as $\exists B \subseteq U$ such that $L^r(A) = A \cup B$ for all $A \subseteq U$.

Then we define $L^{IN-RD}(A) = A \cup \{a \notin A : A \to a \in IN_r\} = A \cup \{a \in A : A \to a \notin RD_r\}$. Then it is easy to see that $L^{IN-RD}(A)$ is a closure operation, and we call it a hybrid closure operation of $RD$ and $IN$. And if $A$ is a minimal winning coalition of a hybrid closure operation, then $A \subseteq L^{IN-RD}(A) = U$, that means maximal losing coalition $K^{-1}_{IN}$ of $A$ does not contain $A$, denoted as $K^{-1}_{IN}(U - A) = \{a \notin A : A \to a \in IN_r\}$. We call such a maximal losing coalition a maximal relational independence losing coalition (MILC). We have the following results.

**Lemma 10.**
- *Out casting (O): If* $K^{-1}_{IN}(X) \subseteq Y \subseteq X$ *then* $K^{-1}_{IN}(X) = K^{-1}_{IN}(Y)$.
- *Heredity (He): If* $X \subseteq Y$ *then* $K^{-1}_{IN}(Y) \cap X \subseteq K^{-1}_{IN}(X)$.

*Proof.* This result comes directly from the nature of MILC, which is not a winning coalition, and it is a CF-II. Thus, properties of $O$ and $He$ holds. ∎

## 4. MINIMAL WINNING COALTION IN RELATIONAL DEPENDENCY COALITIONAL GAMES WITH HYPERGRAPHS AND THEIR PROPERTIES

Hypergraphs are a handy mathematical tool for solving complex combinatorial problems. Thus, describing a relational dependency coalitional game in a hypergraph is useful. The relational dependency game schema can be considered as a set of players and a set of relations on those players, which can easily be converted into a hypergraph model where the set of vertices in the hypergraph present a correspondence to the set of players in the game schema, while each hyperedge corresponding to a set of players included in a relation in the game schema. Then, we must focus on the main parts of the relational dependency game schema, like closure operations, choice functions, minimal winning coalition, and maximal losing coalition, and represent them on the hypergraph.

### 4.1. Hypergraphs and transversals

First, some basic definition is needed here, and for more detail in [21]. Let $U$ be a nonempty finite set and put $P(U)$ as the family of all subsets of $U$ (another name for its power set). And the family $H = \{E_i : E_i \in P(U),\ i = 1, ..., m\}$ is called a hypergraph over $U$ if $E_i$ is nonempty holds for all $i$.

A hypergraph $H$ is simple if $E_i \subseteq E_j$ implies $i = j$. A simple hypergraph is a Sperner hypergraph.

The elements of $U$ are called vertices, and the sets $E_1, ..., E_m$ are the hyperedges of the hypergraph $H$. A graph is a special case of a hypergraph, with $|E_i| = 2$.

A transversal (or "hitting set") of a hypergraph $H$ is a set $T \subseteq U$ having a nonempty intersection with every edge, that means if and only if $E \in H$ implies $T \cap E$ is nonempty.

Transversal $T$ is called minimal if no proper subset of $T$ is a transversal.

The family of all minimal transversals of $H$ (or all minimal keys over $U$) is called the transversal hypergraph of $H$, denoted as $Tr(H)$. Clearly, $Tr(H)$ is a simple hypergraph.

### 4.2. Properties of simple hypergraph and minimal transversals

Claude Berge in [21] determines the following 5 properties of a hypergraph and minimal transversal: Let $H$ and $H'$ be two simple hypergraphs,

1) $H' = Tr(H)$ if and only if $H = Tr(H')$.

2) $Tr(Tr(H)) = H$.

3) $Tr(H) = Tr(H')$ if and only if $H = H'$.

And also,

4) If $H \nsubseteq H'$ iff every edge of $H$ is also an edge of $H'$.

5) If $H \subseteq H'$ and $H' \subseteq H$ then $H = H'$.

And a simple graph $H$ is intersecting iff $H \subseteq Tr(H)$. Thus, we call a simple hypergraph $H$ non-intersecting iff $Tr(H) \subseteq H$.

Examples of intersecting hypergraphs are mentioned above. In other words, a simple hypergraph $H$ is called intersecting if every two hyperedges in $E$ have a vertex in common.

Let $H$ be an intersecting hypergraph over $U$, and we define that a minimal transversal of $H$ be a closure traversal ($L\_Tr$) of $H$ if and only if it satisfies the following conditions:

(1) Extensivity (Ex): $H \subseteq L\_Tr(H)$.

(2) Monotonicity (M): $H \subseteq H'$ implies $L\_Tr(H) \subseteq L\_Tr(H')$.

(3) Idempotency (I): $L\_Tr(L\_Tr((H)) = L\_Tr(H)$.

Thus, a simple intersecting hypergraph $H$ always has its transversal $Tr(H)$ satisfying conditions (1) and (2), but condition (3) is not necessarily satisfied.

As properties of transversal such that $Tr(Tr(H)) = H$ is well-set, so in order to uphold condition (3), which is $Tr(Tr(H)) = Tr(H)$, then $Tr(H) = H$.

Let denote $X(H)$ as the chromatic number of $H$, the minimal number of colors necessary to color the vertices of $H$ such that no edge of cardinality $> 1$ is monochromatic.

By that, we have the following results:

**Lemma 11.** *A simple hypergraph $H$ without loops has its transversals as closure transversals, satisfying all three properties (Ex, M, I) if and only if $X(H) > 2$ and $H$ intersect.*

*Proof.* This lemma could be proved by the following results in [21]: A simple hypergraph $H$ without loops satisfies $X(H) > 2$ iff $Tr(H) \subseteq H$ and a simple hypergraph $H$ without loops satisfies $H = Tr(H)$ iff $X(H) > 2$ and $H$ is intersecting. ■

We denote a simple intersecting hypergraph $H$ satisfying closure transversal as a simple closure intersecting hypergraph $H_L$. Let's investigate the relationship between choice function and non-intersecting hypergraph. Let $H$ be a non-intersecting hypergraph over $U$, and we define that a minimal transversal of $H$ be a choice traversal (C_Tr) of $H$ if and only if it satisfies the following conditions if every $H \in P(U)$ then $C\_Tr(H) \in H$. Choice functions are widely studied in social studies and economics, as well as rational behaviors of individuals and groups. We could understand the choice function as follows: $P(U)$ is interpreted as a set of all alternatives, $H$ as a set of other options given to the decision-maker to choose the best, and $C\_Tr(H)$ as a choice of the best alternatives among $H$. Since the definitions above, we call a simple hypergraph $H$ non-intersecting iff $Tr(H) \subseteq H$, then we have the following results.

**Lemma 12.** *All transversals of simple hypergraph $H$ become choice traversals if and only if $H$ is non-intersecting.*

Choice functions have also been widely studied in connection with the theory of rational behaviors of individuals and groups, see [9–11]. Thus, by Lemma 13, we could link the rational behaviors of individuals and groups with studying traversals of non-intersecting hypergraphs. Based on those properties of choice functions discussed above, we could have the following results:

**Lemma 13.** - *Heredity (He): If $H \subseteq H'$ then $Tr(H') \cap H \subseteq Tr(H)$.*
- *Concordance (C): $Tr(H) \cap Tr(H') \subseteq Tr(H \cap H')$.*
- *Out casting (O): If $Tr(H) \subseteq H' \subseteq H$ then $Tr(H) = Tr(H')$.*
- *Monotonicity (M): If $H \subseteq H'$ then $Tr(H) \subseteq Tr(H')$.*

Let $L$ be a closure operation. Define two choice functions associated with $L$ as follows:

$$C_L(X) = L(U - X) \cap X, \tag{1}$$

$$C(X) = U - L(U - X),\ X \subseteq U. \tag{2}$$

Note that both choice functions related to closure operations above are uniquely determined by the closure $L$; in fact, $L(X) = X \cup C_L(U - X)$ and $L(X) = U - C(U - X)$.

For every $X \subseteq U$, the sets $C_L(X)$ and $C(X)$ form a partition of $X$, which means $C_L(X) \cup C(X) = X$ and $C_L(X) \cap C(X) = \emptyset$.

Based on that, we could have the following results.

**Lemma 14.** *Let $H$ be a simple non-intersecting hypergraph over $U$, and if $L(H) : P(U) \to P(U)$, is defined as follows $L(H) = H \cup Tr_L(U - H)$ and $L(H) = U - Tr(U - H)$. Then, two $L$s defined above are closure operations of $H$, and $Tr_L(U - H)$ and $Tr(U - H)$ form a partition over $(U - H)$.*

Thus, from Lemma 15, we have the following lemmas.

**Lemma 15.** *The mapping $L \to Tr_L$ establishes a one-to-one correspondence between a simple non-intersecting hypergraph's closure operations and traversals.*

**Lemma 16.** *The mapping $L \to Tr$ establishes a one-to-one correspondence between a simple non-intersecting hypergraph's closure operations and traversals. Such mapping is injective.*

*Proof.* Because for two distinct closures $L_1$ and $L_2$ with $L_1(X) \neq L_2(X)$, we will have $Tr_{L_1}(U - X) \neq Tr_{L_2}(U - X)$. Thus, the mapping must be injective. ∎

**Lemma 17.** *Let $H_L$ be a simple closure intersecting hypergraph over $U$, and if two followings are defined as follows: $C_L(H_L) = L\_Tr(U - H_L) \cap H_L$, and $C(H_L) = U - L\_Tr(U - H_L)$, $H_L \subseteq U$. Then $C_L(H_L)$ and $C(H_L)$ defined as above are two choice functions of $H_L$ and they form a partition of $H_L$, which means $C_L(H_L) \cup C(H_L) = H_L$ and $C_L(H_L) \cap C(H_L) = \emptyset$.*

*Then, two Cs defined above are choice functions of $H_L$, and $C_L(H_L)$ and $C(H_L)$ form a partition over $H_L$.*

**Lemma 18.** *The mapping $C_L \to L\_Tr$ establishes a one-to-one correspondence between choice functions and closure traversals of a simple closure intersecting hypergraph.*

**Lemma 19.** *The mapping $C \to L\_Tr$ establishes a one-to-one correspondence between choice functions and closure traversals of a simple closure intersecting hypergraph.*

### 4.3. Anti-transversals of simple closure intersecting hypergraph transversals.

The relationship between closure operations with minimal keys (as minimal winning coalitions in RD coalitional game) and choice functions with maximal anti-keys (maximal losing coalitions in RD coalitional game) leads us to discover the relationship between intersecting hypergraph with closure operations and non-intersecting hypergraph with choice functions. Now, we investigate the relationship between transversals and anti-transversals in a simple closure intersecting hypergraph.

Let $L$ be a closure operation over $U$, and $A \subseteq U$, and $A$ is a key of $L$ if $L(A) = U$. And $A$ is a minimal key of $L$ if $A$ is a key, but $L(B) \neq U$ for any proper subset $B$ of $A$. There fore we can see that a minimal key is a special case of closure operation $A \subseteq L(A) = U$.

Denote $K$ is the set of the minimal keys. It is known that $K$ is the set of minimal keys of any closure operation if and only if $K$ is a nonempty Sperner system, which means $A, B \notin K$: $A \nsubseteq B$.

As we already know in [22], $K$ is a set of minimal transversals, $Tr(H)$, of a simple closure intersecting hypergraph $H$. And $Tr(H)$ is a simple hypergraph too.

We define the set of anti-transversals of $Tr(H)$, denoted $Tr^{-1}$ as follows

$$Tr^{-1} = \{A \subset U : (B \in Tr(H)) \text{ implying } (B \nsubseteq A)$$

$$\text{and } (A \subset C) \text{ implying } (\exists B \in Tr(H))(B \subseteq C)\}.$$

It is easy to see that the anti-transversal $Tr^{-1}$ of $Tr(H)$ are the subsets of $U$ not containing the elements of $Tr(H)$ and which are maximal for this property and $Tr^{-1}$ is a simple hypergraph too. Then we have the following.

**Lemma 20.** *A simple closure intersecting hypergraph $H_L$ has anti-transversal $Tr^{-1}$, which contains CF-II.*

*Proof.* By definitions and results in the above sections, a simple closure intersecting hypergraph has its transversals as closure transversals and anti-transversal $Tr^{-1}$ does not contain any element of transversal so $Tr^{-1}$ contains CF-II. ∎

Based on the properties of choice functions CF-II, we then have the following properties of a set of the maximal anti-transversal $Tr^{-1}$.

**Lemma 21.** *If $|Tr| = 1$ then $Tr^{-1}$ satisfy:*
- *Out casting (O): If $Tr^{-1}(X) \subseteq Y \subseteq X$ then $Tr^{-1}(X) = Tr^{-1}(Y)$.*
- *Heredity (H): If $X \subseteq Y$ then $Tr^{-1}(Y) \cap X \subseteq Tr^{-1}(X)$.*

*Proof*. This result comes directly from the nature of anti-transversals in a simple closure intersecting hypergraph. Thus, properties of O and He holds. ■

Similarly, we have the following result.

**Lemma 22.** *Let $Tr(H)$ be families of minimal transversals of a simple closure intersecting hypergraph over $U$, then $\bigcup Tr(H) = U - \bigcap Tr^{-1}$.*

### 4.4. Describing a minimal winning coalition on hypergraph

As in Sections 2 and 3, let $U = \{a_1, ...a_n\}$ be a nonempty finite set of n players. From [22], we could have Algorithm 3 and Algorithm 4 as follows.

**Algorithm 3** Find a minimal transversal of $H$

**Input**: Let $H$ be a hypergraph over $U$, and let $P = \{p_1, ..., p_k\}$ be a transversal of $H$, and define the recursive procedure (algorithm) as follows.

Step 1: $T(0) := P$ and for the $q = 0, 1, ..., k-1$ define

Step $q+1$ : $T(q+1) = \begin{cases} T(q) - \{l_{q+1}\} & \text{if } \forall E_i \in H : \{T(q) - l_{q+1}\} \cap E_i \neq \emptyset \\ T(q) & \text{otherwise} \end{cases}$

**Output**: $T(k)$ is minimal transversal of $H$

Now, the following proposition can be checked easily.

**Proposition 9.** *The sets $T(1), T(2), ..., T(k)$ are transversal of $H$, and T(k) is a minimal transversal of $H$.*

The time complexity of this algorithm is $O(n^2|H|)$, where $|H| = n$. The next algorithm finds the family of all minimal transversals of a given hypergraph.

**Algorithm 4** Finding the family of all minimal transversals of a given hypergraph

**Input**: Let $H = E_1, ..., E_m$ be a hypergraph over $U$. For every $q = 1, ..., m$ we will construct $tr(\{E_1, ..., E_m\})$ by induction.

Step 1: $L_1 := \{\{a\} : a \in E_1\}$. It is obvious that $L_1 = tr(E_1)$

By the inductive hypothesis, we constructed $L_q = tr(E_1, ...E_q)$ for $q < m$.

Step q: $L_q = S_q \cup \{B_1, ..., B_{tq}\}$, where $B_i \cap E_{q+1} \neq \emptyset, 1 = 1, ..., t_q$ and $S_q = A \in L_q : A \cap E_{q+1} \neq \emptyset$.

For each $i$ $(i = 1, .., t_q)$ construct the set $\{B_i \cup b : b \in E_{q+1}\}$

Denote them by $A^i_1, .., A^i_{ri}$ $(i = 1, .., t_q)$. Let $L_{q+1} = S_q \cup \{A^i_p : A \in S_q \Rightarrow A \not\subset A^i_p, 1 \leq i \leq t_q, 1 \leq p \leq r_i\}$.

**Output**: $L_m = tr\,(\{E_1, ...E_m\})$

Now, the following lemma can be rechecked easily.

**Lemma 23.** *For every $q$ $(1 \leq q \leq m), L_q = tr(\{E_1, ...E_q\})$, which means $L_m = tr(H)$, $S = \langle U, F, v\rangle$ be a coalitional game schema, and $r$ a game over $U$. For every $a \in U$, set $I(A) =$*

$\{a \in U : A \to \{a\} \notin F^+\}$. *Then,* $I(A)$ *is called an independent set of* $S$. *For* $r$, *put* $I(A) = \{a \in U : A \to \{a\} \in F_r\}$.

Denote $I_S$, the family of all independent sets of s. Set $m(S) = \{B \in I_S B \neq \emptyset, \exists C \in I_S : C \subset B\}$, m(s) is called the family of all minimal independent sets of $S$. It can be seen that $A$ is a winning coalition if and only if $I(A) = \emptyset$.

**Lemma 24.** *Let* $S = \langle U, F, v\rangle$ *be a game schema over* $U$. *Then* $Tr(K_S) = m(S)$. *Clearly, m(S) is a simple hypergraph over* $U$, $Tr(K_S) = m(S) = H$. *That means if* $R$ *is a coalitional game schema and* $H$ *is a simple hypergraph over* $U$, *we say that* $R$ *represents* $H$ *if* $Tr(K_S) = H$, *which means* $K_S = Tr(H)$.

Based on the results in Section 2 and Subsection 4.2, the following results could be easily checked.

**Lemma 25.** *A simple hypergraph* $H$ *describing a relational dependency coalitional game* $r$ *over* $U$ *has the following properties:*
*1) If* $r$ *contains no independencies (consistency secured), then the closure operation of transversals of* $H$ *representing gamer is a one-to-one correspondence with choice function CF-I, and* $H$ *is a simple non-intersecting hypergraph.*
*2) If* $r$ *contains both dependencies and independencies (inconsistency occurs), then the closure operation of transversals of* $H$ *representing* $r$ *is a hybrid closure one, and* $H$ *is a simple non-intersecting hypergraph.*

## 5. CONCLUSION

The core is the most attractive and natural way to define stability in coalitional games. When payoff distribution is in the core, no player is incentivized to join a different coalition. This is like the Nash equilibrium of non-cooperative games. Such core of relational dependency coalitional games in the world of complete and incomplete information shall be addressed: disjoint coalitions and overlapping coalitions in the world of complete information; Bayesian relational dependency coalitional games integrated with Pawlak's rough sets in the world of incomplete information or under uncertainty.

Shapley values applied in relational dependency coalitional games and adding "weights" on each relational dependency give us more valuable relationships between minimal winning and maximal losing coalitions along with upper and lower bound properties and algorithms.

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