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NUMERICAL METHOD FOR SOLVING THE DIRICHLET BOUNDARY VALUE PROBLEM FOR NONLINEAR TRIHARMONIC EQUATION

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Abstract. In this work, we consider the Dirichlet boundary value problem for nonlinear triharmonic equation. Due to the reduction of the problem to operator equation for the pair of the right hand side function and the unknown second normal derivative of the function to be sought, we design an iterative method at both continuous and discrete levels for numerical solution of the problem. Some examples demonstrate that the numerical method is of fourth order convergence. When the right hand side function does not depend on the unknown function and its derivatives, the numerical method gives more accurate results in comparison with the results obtained by the interior method of Gudi and Neilan.

Keywords. Nonlinear triharmonic equation; Dirichlet boundary value problem; Iterative method; Fourth order convergence.

1. INTRODUCTION

In this work, we consider the following boundary value problem (BVP) for nonlinear triharmonic equation

$$\Delta^3 u = f(x, u, \Delta u, \Delta^2 u),\ x \in \Omega, \tag{1}$$

$$u = b_0,\ \frac{\partial u}{\partial \nu} = b_1;\ \Delta u = b_2,\ x \in \Gamma, \tag{2}$$

where Ω is a bounded connected domain in $\mathbb{R}^n$ $(n \geq 2)$ with the boundary Γ, Δ is the Laplace operator, ν is outward normal to boundary, $f(x, u, v, w)$ is a continuous function, $b_i = b_i(x),\ (i = 0, 1, 2)$ are continuous functions on the boundary Γ.

Notice that when the domain Ω is a rectangle in $\mathbb{R}^2$ and $b_0 = 0$ then the boundary $\Delta u = b_2$ is the same as the condition $\frac{\partial^2 u}{\partial \nu^2} = b_2$. Therefore, instead the boundary conditions (2) it is possible consider the Dirichlet boundary conditions

$$u = b_0,\ \frac{\partial u}{\partial \nu} = b_1,\ \frac{\partial^2 u}{\partial \nu^2} = b_2. \tag{3}$$

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The triharmonic equation is a particular case of sixth order elliptic equations arising in the modeling of ulcers [20], viscous fluid [9], geometric design [21]. To the best of our knowledge, the nonlinear triharmonic problem (1)-(2) or (1), (3) has not been studied in any works. Meanwhile the nonlinear equation (1) with the boundary conditions

$$u = b_0,\ \Delta u = b_1,\ \Delta^2 u = b_2 \tag{4}$$

has been considered in many works. The great contribution to the numerical solution of the this problem belongs to Mohanty and his colleages in [11]- [15], where the authors constructed compact finite difference schemes with local truncation error of $O(h^2)$ or $O(h^4)$. In result the nonlinear triharmonic problem is reduced to a system of nonlinear algebraic equations, which then is solved by block iterative methods. Numerical examples illustrated the applicability and the effectiveness of the numerical method. However, the authors did not obtain the error estimate of the actually obtained numerical solution.

Recently, in 2018 Ghasemi [7] used the idea of differential quadrature to construct methods to approximate solution of higher elliptic partial differential equations in higher dimensions

$$\Delta^k u(x) = f(x, u, \Delta u, ..., \Delta^{k-1} u)$$

subject to the boundary conditions

$$u = f_1,\ \frac{\partial^{2l} u}{\partial \nu^{2l}} = f_{2l},\ l = 1, 2.$$

As in the Mohanty et al. works, in [7] the author only obtained a local truncation error but not any error estimate for the approximate solution.

It should be said that in all the mentioned above works the authors only considered the numerical methods for solving the problem (1), (4) under the assumptions that the problem has a unique solution with sufficient smoothness although these assumptions are not stated explicitly. Very recently, in [5] we have established the existence and uniqueness of solution of the problem (1), (4) under easily verified conditions and constructed an iterative method having fourth order of convergence.

Now return to the problem (1)-(2). In the case $f = f(x)$, in [2] Dang constructed an iterative method based on the reduction of the problem to a boundary operator equation with completely continuous symmetric positive operator and proposed an iterative method to solve the latter one. The convergence of the method and the acceleration of the convergence was studied. When the equation (1) has the form

$$\Delta^3 u - au = f(x),\ a > 0.$$

Dang in [3] reduced the problem to a domain-boundary operator equation. Using the parametric extrapolation technique the author constructed an iterative method for the problem. It should be noticed that in the two papers mentioned above Dang only constructed iterative methods on continuous level and established their convergence without numerical examples for illustration. After these papers, in 2012 D.Long [4] carried out some numerical experiments for showing the convergence of the problem

$$\begin{aligned} &\Delta^3 u = f(x),\ x \in \Omega, \\ &u = g_0,\ \frac{\partial u}{\partial \nu} = g_1;\ \Delta u = g_2,\ x \in \Gamma. \end{aligned} \tag{5}$$

Differently from [2, 3] some authors constructed approximate solution of the above problem by the direct discretization the differential equation and the boundary conditions. For example, Gudi and Neilan [8] used the cubic Lagrange finite elements to construct an approximation to the solution. The error estimate for the approximate solution u_h is $\|u - u_h\|_{L^2(\Omega)} = O(h^2)$. Recently, in 2018 Abdrabou and El-Gamel applied sinc-Galerkin method to construct a method for the problem (1)- (2), where the solution is sought in the form of an expansion by sinc basis functions. An error estimate for the approximate solution was obtained through two indefinite parameters. Nevertheless, the numerical experiments on some examples show good results.

In this paper, combining the technique for construction of iterative methods for the problem (1), (4) in [5] and the technique for solving the problem (5) in [2] we reduce the problem (1), (2) to a domain-boundary operator equation and construct an iterative method for the latter one. Making discretization for the iterative method on continuous level we obtain an iterative method on discrete level. Numerical experiments on some examples show the convergence of order 4 of the proposed solution method.

2. CONSTRUCTION OF ITERATIVE METHODS

First, we reduce the problem (1)- (2) to an operator equation. For this purpose, we set

$$\varphi(x) = f(x, u(x), \Delta u(x), \Delta^2 u(x)), \tag{6}$$

$$\Delta u = v,\ \Delta v = w,\ w|_\Gamma = g. \tag{7}$$

Then the problem is reduced to the sequence of three second order problems

$$\begin{aligned} \Delta w &= \varphi, \quad x \in \Omega, \\ w &= g, \quad x \in \Gamma, \end{aligned} \tag{8}$$

$$\begin{aligned} \Delta v &= w, \quad x \in \Omega, \\ v &= b_2, \quad x \in \Gamma, \end{aligned} \tag{9}$$

$$\begin{aligned} \Delta u &= v, \quad x \in \Omega, \\ u &= b_0, \quad x \in \Gamma. \end{aligned} \tag{10}$$

The solutions of the above problems depend on the unknown functions φ in Ω and g on Γ, i.e., $w = w_{\varphi g},\ v = v_{\varphi g},\ u = u_{\varphi g}$. These solutions must satisfy the conditions

$$\begin{aligned} f(x, u_{\varphi g}, v_{\varphi g}, w_{\varphi g}) &= \varphi(x), \quad x \in \Omega, \\ \frac{\partial u_{\varphi g}}{\partial \nu} &= b_1, \quad x \in \Gamma. \end{aligned} \tag{11}$$

This is the system of equations for determining φ and g.
Denote

$$Z = \begin{bmatrix} \varphi \\ g \end{bmatrix} \tag{12}$$

and introduce the operator A defined on elements Z by the formula

$$AZ = \begin{bmatrix} f(., u_{\varphi g}, v_{\varphi g}, w_{\varphi g}) \\ g - \tau\big(\dfrac{\partial u_{\varphi g}}{\partial \nu} - b_1\big) \end{bmatrix}, \tag{13}$$

where $w = w_{\varphi g},\ v = v_{\varphi g},\ u = u_{\varphi g}$ are the solutions to the problems (8)-(10) and τ is a positive parameter. Then the system (11) is equivalent to the operator equation

$$AZ = Z. \tag{14}$$

For this equation we shall apply the successive iteration method, which has the form

$$\begin{aligned} Z_{k+1} &= AZ_k,\ k = 0, 1, ... \\ Z_0 &\text{ is given.} \end{aligned} \tag{15}$$

This iterative method is realized by the following iterative process:
i) Given an initial approximation φ_0, g_0, for example,

$$\varphi_0(x) = f(x, 0, 0, 0), \quad x \in \Omega;\ g_0 = 0, \quad x \in \Gamma. \tag{16}$$

ii) Knowing $\varphi_k,\ g_k\ (k = 0, 1, 2, ...)$ solve sequentially three second order problems

$$\begin{aligned} \Delta w_k &= \varphi_k, \quad x \in \Omega, \\ w_k &= g_k, \quad x \in \Gamma, \end{aligned} \tag{17}$$

$$\begin{aligned} \Delta v_k &= w_k, \quad x \in \Omega, \\ v_k &= b_2, \quad x \in \Gamma, \end{aligned} \tag{18}$$

$$\begin{aligned} \Delta u_k &= v_k, \quad x \in \Omega, \\ u_k &= b_0, \quad x \in \Gamma. \end{aligned} \tag{19}$$

iii) Calculate the new approximation

$$\begin{aligned} \varphi_{k+1}(x) &= f(x, u_k(x), v_k(x), w_k(x)), \\ g_{k+1} &= g_k - \tau\big(\frac{\partial u_k}{\partial \nu} - b_1\big). \end{aligned} \tag{20}$$

In order to numerically realize the above iterative method on continuous level we propose the discrete iterative method as follows.

We limit to consider the problem (1)-(2) in the rectangle $\overline{\Omega} = [0, l_1] \times [0, l_2]$. On this domain introduce the uniform grid

$$\overline{\omega}_h = \Big\{(x_1, x_2)|\ x_1 = ih_1,\ x_2 = jh_2,\ i = \overline{0, m},\ j = \overline{0, n}\Big\},$$

where $h_1 = l_1/m,\ h_2 = l_2/n$. Denote by Ω_h and γ_h the set of interior points and the set of boundary points of $\overline{\omega}_h$, respectively. For grid function $Y(x)$ defined on the grid $\overline{\omega}_h$ we use the max-norm $\|Y\|_h = \max_{x \in \overline{\omega}_h} |Y(x)|$.

To solve the Poisson problems (17)-(19) at each iterative step we shall use finite difference schemes of fourth order of accuracy. For this purpose, denote by $\Phi_k(x), W_k(x), V_k(x), U_k(x)$ the grid functions defined on the grid $\overline{\omega}_h$ and approximating the functions $\varphi_k(x), w_k(x), v_k(x)$, $u_k(x)$ on this grid. Besides, we denote by $G_k(x)$ the grid function defined on the boundary nodes γ_h and approximating the function $g_k(x)$ on Γ. The discrete iterative process is described as follows:

1. Given
$$\Phi_0(x) = f(x,0,0,0),\ x \in \omega_h;\ G_0(x) = 0,\ x \in \gamma_h. \tag{21}$$

2. Knowing Φ_k in ω_h and G_k on γ_h $(k = 0,1,...)$ solve consecutively three difference problems
$$\begin{aligned} \Lambda^* W_k &= \Phi_k{}^*, \quad x \in \omega_h, \\ W_k &= G_k, \quad x \in \gamma_h, \end{aligned} \tag{22}$$
$$\begin{aligned} \Lambda^* V_k &= W_k{}^*, \quad x \in \omega_h, \\ V_k &= b_2, \quad x \in \gamma_h, \end{aligned} \tag{23}$$
$$\begin{aligned} \Lambda^* U_k &= V_k{}^*, \quad x \in \omega_h, \\ U_k &= b_0, \quad x \in \gamma_h. \end{aligned} \tag{24}$$

3. Compute the new approximation
$$\begin{aligned} \Phi_{k+1}(x) &= f(x, U_k, V_k, W_k), \quad x \in \omega_h, \\ G_{k+1}(x) &= G_k(x) - \tau\big(D_\nu U_k - b_1\big), \quad x \in \gamma_h. \end{aligned} \tag{25}$$

Here we adopt the following notations for grid function Y defined on the grid $\overline{\omega_h}$ (see [17])
$$\begin{aligned} \Lambda^* Y &= \Lambda Y + \frac{h_1^2 + h_2^2}{12}\Lambda_1\Lambda_2 Y,\ \Lambda Y = (\Lambda_1 + \Lambda_2)Y, \\ \Lambda_1 Y &= \frac{Y_{i-1,j} - 2Y_{ij} + Y_{i+1,j}}{h_1^2},\ \Lambda_2 Y = \frac{Y_{i,j-1} - 2Y_{ij} + Y_{i,j+1}}{h_2^2}, \\ \psi^* &= \psi + \frac{h_1^2}{12}\Lambda_1\psi + \frac{h_2^2}{12}\Lambda_2\psi, \end{aligned}$$
where $Y_{ij} = Y(ih_1, jh_2)$. Besides, we use the following notation for discrete normal derivative
$$D_\nu U = \begin{cases} \dfrac{1}{12h_1}\Big(- 25U_{0j} + 48U_{1j} - 36U_{2j} + 16U_{3j} - 3U_{4j}\Big),\ x = (0, jh_2) \\ \dfrac{1}{12h_1}\Big(25U_{nj} - 48U_{n-1,j} + 36U_{n-2,j} - 16U_{n-3,j} + 3U_{n-4,j}\Big),\ x = (l_1, jh_2) \\ \dfrac{1}{12h_2}\Big(- 25U_{i0} + 48U_{i1} - 36U_{i2} + 16U_{i3} - 3U_{i4}\Big),\ x = (ih_1, 0) \\ \dfrac{1}{12h_2}\Big(25U_{im} - 48U_{i,m-1} + 36U_{i,m-2} - 16U_{i,m-3} + 3U_{i,m-4}\Big),\ x = (ih_1, l_2). \end{cases}$$

This formula of numerical differentiation has fourth order accuracy (see [10]).

3. NUMERICAL EXAMPLES

To demonstrate the effectiveness of the iterative method in the previous section we shall consider several examples. All examples will be considered in the computational domain $\Omega = [0,1] \times [0,1]$ with the boundary Γ. On this domain we use an uniform grid with the grid size h: $\overline{\omega}_h = \{(x_1,x_2)|\ x_1 = ih,\ x_2 = jh;\ i,j = \overline{0,n}\}$. In order to test the convergence of the proposed iterative method we perform some experiments for the cases, where exact solutions are known and for the cases where exact solutions are not known. The criterion for stopping the iterative process (21)-(25) is $\|U_k - U_{k-1}\|_h \leq TOL$, where TOL is a given accuracy. All numerical experiments are performed on computer LENOVO, 64-bit Operating System (Win 10), Intel Core I5, 1.8 GHz, 8 GB RAM. The iterative parameter τ is chosen as $\tau = 100$. For solving the discrete problems (22)-(24) we use the cyclic reduction method [18], which is one of the efficient direct methods for grid equations.

Example 1. (Example with exact solution).
Consider the equation

$$\Delta^3 u = \Delta^3 u^* + \sin(u - u^*) - \cos(\Delta u - \Delta u^*) + \Delta^2 u - \Delta^2 u^* + 1,$$

where $u^* = e^{x_1}\sin(x_2)$. Obviously, this function u^* is the exact solution of the above equation. The associated boundary conditions are calculated from this exact solution. Below we report the results of computation for some different TOL and make a remark on them. First, for $TOL = 10^{-8}$ the results are given in Table 1, where K is the number of iterations performed, $E^h(K) = \|U_K^h - u^*\|_h$, $Rate$ is the rate of convergence calculated by the formula

$$Rate = \log_2 \frac{\|U_K^h - u^*\|_h}{\|U_K^{h/2} - u^*\|_{h/2}}.$$

In the above formula the superscripts h and $h/2$ of U mean that U is computed on the grid with the corresponding grid sizes. The total running time for obtaining the results of Table 1 is 41.2656 seconds.

Table 1: The results of computation of Example 1 for $TOL = 10^{-8}$

h	h^4	K	$E(K)$	$Rate$
1/8	2.4414e-04	117	1.1083e-05	3.7394
1/16	1.5259e-05	12	8.0959e-07	3.0400
1/32	9.3567e-07	15	9.8433e-08	0.8380
1/64	5.9605e-08	15	5.5103e-08	0.0676
1/128	3.7253e-09	15	5.2581e-08	0.0041
1/256	2.3283e-10	15	5.2431e-08	

From Table 1 it is observed that with the criterion of stopping $TOL = 10^{-8}$, firstly with the decrease of the grid size h the accuracy of the approximate solution increases until it reaches the amplitude of h^4 as the same order of TOL for $h = 1/64$. After that the decrease of the grid size does not improve the accuracy of the approximate solution, and simultaneously, the number of iterations needed remains unchanged. The above phenomenon is also reflected in values of the rate of convergence. It can be said that the grid size $h = 1/64$ is consistent with the accuracy 10^{-8}.

The results of computation for another $TOL = 10^{-6}$, which are presented in Table 2, support the above observation. The total running time for obtaining these results is 22.3750 seconds.

From Table 2 it is seen that the grid size $h = 1/32$ is consistent with the accuracy 10^{-6}.

Table 2: The results of computation of Example 1 for $TOL = 10^{-6}$

h	h^4	K	$E(K)$	$Rate$
1/8	2.4414e-04	8	9.5729e-06	3.5217
1/16	1.5259e-05	8	8.3351e-07	1.6258
1/32	9.3567e-07	8	2.7009e-07	0.1920
1/64	5.9605e-08	8	2.3643e-07	0.0128
1/128	3.7253e-09	8	2.3434e-07	0.0006
1/256	2.3283e-10	8	2.3445e-07	

Example 2 (Exact solution is not known). Consider the problem (1)-(2) with the right hand side function

$$f = x_1^6 + x_2^6 + \sin(\Delta u)\sin(\Delta^2 u)(e^{\Delta u - 1})$$

and the homogeneous boundary conditions. The exact solution of the problem is not known. The results of computation for $TOL = 10^{-6}$ are given in Table 3. Here, in the case when the exact solution is unknown, the deviation between two successive iterations $e^h(K)$ and $Rate$ of convergence are calculated by the formulas

Table 3: The results of computation of Example 2 for $TOL = 10^{-6}$

N	K	$e^h(K)$	$Rate$
1/8	11	7.0714 e-07	4.5708
1/16	10	9.8720e-07	3.1241
1/32	10	9.7352e-07	3.6005
1/64	10	9.7317e-07	3.8079
1/128	10	9.7315e-07	3.9053
1/256	10	9.7324e-07	
1/512	10	9.7324e-07	

$$e^h(K) = \|U_K^h - U_{K-1}^h\|_h,$$
$$Rate = \log_2 \frac{\|U_K^h - U_K^{h/2}\|_h}{\|U_K^{h/2} - U_K^{h/4}\|_{h/2}}.$$

Example 3 (Exact solution is not known). Consider the problem (1)-(2) with the right hand side function

$$f = 2u^2 - \Delta u\, \Delta^2 u + 30$$

and the homogeneous boundary conditions. The results of convergence for $TOL = 10^{-6}$ are given in Table 4.

The total running time for obtaining these results is 66.6250 seconds.

Remark 1. In the case when the right-hand side of the equation (1) does not depend on $u, \Delta u, \Delta^2 u$, i.e., $f = f(x)$, the iterative method (21)- (25) becomes:

1. Given

$$G_0(x) = 0,\ x \in \gamma_h$$

Table 4: The results of computation of Example 3 for $TOL = 10^{-6}$

N	K	$e^h(K)$	$Rate$
1/8	5	3.7968 e-07	2.9323
1/16	5	6.6379-07	3.4956
1/32	5	7.2627e-07	3.7545
1/64	5	7.3161e-07	3.8788
1/128	5	7.3199e-07	3.9398
1/256	5	7.3202e-07	
1/512	5	7.3202e-07	

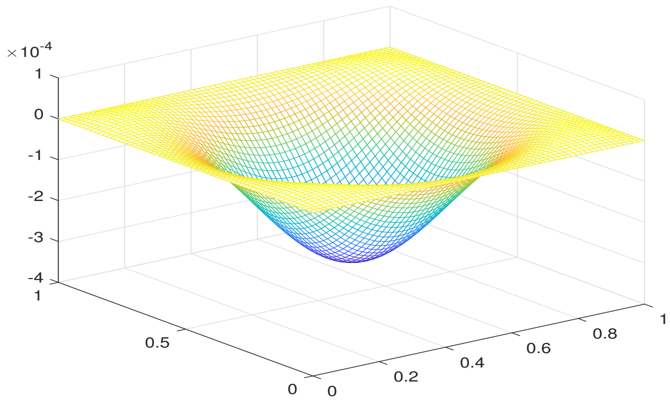

Figure 1: The graph of the approximate solution in Example 3

2. Knowing G_k on γ_h $(k = 0, 1, ...)$ solve consecutively three difference problems

$$\Lambda^* W_k = f^*, \quad x \in \omega_h,$$
$$W_k = G_k, \quad x \in \gamma_h,$$

$$\Lambda^* V_k = {W_k}^*, \quad x \in \omega_h,$$
$$V_k = b_2, \quad x \in \gamma_h,$$

$$\Lambda^* U_k = {V_k}^*, \quad x \in \omega_h,$$
$$U_k = b_0, \quad x \in \gamma_h.$$

3. Compute the new approximation

$$G_{k+1}(x) = G_k(x) - \tau\big(D_\nu U_k - b_1\big), \quad x \in \gamma_h.$$

Below, we use the above discrete iterative method for an example, for which the exact solution is known and compare the results with the ones obtained by the interior penalty method in [8].

Example 4 (Example 1 in [8]). Consider the problem

$$\Delta^3 u = \Delta^3 u^*, \quad x \in \Omega, \tag{26}$$

$$u = 0, \ \frac{\partial u}{\partial \nu} = 0; \ \Delta u = 0, \quad x \in \Gamma, \tag{27}$$

where

$$u^* = (x_1 - {x_1}^2)^3 (x_2 - {x_2}^2)^3.$$

It is easy verify that the function u^* is an exact solution of the problem. Below we report the accuracy of the approximate solution $\|u_h - u^*\|$ of the above problem obtained by the iterative method in Remark 1 for $TOL = 10^{-6}$ and $TOL = 10^{-9}$ compared with the ones of Gudi and Neilan in [8].

Table 5: Comparison of the accuracy of our results with the ones in [8] for Example 4

h	$TOL = 10^{-6}$	$TOL = 10^{-9}$	In [8]
1/8	5.0535e-05	4.3624e-05	4.2564e-02
1/16	1.5678e-05	3.8348e-06	2.2396e-02
1/32	1.2838e-05	3.1764e-07	1.1501e-02

Example 5 (Example 2 in [8]). Consider the problem (26)-(27) with the exact solution

$$u^* = (x_1 - {x_1}^2)^3 \sin(4\pi x_1)(x_2 - {x_2}^2)^3 \sin(4\pi x_2).$$

The comparison of accuracy of the approximate solution $\|u_h - u^*\|$ of the above problem obtained by our iterative method for $TOL = 10^{-6}$ and $TOL = 10^{-9}$ compared with the ones in [8] is given in Table 6.

Table 6: Comparison of the accuracy of our results with the ones in [8] for Example 5.

h	$TOL = 10^{-6}$	$TOL = 10^{-9}$	In [8]
1/8	1.4638e-04	1.7966e-04	7.7198e-01
1/16	1.6648e-05	2.6336e-06	3.9348e-01
1/32	1.9535e-05	1.4857e-06	2.6287e-01

Example 6 (Example 3 in [8]). Consider the problem (26)-(27) with the exact solution

$$u^* = (x_1^2 + x_2^2)^{7.1/4} (x_1 - {x_1}^2)^3 (x_2 - {x_2}^2)^3.$$

The comparison of accuracy of the approximate solution $\|u_h - u^*\|$ of the above problem obtained by our iterative method for $TOL = 10^{-6}$ and $TOL = 10^{-9}$ compared with the ones in [8] is given in Table 6.

4. CONCLUSIONS

In this work, by reducing the original boundary value problem of nonlinear triharmonic equation with Dirichlet boundary conditions to a domain-boundary operator equation for

Table 7: Comparison of the accuracy of our results with the ones in [8] for Example 6

h	$TOL = 10^{-6}$	$TOL = 10^{-9}$	In [8]
1/8	9.4129e-05	8.0619e-05	5.0400e-02
1/16	2.7840e-05	1.3848e-05	2.5500e-02
1/32	1.8612e-05	3.3878e-06	1.3200e-02

the nonlinear term and boundary value of the second normal derivative we have designed a numerical iterative method consisting of solving consecutively three BVPs for Poisson equations by difference schemes of fourth order approximation and computing normal derivative by a formula of fourth order accuracy at each iteration. This iterative method combines the ideas of the iterative method for BVP for triharmonic equation with the Dirichlet boundary conditions and the iterative method for nonlinear triharmonic equation with boundary conditions for the even order derivatives. Both of them were developed by ourselves before. Numerical experiments on some examples, where the exact solutions are known or are not known, show that the method is of fourth order convergence. When the right hand side function does not depend on the function to be sought and its derivatives the numerical method gives more accurate results compared with the results obtained by the interior method of Gudi and Neilan in [8].

The proposed method can be applied to boundary value problems for nonlinear triharmonic equation with more complicated right hand side functions. This is the direction of our research in the future.

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