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DERIVED TERMS WITHOUT DERIVATION
A SHIFTED PERSPECTIVE ON THE DERIVED-TERM AUTOMATON

SYLVAIN LOMBARDY[1],*, JACQUES SAKAROVITCH[2]

[1]*LaBRI - UMR 5800 - Bordeaux INP - Bordeaux University - CNRS, France*
[2]*IRIF - CNRS/Paris University and Telecom Paris, IPP, France*

Abstract. We present here a construction for the derived term automaton (aka partial derivative, or Antimirov, automaton) of a rational (or regular) expression based on a sole induction on the depth of the expression and without making reference to an operation of derivation of the expression. It is particularly well-suited to the case of weighted rational expressions and the case of expressions over non free monoids.

Keywords. Regular expressions; Rational expressions; Finite weighted automata; Finite transducers; Position automaton; Derivation of expressions.

1. INTRODUCTION

In this paper, we address once again the laboured problem of the transformation of a rational (regular) expression into a finite automaton that accepts the language, or the series, denoted by the expression.

In the Handbook of Automata Theory that appeared recently [16], we have given a survey on the many aspects of the transformation of an automaton into an expression and vice-versa, together with a comprehensive bibliography [20]. We explain in this chapter that the equivalence between automata and expressions may be generalised from 'classical' automata and expressions to *weighted* automata and *weighted* expressions, to automata and expressions over monoids that are not necessarily free monoids, and even to *weighted* automata and *weighted* expressions over monoids that are not necessarily free monoids (but still *graded*). This generalisation makes on one hand-side the relationship between automata and expressions tighter and leads on the other hand-side to 'split' Kleene Theorem into two parts: the first one is the correspondence between automata and expressions, and the second the equality between the family of rational (or regular) languages or series and the family of recognisable languages or series, an equality which holds in the case of languages or series over free monoids only.

As we describe in that survey [20], there are two main methods for computing an automaton from an expression which yield two distinct, even though related, automata: the *position automaton* and the *derived-term automaton* of the expression.

The first method can be credited to Glushkov [11]. It associates with an expression of litteral length n a (non-deterministic) automaton with $n + 1$ states — often called the *Glushkov* or *position* automaton

Dedicated to Professor Phan Dinh Dieu on the occasion of his 85th birth anniversary.

*Corresponding author.
E-mail addresses: sylvain.lombardy@labri.fr (S. Lombardy); sakarovitch@enst.fr (J. Sakarovitch).

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of the expression. As we recall in Section 3 below the position automaton may be inductively defined by means of operations on automata of a certain kind that we call *standard automata*. The definition and computation of the position automaton readily generalises to *weighted expressions* [5] and, even more easily as there is nothing to change, to expressions over *non free monoids*.

It takes some more lines to sketch the second method. It is well-known that a language (subset of a free monoid) is accepted by a finite automaton if and only if it has a *finite number* of (left) quotients. The starting point of the second method is the idea, due to Brzozowski, to lift this property of recognisable languages at the symbolic level of rational (or regular) expressions. In [4], Brzozowski defined the *derivatives* of a rational expression and turned them into the states of a deterministic automaton that recognises the language denoted by the expression. He then showed that modulo the axioms of associativity, commutativity, and idempotency of the addition (on the set of languages), the ACI-properties, the set of derivatives of an expression is finite. Under this form, it is clear that this 'derivative' method is essentially different from the first one as it cannot be generalised to weighted rational expressions since weighted finite automata cannot be determinised nor to expressions over non free monoids since subsets accepted by finite automata over such monoids have not necessarily a finite number of (left) quotients.

Thirty years later, Antimirov made another fundamental contribution to this theory and proposed a new derivation process [1]. Antimirov's derivation breaks Brzozowski's derivatives into parts — hence the name 'partial derivatives' given to these parts, a terminology we find unfortunate and we call them *derived terms*. As before, derived terms are turned into the states of a finite automaton which we call *derived-term automaton* and which accepts the language denoted by the expression. This construction has several outcomes. The number of derived terms of an expression is not only finite but also 'small', smaller than, or equal to, the litteral length of the expression. Derived terms are defined without the usage of ACI-properties, which makes them easier to compute than the derivatives.

Finally, a link between the two methods was established somewhat later by Champarnaud and Ziadi in [7], and the derived-term automaton of an expression E was shown to be a *morphic image*[1] of the position automaton of E, a result which we refer to as the *morphism theorem* in the sequel.

In [13], we have extended the construction of the derived-term automaton to *weighted expressions*. Of course, the relationship with 'derivatives' has disappeared in this generalisation, but the link the derivation of an expression and the *quotient* of series is as strong as in the Boolean case. The 'weighted version' of the characterisation of recognisability with the quotients is due to Jacob in full generality and reads as follows: *a series is recognisable if and only if it belongs to a finitely generated submodule stable by quotient* (see [3] or [18] for instance). And the derived terms of a weighted expression are a set of generators of a module that contains the series denoted by the expression and that is stable by quotient. The construction of the same automaton has also been given by Rutten as a byproduct of his theory of conduction on series which puts the quotient operation on series at the first place [17].

We also showed, in the same paper, that the result quoted above could be generalised to the weighted case, and, with the adequate generalisation of the notion of quotient to weighted automata, that the derived-term automaton of a weighted expression E is a *quotient* of the position automaton of E.

It must be noted however that a difficulty arose in the proof of this last result. In the derivation

[1]Usually, one says that an automaton $\mathcal{A}$ is a *quotient* of an automaton $\mathcal{B}$ if there exists a morpism from $\mathcal{B}$ onto $\mathcal{A}$, that is, if $\mathcal{A}$ is a morphic image of $\mathcal{B}$. In this introduction, we prefer this latter terminology as it does not collide with the (left) *quotient* of a language, or of a series (by a word).

process and, if the weight semiring contains such elements, some terms may vanish from the set of derived terms by the interplay of 'positive' and 'negative' coefficients. In such cases, they will be 'missing' and the derived-term automaton will not be a quotient of the position automaton but only a sub-automaton of a quotient of the position automaton. Instead of contenting ourselves with this weaker statement, we proved that the definition of the derived terms could be decorrelated from the derivation itself and obtained by induction on the expression and that the 'quotient theorem' would then hold in full generality. The proofs of the various properties of the derived-term automaton however relied on the connection with the derivation of the expression and the quotient of the series.

This long presentation was necessary to set up the framework in which this work takes place and to state the new ideas it brings to this much walked subject.

We present here a definition of the derived-term automaton of an expression E by induction on the formation of E, in parallel with the construction of the position automaton of E and with no reference whatsoever to the quotients of the series denoted by E nor to a derivation operation defined on expressions. This new prespective shows that the derived-term automaton is indeed intrinsically attached to the structure, or to the syntactic tree, of the expression, in the same way as the position automaton is.

The first consequence, or outcome, of the decorrelation between the construction of the derived-term automaton and the derivation, and thus the quotient of series, is that it can be achieved on expressions over *non free* monoids in which the rational languages or series no longer coincide with the recognisable ones, and hence are not characterised by the Jacob's theorem quoted above any longer.

The second outcome is that the 'morphism theorem' which was somewhat tedious to establish comes for free with this new point of view as it is an intrinsic property of the construction: at every step of the induction, the derived-term automaton is built as a morphic image of an automaton which is already a morphic image of the position automaton.

Of course, the connection with the derivation of expressions, and the quotient of the denoted series remains when the expression are over free monoids, since the new construction yields the same automata as the old one.

The fact that the derived-term automaton is indeed related to the structure of the expression is not completely new. As we have explained above,and in our own work [13], we have defined the derived terms by means of an induction rather than by the plain derivation. In [6], and in order to describe an *efficient algorithm* for the construction of the derived-term automaton, a link between the *positions* of the letters in the rational expression and the derived terms is made through in-between objects called *c-continuations*. This allows to define the derived-term automaton as a morphic image of the position automaton. This construction is different from ours, since we apply morphisms at every step of our inductive construction.

There are also been several attempts to apply the derivation techniques to expressions over non free monoids, namely direct products of free monoids, for dealing with *rational relations*. In [8], the extension is made through a new operator that represents the direct product of two languages (or relations), while in [12] the atoms of the expression, that are letters in the classical case, are replaced by pairs of letters or special symbols (expressing constraints over letters or pairs). The formalism used in both papers bears some similarities with ours, but they both use it to define an analogue of the derivation to bring the construction of the transducer back to the usual construction of the derived-term automaton. Notice also that only Boolean transducers are considered in [12].

The essence of the new perspective we take on the derived-term automaton of an expression could be described in the classical case of the rational expressions on a free monoid. But we discovered this new point of view when we were dealing with *weighted rational expressions on non free monoids*. Even though it makes the exposition somewhat longer and burdensome, we have chosen to present it in its full generality.

In Section 2, we fix the notation for weighted expressions and weighted automata and define the morphisms of weighted automoata *via* the notion of *conjugacy* which proves to be efficient. In Section 3, we define the restricted class of *standard automata* on which one can lift the rational operators and the position automaton, which we prefer to call the *standard automaton* of the expression.

The core of the paper lays in Section 4 where the new definition of the derived-term automaton is presented and its consistency proved. Even though the definition goes purely by induction on the formation of the expression, we have chosen to keep the old terminology which bears the weight of history and reconnects with it in the prevalent case of expressions on a free monoid. In Section 5, we show, via the notion of *differential* of an expression, that the new definition of derived-term automaton coincides with the one given in the previous works on the subject, in the case of expressions on a free monoid.

2. PRELIMINARIES AND NOTATIONS

The definition of usual notions in theoretical computer science, such as free monoids, languages, expressions, automata, rational (or regular) sets, recognisable sets, etc. may be found in numerous textbooks.

The corresponding notions of multiplicity (or weight) semirings, (formal power) series, weighted automata, etc. are probably less common knowledge but are still presented in quite a few books [2, 3, 10, 21] to which we refer the reader. For the notation, we follow [18, 20]. Let us be more explicit for the two notions we study: the *weighted rational expressions* and the *weighted finite automata*. Before, we recall the notions of *graded* monoids and of *starrable element* in a semiring. And then, we define the notion of *morphism* of weighted automata that will be instrumenal in this work.

The purpose of the paper, is the construction of automata over a monoid M which is *not necessarily free*, for instance automata over $A^* \times B^*$ which are transducers — and this is a key feature of this work. At the same time we want the automata possibly be weighted with coefficients taken in a semiring $\mathbb{K}$, thus realising maps from M to $\mathbb{K}$, that is, *series* in $\mathbb{K}\langle\langle M\rangle\rangle$. The required hypothesis on M for $\mathbb{K}\langle\langle M\rangle\rangle$ to be closed under (Cauchy) product is that M be a *graded monoid* (i.e. endowed with an additive length function) — which is the case for $A^* \times B^*$ for instance, or more generally for all trace monoids [9].

If k is an element of a semiring $\mathbb{K}$, k^* is the sum of all powers of k: $k^* = \sum_{n\in\mathbb{N}} k^n$. This infinite sum may be defined — k is said to be *starrable* — or not defined — k is said to be *non starrable.* We are not interested in the problem of determining whether an element of $\mathbb{K}$ is starrable or not. Somehow, we consider that $\mathbb{K}$ is equipped with this operator $*$, which is defined on a known subset of $\mathbb{K}$. But the question arises to know if we are able, given $\mathbb{K}$ and M, to determine whether a series of $\mathbb{K}\langle\langle M\rangle\rangle$ is starrable or not. The answer is positive, *via* the notion of strong semiring (see [13, 18]), and some additional notation.

The *identity element* of M is denoted by 1_M. We write $M_\bullet$ for the set of elements of M different from 1_M, that is, the set of elements with a (stricly) positive length: $M_\bullet = M \setminus \{1_M\}$.

The *constant term* $\mathsf{c}(s)$ of a series s is the coefficient of 1_M in s (that is, the image of 1_M in s). A series is *proper* if its constant term in $0_{\mathbb{K}}$. The *proper part* s_{p} of a series s is the series obtained from s by zeroing the coefficient of 1_M and keeping all other coefficients unchanged.

Definition 1. A topological semiring is *strong* if the product of two summable families is a summable family.

The definition is taken in view of the following statement.

Theorem 2. *Let* $\mathbb{K}$ *be a strong semiring and* M *a graded monoid. Let* s *be a series of* $\mathbb{K}\langle\langle M\rangle\rangle$, $s_0 = \mathsf{c}(s)$ *its constant term and* s_{p} *its proper part. Then* s^* *is defined if and only if* s_0 *is starrable and in this case we have*

$$s^* = (s_0^* s_{\mathsf{p}})^* s_0^* = s_0^*(s_{\mathsf{p}} s_0^*)^* .$$

The details are not of interest here. It is enough for us to know that all usual semirings such as $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $(\mathbb{Z}, \mathsf{min}, +)$, etc. are strong topological semirings. And may be that not all topological semirings are strong (c.f. [15]). In the sequel, the semirings are supposed to be strong, and the monoids to be graded, without always stating it explicitely.

2.1. Weighted rational expressions

Definition 3. A *rational expression over a monoid* M *with weight in a semiring* $\mathbb{K}$ is a well-formed formula built inductively from the *constants* $\mathsf{0}$ and $\mathsf{1}$ and the elements m in $M_\bullet$ as *atomic formulas*, using two binary operators $+$ and $\cdot$, one unary operator $*$ and two operators for every k in $\mathbb{K}$: if E and F are expressions, so are $(k\,\mathsf{E})$, $(\mathsf{E}\,k)$, $(\mathsf{E}+\mathsf{F})$, $(\mathsf{E}\cdot\mathsf{F})$, and (E^*). We denote by $\mathbb{K}\mathsf{RatE}\,M$ the set of rational expressions over M with weight in $\mathbb{K}$ and often call them $\mathbb{K}$-*expressions* or even simply *expressions*.

Expressions are thus given by the following grammar

$$\mathsf{E} \to \mathsf{0} \mid \mathsf{1} \mid m \mid (k\,\mathsf{E}) \;\mid (\mathsf{E}\,k) \;\mid (\mathsf{E}+\mathsf{E}) \mid (\mathsf{E}\cdot\mathsf{E}) \mid (\mathsf{E}^*) \quad \forall m \in M_\bullet,\ \forall k \in \mathbb{K}.$$

Definition 4. The *constant term* of an expression E in $\mathbb{K}\mathsf{RatE}\,M$ — if it exists — is the element of $\mathbb{K}$, written $\mathsf{c}(\mathsf{E})$, and inductively computed using the following equations

$$\mathsf{c}(\mathsf{0}) = 0\,, \quad \mathsf{c}(\mathsf{1}) = 1\,, \quad \mathsf{c}(m) = 0 \quad \forall m \in M_\bullet\,,$$
$$\mathsf{c}(k\,\mathsf{E}) = k\,\mathsf{c}(\mathsf{E})\,, \quad \mathsf{c}(\mathsf{E}\,k) = \mathsf{c}(\mathsf{E})\,k \quad \forall k \in \mathbb{K}\,,$$
$$\mathsf{c}(\mathsf{F}+\mathsf{G}) = \mathsf{c}(\mathsf{F}) + \mathsf{c}(\mathsf{G})\,, \quad \mathsf{c}(\mathsf{F}\cdot\mathsf{G}) = \mathsf{c}(\mathsf{F})\,\mathsf{c}(\mathsf{G})\,,$$
$$\mathsf{c}(\mathsf{F}^*) = (\mathsf{c}(\mathsf{F}))^* \quad \text{if } \mathsf{c}(\mathsf{F}) \text{ is starrable.}$$

If the constant term of a subexpression F of E is not starrable, $\mathsf{c}(\mathsf{E})$ is *undefined*, and E is said to be *non valid*; otherwise, E is a *valid* expression.

Definition 5. With every *valid* expression E in $\mathbb{K}\mathsf{RatE}\,M$ is associated a series of $\mathbb{K}\langle\langle M\rangle\rangle$, which is called *the series denoted by* E, and which we write $|\mathsf{E}|$.

The series $|\mathsf{E}|$ is inductively defined by

$$|\mathsf{0}| = 0_{\mathbb{K}},\ |\mathsf{1}| = 1_M,\ |m| = m \quad \forall m \in M_\bullet\,, \quad |k\,\mathsf{E}| = k\,|\mathsf{E}|,\ |\mathsf{E}\,k| = |\mathsf{E}|\,k \quad \forall k \in \mathbb{K},$$
$$|\mathsf{F}+\mathsf{G}| = |\mathsf{F}| + |\mathsf{G}|,\ |\mathsf{F}\cdot\mathsf{G}| = |\mathsf{F}|\,|\mathsf{G}|, \quad \text{and}$$
$$|\mathsf{F}^*| = (|\mathsf{F}|)^* \quad (|\mathsf{F}| \text{ is starrable by the validity of } \mathsf{E} \text{ and Theorem 2}).$$

Two expressions are *equivalent* if they denote the same series.

It directly follows from Definitions 4 and 5 that the constant term of an expression is equal to the constant term of the series denoted by the expression.

Proposition 6. $\mathsf{c}(\mathsf{E}) = \mathsf{c}(|\mathsf{E}|)$.

Example 7. The $\mathbb{Z}$-expression E_1 over the monoid $\{a,b\}^*$, $\mathsf{E}_1 = a^* \cdot (a^* + (-1)b^*)^*$, is valid: $\mathsf{c}(\mathsf{E}_1) = 1$.

Even though they do not play a role in this work, the definition of the set of *rational series* and its characterisation with expressions build its background.

Definition 8. The set of $\mathbb{K}$-*rational series over* M is the smallest subalgebra of $\mathbb{K}\langle\langle M\rangle\rangle$ which contains the polynomials and is closed under star. It is denoted by $\mathbb{K}\mathrm{Rat}\, M$.

Proposition 9. *A series of* $\mathbb{K}\langle\langle M\rangle\rangle$ *is rational if and only if it is denoted by a valid expression in* $\mathbb{K}\mathsf{RatE}\, M$.

2.2. Weighted finite automata

An automaton $\mathcal{A}$ over a monoid M with weights in a semiring $\mathbb{K}$ is a *labelled directed graph* (Q, E), together with two functions I and T from the set Q of vertices – called *states* – into $\mathbb{K}$. The set E of edges – called *transitions* – is contained in $Q \times \mathbb{K} \times M_\bullet \times Q$, that is, every transition is labelled with a monomial $k\,m$ – the *weighted label* of the transition – where k is the *weight* of the transition and m its *label*. The automaton $\mathcal{A}$ is *finite* if E is finite.

The weighted label of a path in $\mathcal{A}$ is the product of the weighted labels of the transitions that form the path, hence a monomial $h\,x$, where h is the product of the weights of the transitions and x the product of their labels.

The automaton $\mathcal{A}$ determines a map from M to $\mathbb{K}$, that is a series in $\mathbb{K}\langle\langle M\rangle\rangle$, called the *behaviour* of $\mathcal{A}$ and denoted by $|\mathcal{A}|$. The series $|\mathcal{A}|$ maps every x in M to the *sum* of all elements $I(p)\,h\,T(q)$ where h is the weight of a path π with label x, for all such paths π from p to q, and all pairs of states (p,q). The definition of $|\mathcal{A}|$ takes a handier form in an algebraic setting. The set E of transitions of $\mathcal{A}$ is conveniently described by the *transition matrix* of $\mathcal{A}$, which we also denote by E (as it will be indeed the unique way we deal with this set in the sequel), and which is thus a matrix of dimension $Q \times Q$ whose (p,q)-entry is the sum of the weighted labels of the transitions that go from p to q, a linear combination of elements of $M_\bullet$ when $\mathcal{A}$ is finite. We write $\mathcal{A} = \langle\, I, E, T\,\rangle$ where the function I is written as a row-vector of dimension Q whose pth entry is $I(p)$ and the function T is written as a column-vector of dimension Q whose qth entry is $T(q)$. Since the formation of paths corresponds to the multiplication of the transition matrix, the behaviour of $\mathcal{A}$ may then be written as

$$|\mathcal{A}| = I \cdot E^* \cdot T.$$

Two automata are *equivalent* if they have the same behaviour. Finite automata and rational expressions have the same computational power, as expressed by the following statement.

Theorem 10. *Let* M *be a graded monoid. A series of* $\mathbb{K}\langle\langle M\rangle\rangle$ *is rational if and only if it is the behaviour of a finite* $\mathbb{K}$*-automaton over* M.

The subject of this work is the study of a particular proof of the sufficient condition of this statement.

2.3. Morphisms and quotient of weighted automata

Automata are structures; one can thus define *morphisms* between them. We choose to define the morphisms of weighted automata via the notion of *conjugacy*, borrowed from the theory of symbolic dynamical systems. It is the most concise way, and ideally suited for the sequel.

Definition 11. A $\mathbb{K}$-automaton $\mathcal{A} = \langle I, E, T \rangle$ *is conjugate to* a $\mathbb{K}$-automaton $\mathcal{B} = \langle J, F, U \rangle$ if there exists a matrix X with entries in $\mathbb{K}$ such that

$$I\,X = J, \qquad E\,X = X\,F, \quad \text{and} \quad T = X\,U. \tag{1}$$

The matrix X is the *transfer matrix* of the conjugacy and we write $\mathcal{A} \overset{X}{\Longrightarrow} \mathcal{B}$.

If $\mathcal{A}$ is conjugate to $\mathcal{B}$, then, for every n, the series of equalities holds

$$I\,E^n\,T = I\,E^n\,X\,U = I\,E^{n-1}\,X\,F\,U = \ldots = I\,X\,F^n\,U = J\,F^n\,U,$$

from which $I\,E^*\,T = J\,F^*\,U$ directly follows.

Proposition 12. *If $\mathcal{A}$ is conjugate to $\mathcal{B}$, then $\mathcal{A}$ and $\mathcal{B}$ are equivalent.*

Let $\varphi\colon Q \to R$ be a *surjective* map and X_φ the $Q{\times}R$-matrix where the (q,r)-th entry is 1 if $\varphi(q) = r$, and 0 otherwise. Since φ is a map, every row of X_φ contains exactly one 1 and since φ is surjective, every column of X_φ contains at least one 1. Such a matrix is called an *amalgamation matrix* in the setting of symbolic dynamics. By convention, if we deal with $\mathbb{K}$-automata, an amalgamation matrix is silently assumed to be a $\mathbb{K}$-matrix, that is, the null entries are equal to $0_{\mathbb{K}}$ and the non zero entries to $1_{\mathbb{K}}$.

Definition 13. Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathbb{K}$-automata of dimension Q and R respectively. We say that a surjective map $\varphi\colon Q \to R$ is a *morphism*[2] (from $\mathcal{A}$ onto $\mathcal{B}$) if $\mathcal{A}$ is conjugate to $\mathcal{B}$ by X_φ, i.e. if $\mathcal{A} \overset{X_\varphi}{\Longrightarrow} \mathcal{B}$, and we write $\varphi\colon \mathcal{A} \to \mathcal{B}$.

We also say that $\mathcal{B}$ is a *quotient* of $\mathcal{A}$, if there exists a morphism $\varphi\colon \mathcal{A} \to \mathcal{B}$.

The composition of two morphisms is a morphism. From Proposition 12 follows that any quotient of $\mathcal{A}$ is equivalent to $\mathcal{A}$.

On the other hand, we can determine whether a surjective map $\varphi\colon Q \to R$ is a morphism or not, without reference to any automaton $\mathcal{B}$. From X_φ we construct a *selection matrix* Y_φ by transposing X_φ and by zeroing some of its non zero entries in such a way that Y_φ is row-monomial, with *exactly one* 1 per row. A matrix Y_φ is not uniquely determined by φ but also depends on the choice of a 'representative' in each class of the map equivalence of φ.

Proposition 14. *Let $\mathcal{A} = \langle I, E, T \rangle$ be a $\mathbb{K}$-automaton of dimension Q. Let $\varphi\colon Q \to R$ be a surjective map, X_φ its amalgamation matrix, and Y_φ a selection matrix. Then φ is a* morphism *if $\mathcal{A}$ is conjugate by X_φ to the automaton $\varphi(\mathcal{A})$ of dimension R: $\varphi(\mathcal{A}) = \langle I \cdot X_\varphi,\ Y_\varphi \cdot E \cdot X_\varphi,\ Y_\varphi \cdot T \rangle$ (in which case $\varphi(\mathcal{A})$ does not depend on the choice of Y_φ).*

This proposition points out that the image of φ is indeed immaterial and what only counts, and makes it a morphism of automata or not, is the map equivalence of φ.

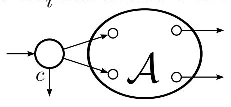
In the proofs at Sec. 4., we use indeed a more intuitive description of morphisms. With the same notation as above, the map φ is a morphism if and only if the rows of $E \cdot X_\varphi$ whose indices are equivalent modulo φ are equal and the entries of T whose indices are equivalent modulo φ are equal.

[2]The morphisms of weighted automata are 'more constrained' than those of Boolean automata. They correspond to what is often called *simulation*. Moreover, they are *directed* and for a same map φ one should distinguish between *Out*-morphism and *In*-morphism. But in this work we only deal with Out-morphisms, which we simply call morphisms.

3. STANDARD AUTOMATA

We define a restricted class of automata, and then show that rational operations on series can be lifted on the automata of that class. They are thus well-suited for the constructions we build by induction on the formation of the expressions.

An automaton is *standard* if it has only one initial state, which is the end of no transition. Figure 1 shows a standard automaton, both as a sketch, and under the matrix form. The definition does not forbid the initial state i from also being final and the scalar c in $\mathbb{K}$, is the *constant term* of $|\mathcal{A}|$.



$$\mathcal{A} = \left\langle \begin{pmatrix} 1 & \boxed{0} \end{pmatrix}, \begin{pmatrix} 0 & \boxed{J} \\ \boxed{0} & \boxed{F} \end{pmatrix}, \begin{pmatrix} c \\ \boxed{U} \end{pmatrix} \right\rangle .$$

Figure 1: A standard automaton

Elementary matrix computations show

$$|\mathcal{A}| = c + J\,F^*\,U, \tag{2}$$

where $c = \mathsf{c}(|\mathcal{A}|)$ is the *constant term* of $|\mathcal{A}|$ and $J\,F^*\,U$ is the proper part of $|\mathcal{A}|$.

It is convenient to say — when there is no ambiguity — that the *dimension* of the standard automaton $\mathcal{A}$ is is the dimension of the vector J (or of the matrix F).

It is rather obvious that every automaton is equivalent to a standard one, but this will not be used here.

3.1. Operations on standard automata

Their special form allows to define *operations* on standard automata that are parallel to the *rational operations*. Let $\mathcal{A}$ (as in Figure 1) and $\mathcal{B}$ (with obvious notation) be two standard ($\mathbb{K}$-)automata; let k be in $\mathbb{K}$. Then we define the following standard automata

- $$k\,\mathcal{A} = \left\langle \begin{pmatrix} 1 & \boxed{0} \end{pmatrix}, \begin{pmatrix} 0 & \boxed{k\,J} \\ \boxed{0} & \boxed{F} \end{pmatrix}, \begin{pmatrix} k\,c \\ \boxed{U} \end{pmatrix} \right\rangle, \tag{3}$$

- $$\mathcal{A}\,k = \left\langle \begin{pmatrix} 1 & \boxed{0} \end{pmatrix}, \begin{pmatrix} 0 & \boxed{J} \\ \boxed{0} & \boxed{F} \end{pmatrix}, \begin{pmatrix} c\,k \\ \boxed{U\,k} \end{pmatrix} \right\rangle, \tag{4}$$

- $$\mathcal{A} + \mathcal{B} = \left\langle \begin{pmatrix} 1 & \boxed{0} & \boxed{0} \end{pmatrix}, \begin{pmatrix} 0 & \boxed{J} & \boxed{K} \\ \boxed{0} & \boxed{F} & \boxed{0} \\ \boxed{0} & \boxed{0} & \boxed{G} \end{pmatrix}, \begin{pmatrix} c+d \\ \boxed{U} \\ \boxed{V} \end{pmatrix} \right\rangle, \tag{5}$$

- $$\mathcal{A} \cdot \mathcal{B} = \left\langle \begin{pmatrix} 1 & \boxed{0} & \boxed{0} \end{pmatrix}, \begin{pmatrix} 0 & \boxed{J} & \boxed{c\,K} \\ \boxed{0} & \boxed{F} & \boxed{U \cdot K} \\ \boxed{0} & \boxed{0} & \boxed{G} \end{pmatrix}, \begin{pmatrix} c\,d \\ \boxed{U\,d} \\ \boxed{V} \end{pmatrix} \right\rangle, \tag{6}$$

and finally $\mathcal{A}^*$, which is defined when c^* is defined

- $$\mathcal{A}^* = \left\langle \begin{pmatrix} 1 & \boxed{\quad 0 \quad} \end{pmatrix}, \begin{pmatrix} 0 & \boxed{c^* J} \\ \boxed{0} & \boxed{\quad H \quad} \end{pmatrix}, \begin{pmatrix} c^* \\ \boxed{U c^*} \end{pmatrix} \right\rangle, \tag{7}$$

where $H = U \cdot c^* J + F$.

After these definitions, it is rather natural to say that the two exterior multiplications and the star are the *dimension invariant* operations. Elementary matrix computations then establish the following.

Proposition 15. *Let $\mathcal{A}$ and $\mathcal{B}$ be two standard $\mathbb{K}$-automata, and let k be an element in $\mathbb{K}$. It then holds: $|k\,\mathcal{A}| = k|\mathcal{A}|$, $|\mathcal{A}\,k| = |\mathcal{A}|k$, $|\mathcal{A} + \mathcal{B}| = |\mathcal{A}| + |\mathcal{B}|$, and $|\mathcal{A} \cdot \mathcal{B}| = |\mathcal{A}|\,|\mathcal{B}|$.*

The case of the star operation is significantly more involved and requires that Theorem 2 is first established. Then, the following statement holds.

Proposition 16. *Let $\mathbb{K}$ be a strong semiring. If $\mathcal{A}$ is a standard $\mathbb{K}$-automaton, it then holds $|\mathcal{A}^*| = (|\mathcal{A}|)^*$.*

3.2. The standard automaton of an expression

The definition of the 'rational' operations on standard automata immediately induced the definition of a standard automaton canonically associated with every rational expression. It coincides with the automaton first defined by Glushkov in [11].

Proposition 17. *For every valid rational $\mathbb{K}$-expression E, there exists a canonical standard $\mathbb{K}$-automaton $\mathcal{S}_{\mathsf{E}}$ that realises the series denoted by E, that is, $|\mathcal{S}_{\mathsf{E}}| = |\mathsf{E}|$.*

Proof. The definition of $\mathcal{S}_{\mathsf{E}}$ starts with the definitions of standard automata for the atomic formulas.

- $\mathsf{E} = 0$ then $\mathcal{S}_0 = \left\langle \begin{pmatrix} 1 \end{pmatrix}, \begin{pmatrix} 0 \end{pmatrix}, \begin{pmatrix} 0 \end{pmatrix} \right\rangle$.
- $\mathsf{E} = 1$ then $\mathcal{S}_1 = \left\langle \begin{pmatrix} 1 \end{pmatrix}, \begin{pmatrix} 0 \end{pmatrix}, \begin{pmatrix} 1 \end{pmatrix} \right\rangle$.
- $\mathsf{E} = m \in M$ then $\mathcal{S}_m = \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$.

It is clear that $|\mathcal{S}_0| = 0 = |0|$, $|\mathcal{S}_1| = 1 = |1|$, and $|\mathcal{S}_m| = m = |m|$.

The natural definitions: $\mathcal{S}_{k\,\mathsf{E}} = k\,\mathcal{S}_{\mathsf{E}}$, $\mathcal{S}_{\mathsf{E}\,k} = \mathcal{S}_{\mathsf{E}}\,k$, $\mathcal{S}_{\mathsf{F}+\mathsf{G}} = \mathcal{S}_{\mathsf{F}} + \mathcal{S}_{\mathsf{G}}$, $\mathcal{S}_{\mathsf{F}\cdot\mathsf{G}} = \mathcal{S}_{\mathsf{F}} \cdot \mathcal{S}_{\mathsf{G}}$, and $\mathcal{S}_{\mathsf{F}^*} = (\mathcal{S}_{\mathsf{F}})^*$, allow the construction of $\mathcal{S}_{\mathsf{E}}$ for every valid $\mathbb{K}$-rational expression E, by induction on the formation of the expression, whereas Propositions 15 and 16 insure that $|\mathcal{S}_{\mathsf{E}}| = |\mathsf{E}|$. ∎

This automaton $\mathcal{S}_{\mathsf{E}}$ will be indifferently called '*the* standard automaton' or 'the position automaton' of E.

Example 7 (Continued)**.** Let us write $\mathsf{E}_1 = \mathsf{F}_1 \cdot \mathsf{G}_1$ with $\mathsf{F}_1 = a^*$ and $\mathsf{G}_1 = (a^* + (-1)b^*)^*$. It comes

$$\mathcal{S}_{\mathsf{F}_1} = \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a \\ 0 & a \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle, \qquad \mathcal{S}_{\mathsf{G}_1} = \left\langle \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a & -b \\ 0 & 2a & -b \\ 0 & a & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle,$$

$$\text{and} \qquad \mathcal{S}_{\mathsf{E}_1} = \mathcal{S}_{\mathsf{F}_1} \cdot \mathcal{S}_{\mathsf{G}_1} = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a & a & -b \\ 0 & a & a & -b \\ 0 & 0 & 2a & -b \\ 0 & 0 & a & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle.$$

3.3. Morphisms of standard automata

Let $\mathcal{A}$ and $\mathcal{A}'$ be two standard automata

$$\mathcal{A} = \left\langle \begin{pmatrix} 1 & \boxed{0} \end{pmatrix}, \begin{pmatrix} 0 & \boxed{J} \\ \boxed{0} & \boxed{F} \end{pmatrix}, \begin{pmatrix} c \\ \boxed{U} \end{pmatrix} \right\rangle, \text{ and } \mathcal{A}' = \left\langle \begin{pmatrix} 1 & \boxed{0} \end{pmatrix}, \begin{pmatrix} 0 & \boxed{J'} \\ \boxed{0} & \boxed{F'} \end{pmatrix}, \begin{pmatrix} c' \\ \boxed{U'} \end{pmatrix} \right\rangle,$$

and $\varphi\colon \mathcal{A} \to \mathcal{A}'$ a morphism of automata. The image by φ of the inital state i of $\mathcal{A}$ is necessarily the initial state i' of $\mathcal{A}'$ and no other state q of $\mathcal{A}$ is mapped onto i' for otherwise q would not be accessible.

Hence the transfer matrix of φ is of the form

$$\begin{pmatrix} 1 & \boxed{0} \\ \boxed{0} & \boxed{X_\varphi} \end{pmatrix},$$

and the conjugacy relation (1) passes to the 'core' of the automata

$$c = c', \;\; J\,X_\varphi = J', \;\; F\,X_\varphi = X_\varphi\,F', \text{ and } \;\; U = X_\varphi U'.$$

The operations on standard automata that we have defined above are consistant with morphisms.

Proposition 18. *Let $\mathcal{A}$ and $\mathcal{A}'$, $\mathcal{B}$ and $\mathcal{B}'$ be four standard automata, and let $\varphi\colon \mathcal{A} \to \mathcal{A}'$ and $\psi\colon \mathcal{B} \to \mathcal{B}'$ be two automata morphisms. Then $\varphi{\times}\psi\colon \mathcal{A}{+}\mathcal{B} \to \mathcal{A}'{+}\mathcal{B}'$, $\varphi{\times}\psi\colon \mathcal{A}{\cdot}\mathcal{B} \to \mathcal{A}'{\cdot}\mathcal{B}'$, and $\varphi\colon \mathcal{A}^* \to (\mathcal{A}')^*$ are automata morphisms.*

Proof. The statement for the addition is obvious. The computations for the other two operations are hardly more complex. For the product we have

$$\begin{pmatrix} 0 & \boxed{J} & \boxed{c\,K} \\ \boxed{0} & \boxed{F} & \boxed{U\,K} \\ \boxed{0} & \boxed{0} & \boxed{G} \end{pmatrix} \cdot \begin{pmatrix} 1 & \boxed{0} & \boxed{0} \\ \boxed{0} & \boxed{X_\varphi} & \boxed{0} \\ \boxed{0} & \boxed{0} & \boxed{X_\psi} \end{pmatrix} = \begin{pmatrix} 0 & \boxed{J\,X_\varphi} & \boxed{c\,K\,X_\psi} \\ \boxed{0} & \boxed{F\,X_\varphi} & \boxed{U\,K\,X_\psi} \\ \boxed{0} & \boxed{0} & \boxed{G\,X_\psi} \end{pmatrix} =$$

$$\begin{pmatrix} 0 & \boxed{J'} & \boxed{c\,K'} \\ \boxed{0} & \boxed{X_\varphi F} & \boxed{X_\varphi U'\,K'} \\ \boxed{0} & \boxed{0} & \boxed{X_\psi G'} \end{pmatrix} = \begin{pmatrix} 1 & \boxed{0} & \boxed{0} \\ \boxed{0} & \boxed{X_\varphi} & \boxed{0} \\ \boxed{0} & \boxed{0} & \boxed{X_\psi} \end{pmatrix} \cdot \begin{pmatrix} 0 & \boxed{J'} & \boxed{c\,K'} \\ \boxed{0} & \boxed{F'} & \boxed{U'\,K'} \\ \boxed{0} & \boxed{0} & \boxed{G'} \end{pmatrix} .$$

And for the star, the sequence of equalities:

$$(U\,c^*\,J + F)\,X_\varphi = (U\,c^*\,J)\,X_\varphi + F\,X_\varphi = X_\varphi\,(U'\,c^*\,J') + X_\varphi\,F = X_\varphi\,(U'\,c^*\,J' + F')$$

yields the result. ■

4. THE STANDARD DERIVED-TERM AUTOMATON OF AN EXPRESSION

By a process similar to the construction of $\mathcal{S}_{\mathsf{E}}$, though more involved, we associate now with every $\mathbb{K}$-expression E *another* standard automaton, the *standard derived-term automaton* $\mathcal{T}_{\mathsf{E}}$. We begin with the definition of the set $\mathrm{D}(\mathsf{E})$ of *derived terms* of E.

4.1. The derived terms of an expression

The set of derived terms is defined by induction on the formation of the expression.

Definition 19. The set $\mathrm{D}(\mathsf{E})$ of derived terms[3] of a $\mathbb{K}$-expression E over M is a set of $\mathbb{K}$-expressions defined inductively by:

Base cases

- $\mathsf{E} = 0$ or $\mathsf{E} = 1$ $\qquad \mathrm{D}(\mathsf{E}) = \emptyset.$ (8)
- $\mathsf{E} = m \quad m \in M \setminus 1_M$ $\qquad \mathrm{D}(\mathsf{E}) = \{1\}.$ (9)

Induction

- $\mathsf{E} = k\,\mathsf{F} \qquad \mathrm{D}(\mathsf{E}) = \mathrm{D}(\mathsf{F})\,.$ (10)
- $\mathsf{E} = \mathsf{F}\,k \qquad \mathrm{D}(\mathsf{E}) = \mathrm{D}(\mathsf{F})\;k = \{\mathsf{K}\,k \mid \mathsf{K} \in \mathrm{D}(\mathsf{F})\}\,.$ (11)
- $\mathsf{E} = \mathsf{F} + \mathsf{G} \qquad \mathrm{D}(\mathsf{E}) = \mathrm{D}(\mathsf{F}) \cup \mathrm{D}(\mathsf{G})\,.$ (12)
- $\mathsf{E} = \mathsf{F} \cdot \mathsf{G} \qquad \mathrm{D}(\mathsf{E}) = \mathrm{D}(\mathsf{F}) \cdot \mathsf{G} \cup \mathrm{D}(\mathsf{G}) = \{\mathsf{K} \cdot \mathsf{G} \mid \mathsf{K} \in \mathrm{D}(\mathsf{F})\} \cup \mathrm{D}(\mathsf{G})\,.$ (13)
- $\mathsf{E} = \mathsf{F}^* \qquad \mathrm{D}(\mathsf{E}) = \mathrm{D}(\mathsf{F}) \cdot \mathsf{F}^* = \{\mathsf{K} \cdot \mathsf{F}^* \mid \mathsf{K} \in \mathrm{D}(\mathsf{F})\}\,.$ (14)

Lemma 20. *Let* E *be a* $\mathbb{K}$*-expression over* M*. Then* $\mathrm{Card}(\mathrm{D}(\mathsf{E})) \leq \ell_{\mathsf{E}}\,.$

Proof. The equalityt holds for the base cases. Both litteral length and number of derived terms are invariant for dimension invariant operations. For the addition and product operations, $\ell_{\mathsf{F}+\mathsf{G}} = \ell_{\mathsf{F}\cdot\mathsf{G}} = \ell_{\mathsf{F}} + \ell_{\mathsf{G}}$ and $\mathrm{D}(\mathsf{F}+\mathsf{G})$ and $\mathrm{D}(\mathsf{F}\cdot\mathsf{G})$ are the union of sets each of which satisfies the inequality: they also satisfy the inequality, *all the more that the union may not be disjoint.* ■

Indeed, the interest, the subtility, and the difficulty, of the construction to come arise from the fact that the union in the definition of $\mathrm{D}(\mathsf{F}+\mathsf{G})$ and $\mathrm{D}(\mathsf{F}\cdot\mathsf{G})$ happens not to be disjoint.

Example 7 (Continued)**.** Let $\mathsf{F}_1 = a^*$, $\mathsf{G}_1 = (a^* + (-1)b^*)^*$ and $\mathsf{E}_1 = \mathsf{F}_1 \cdot \mathsf{G}_1$. It holds: $\mathrm{D}(\mathsf{F}_1) = \{a^*\}$, $\mathrm{D}(\mathsf{G}_1) = \{a^* \cdot \mathsf{G}_1, b^* \cdot \mathsf{G}_1\}$ and $\mathrm{D}(\mathsf{E}_1) = \mathrm{D}(\mathsf{G}_1)$.

4.2. The inductive definition of the standard derived-term automaton

With every $\mathbb{K}$-expression E, and by induction on the formation of E, we associate a standard automaton $\mathcal{T}_{\mathsf{E}}$ of dimension $\mathrm{D}(\mathsf{E})$, which we call the *standard derived-term automaton* of E.

Base cases

$\mathcal{T}_0 = \mathcal{S}_0$, $\mathcal{T}_1 = \mathcal{S}_1$, and $\mathcal{T}_m = \mathcal{S}_m$ for every m in M.

Dimension invariant operations

$\mathcal{T}_{k\,\mathsf{F}} = k\,\mathcal{T}_{\mathsf{F}}$, $\mathcal{T}_{\mathsf{F}\,k} = \mathcal{T}_{\mathsf{F}}\,k$, and $\mathcal{T}_{\mathsf{F}^*} = (\mathcal{T}_{\mathsf{F}})^*$.

Addition and product

- $\mathcal{T}_{\mathsf{F}} + \mathcal{T}_{\mathsf{G}}$ is a standard automaton of dimension $\mathrm{D}(\mathsf{F}) \sqcup \mathrm{D}(\mathsf{G})$. Let φ be the 'natural' map

$$\varphi\colon \mathrm{D}(\mathsf{F}) \sqcup \mathrm{D}(\mathsf{G}) \to \mathrm{D}(\mathsf{F}) \cup \mathrm{D}(\mathsf{G})\,,$$

that is, φ maps two terms K and K' of $\mathrm{D}(\mathsf{F}) \sqcup \mathrm{D}(\mathsf{G})$ onto one if they are *equal*, hence $\mathsf{K} \in \mathrm{D}(\mathsf{F})$, $\mathsf{K}' \in \mathrm{D}(\mathsf{G})$ and $\mathsf{K} = \mathsf{K}'$, or, to state it otherwise, if $\mathsf{K} \in \mathrm{D}(\mathsf{F}) \cap \mathrm{D}(\mathsf{G})$.

[3]The definition is the same as in [13] and all subsequent works of ours. We have changed the name from *true derived term* to *derived term* and the notation from $\mathrm{TD}\,(\mathsf{E})$ to $\mathrm{D}(\mathsf{E})$ for simplification, as the new presentation allows it.

Proposition 21. *The map φ is a morphism of automata.*

And we define

$$\mathcal{T}_{\mathsf{F+G}} = \varphi(\mathcal{T}_{\mathsf{F}} + \mathcal{T}_{\mathsf{G}}).$$

- $\mathcal{T}_{\mathsf{F}} \cdot \mathcal{T}_{\mathsf{G}}$ is a standard automaton of dimension $\mathrm{D}(\mathsf{F}) \sqcup \mathrm{D}(\mathsf{G})$ in bijection with $\mathrm{D}(\mathsf{F}) \cdot \mathsf{G} \sqcup \mathrm{D}(\mathsf{G})$. Let ψ be the 'natural' map

$$\psi \colon \mathrm{D}(\mathsf{F}) \cdot \mathsf{G} \sqcup \mathrm{D}(\mathsf{G}) \to \mathrm{D}(\mathsf{F}) \cdot \mathsf{G} \cup \mathrm{D}(\mathsf{G}).$$

that is, φ maps two terms K and K' of $\mathrm{D}(\mathsf{F}) \cdot \mathsf{G} \sqcup \mathrm{D}(\mathsf{G})$ onto one if they are *equal*, hence $\mathsf{K} \in \mathrm{D}(\mathsf{F}) \cdot \mathsf{G}$, $\mathsf{K}' \in \mathrm{D}(\mathsf{G})$ and $\mathsf{K} = \mathsf{K}'$, or, to state it otherwise, $\mathsf{K} \in \mathrm{D}(\mathsf{F}) \cdot \mathsf{G} \cap \mathrm{D}(\mathsf{G})$.

Proposition 22. *The map ψ is a morphism of automata.*

And we define

$$\mathcal{T}_{\mathsf{F \cdot G}} = \psi(\mathcal{T}_{\mathsf{F}} \cdot \mathcal{T}_{\mathsf{G}}).$$

This ends the inductive definition of $\mathcal{T}_{\mathsf{E}}$. Modulo the proof of Propositions 21 and 22 which is given below, this definition directly implies, by induction on the formation of the expression E and as a consequence of Propositions 17, 18, and 12.

Theorem 23. *For every valid $\mathbb{K}$-rational expression E, the standard $\mathbb{K}$-automaton $\mathcal{T}_{\mathsf{E}}$ realises the series denoted by E and is a quotient of $\mathcal{S}_{\mathsf{E}}$.*

Example 7 (Continued)**.** Let $\mathsf{F}_1 = a^*$, $\mathsf{G}_1 = (a^* + (-1)b^*)^*$ and $\mathsf{E}_1 = \mathsf{F}_1 \cdot \mathsf{G}_1$. We have seen that: $\mathrm{D}(\mathsf{F}_1) = \{a^*\}$, $\mathrm{D}(\mathsf{G}_1) = \{a^* \cdot \mathsf{G}_1, b^* \cdot \mathsf{G}_1\}$ and $\mathrm{D}(\mathsf{E}_1) = \mathrm{D}(\mathsf{G}_1)$.

It then comes $\mathcal{T}_{\mathsf{F}_1} = \mathcal{S}_{\mathsf{F}_1}$, $\mathcal{T}_{\mathsf{G}_1} = \mathcal{S}_{\mathsf{G}_1}$ and

$$\mathcal{T}_{\mathsf{F}_1} \cdot \mathcal{T}_{\mathsf{G}_1} = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a & a & -b \\ 0 & a & a & -b \\ 0 & 0 & 2a & -b \\ 0 & 0 & a & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle.$$

The derived term of F_1, a^*, multiplied by G_1, is equal to the first derived term of G_1, $a^* \cdot \mathsf{G}_1$. They index respectively the second and third rows and columns of the matrix above. If we add the second and third columns, we get a matrix whose second and third rows are equal, and the second and third entries of the final vector are also equal (instance of Proposition 22). These two states may then be merged to build the quotient and we get

$$\mathcal{T}_{\mathsf{E}_1} = \left\langle \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2a & -b \\ 0 & 2a & -b \\ 0 & a & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle.$$

4.3. Proof of Propositions 21 and 22

The construction of $\mathcal{T}_{\mathsf{E}}$ starts with the same automata as $\mathcal{S}_{\mathsf{E}}$ for the base cases. At every step, it uses an operation on standard automata, and possibly a morphism. Let us be more precise in the definition and notation for the term automaton.

4.3.1. Definitions and notation

Let F be a $\mathbb{K}$-expression over M. As we have seen, the standard automaton $\mathcal{T}_{\mathsf{F}}$ has dimension $\mathrm{D}(\mathsf{F})$ and we write

$$\mathcal{T}_{\mathsf{F}} = \left\langle \begin{pmatrix} 1 & \boxed{\;0\;} \end{pmatrix}, \begin{pmatrix} 0 & \boxed{\;J\;} \\ \boxed{0} & \boxed{\;F\;} \end{pmatrix}, \begin{pmatrix} x \\ \boxed{U} \end{pmatrix} \right\rangle . \tag{15}$$

The mere equation (15) implies that the scalar x, the vectors J and U of dimension $\mathrm{D}(\mathsf{F})$, as well as the matrix F of dimension $\mathrm{D}(\mathsf{F})\times\mathrm{D}(\mathsf{F})$, are also *associated with* F even though it does appear explicitely in the writing. When we need to make it more explicit, we write

$$J = \mathcal{J}(\mathsf{F}), \ F = \mathcal{F}(\mathsf{F}), \text{ and } U = \mathcal{U}(\mathsf{F}). \tag{16}$$

By (2) and Proposition 6, the scalar x is the constant term of F. By convention, we consider that the vectors $\mathcal{J}(\mathsf{F})$ and $\mathcal{U}(\mathsf{F})$ of dimension $\mathrm{D}(\mathsf{F})$ are also of dimension D, for any finite $\mathrm{D} \subset \mathbb{K}\mathsf{RatE}\,M$ that contains $\mathrm{D}(\mathsf{F})$, or that $\mathcal{F}(\mathsf{F})$ is a matrix of dimension $\mathrm{D}\times\mathrm{D}$, the 'missing' entries being set to $0_{\mathbb{K}}$.

4.3.2. The running claims and the preparatory lemmas

In order to be able to establish Propositions 21 and 22, that is, to settle the cases of addition and product operators, we have to maintain properties, 'the claims', all along the inductive process, hence for *all* operators. To this end, we also introduce another function $\mathcal{I}(\mathsf{F})$, which is a vector of dimension $\mathrm{D}(\mathsf{F})$ defined inductively as follow.

Base cases

- $\mathsf{E} = 0$ or $\mathsf{E} = 1$ $\qquad \mathcal{I}(\mathsf{E}) = \emptyset \qquad$ vector of dimension 0. (17)
- $\mathsf{E} = m \quad m \in M \setminus 1_M \qquad \mathcal{I}(\mathsf{E}) = (m).$ (18)

Dimension invariant operators

- $\mathsf{E} = k\,\mathsf{F} \qquad \mathcal{I}(\mathsf{E}) = k\,\mathcal{I}(\mathsf{F})\,.$ (19)
- $\mathsf{E} = \mathsf{F}\,k \qquad \mathcal{I}(\mathsf{E}) = \mathcal{I}(\mathsf{F})\,,$ (20)

  more precisely $\qquad \mathcal{I}(\mathsf{E})_{\mathsf{K}\,k} = \mathcal{I}(\mathsf{F})_{\mathsf{K}} \qquad \forall \mathsf{K} \in \mathrm{D}(\mathsf{F})\,.$
- $\mathsf{E} = \mathsf{F}^{*} \qquad \mathcal{I}(\mathsf{E}) = (\mathsf{c}(\mathsf{F}))^{*}\,\mathcal{I}(\mathsf{F})\,,$ (21)

  more precisely $\qquad \mathcal{I}(\mathsf{E})_{\mathsf{K}\,\mathsf{F}^{*}} = (\mathsf{c}(\mathsf{F}))^{*}\,\mathcal{I}(\mathsf{F})_{\mathsf{K}} \qquad \forall \mathsf{K} \in \mathrm{D}(\mathsf{F})\,.$

Addition and product

- $\mathsf{E} = \mathsf{F} + \mathsf{G} \qquad \mathcal{I}(\mathsf{E}) = \mathcal{I}(\mathsf{F}) + \mathcal{I}(\mathsf{G})\,,$ (22)

  i.e. $\qquad \mathcal{I}(\mathsf{E})_{\mathsf{K}} = \mathcal{I}(\mathsf{F})_{\mathsf{K}} + \mathcal{I}(\mathsf{G})_{\mathsf{K}} \qquad \forall \mathsf{K} \in \mathrm{D}(\mathsf{F}) \cup \mathrm{D}(\mathsf{G})\,.$
- $\mathsf{E} = \mathsf{F} \cdot \mathsf{G} \qquad \mathcal{I}(\mathsf{E}) = \mathcal{I}(\mathsf{F}) + \mathsf{c}(\mathsf{F})\,\mathcal{I}(\mathsf{G})\,,$ (23)

  i.e. $\qquad \mathcal{I}(\mathsf{E})_{\mathsf{K}} = \mathcal{I}(\mathsf{F})_{\mathsf{K}} + \mathsf{c}(\mathsf{F})\,\mathcal{I}(\mathsf{G})_{\mathsf{K}} \qquad \forall \mathsf{K} \in \mathrm{D}(\mathsf{F}) \cdot \mathsf{G} \cup \mathrm{D}(\mathsf{G})\,.$

The construction of $\mathcal{T}_{\mathsf{E}}$ goes with the verification, at every step of the induction, of the following properties.

Claim 1. *$J = \mathcal{J}(\mathsf{E}) = \mathcal{I}(\mathsf{E})$, that is,* $\forall \mathsf{K} \in \mathrm{D}(\mathsf{E}) \quad J_{\mathsf{K}} = \mathcal{I}(\mathsf{E})_{\mathsf{K}}$.

Claim 2. *$U = \mathcal{U}(\mathsf{E}) = \mathsf{c}(\mathrm{D}(\mathsf{E}))$, that is, $\forall \mathsf{K} \in \mathrm{D}(\mathsf{E}) \quad U_{\mathsf{K}} = \mathsf{c}(\mathsf{K})$.*

Claim 3. *For any K in $\mathrm{D}(\mathsf{E})$, the* row *of index K of $F = \mathcal{F}(\mathsf{E})$ is equal to $\mathcal{I}(\mathsf{K})$, that is,*

$$\forall \mathsf{K}, \mathsf{H} \in \mathrm{D}(\mathsf{E}) \qquad \mathcal{F}(\mathsf{E})_{\mathsf{K},\mathsf{H}} = \mathcal{I}(K)_{\mathsf{H}}\,. \tag{24}$$

The idea behind the definition of $\mathcal{I}(\mathsf{E})$ and the claims is that we have, at every step of the induction, the knowledge on $\mathcal{T}_{\mathsf{E}}$ necessary to prove that the maps φ or ψ are morphisms when the operators addition or product come into play. Before getting to the induction itself, we state some preparatory lemmas.

Lemma 24. *Let E be a $\mathbb{K}$-expression over M. If $\mathsf{K} \in \mathrm{D}(\mathsf{E})$, then $\mathrm{D}(\mathsf{K}) \subseteq \mathrm{D}(\mathsf{E})$.*

Lemma 24 will be used under the following form.

Lemma 25. *Let F be a $\mathbb{K}$-expression over M. If $\mathsf{H} \in \mathrm{D}(\mathsf{F})$, then $\mathrm{D}(\mathsf{H} \cdot \mathsf{F}^*) \subseteq \mathrm{D}(\mathsf{F}^*)$.*

4.3.3. The induction: The base cases

- $\mathsf{E} = 0$ then $\mathcal{T}_0 = \left\langle \begin{pmatrix}1\end{pmatrix}, \begin{pmatrix}0\end{pmatrix}, \begin{pmatrix}0\end{pmatrix} \right\rangle$.
- $\mathsf{E} = 0$ then $\mathcal{T}_1 = \left\langle \begin{pmatrix}1\end{pmatrix}, \begin{pmatrix}0\end{pmatrix}, \begin{pmatrix}1\end{pmatrix} \right\rangle$.

  In both cases, Claims 1, 2, and 3 are obvious by the emptyness of $\mathrm{D}(0)$ and $\mathrm{D}(1)$.
- $\mathsf{E} = m \in M$ then $\mathcal{T}_m = \left\langle \begin{pmatrix}1 & 0\end{pmatrix}, \begin{pmatrix}0 & m \\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 \\ 1\end{pmatrix} \right\rangle$.

  Claim 1 holds by (18), Claim 2 since $\mathrm{D}(m) = 1$. Since $\mathrm{D}(1)$ is empty, it follows from our convention that $\mathcal{I}(1)$ is the null vector of any dimension and we have here $\mathcal{F}(1)_{1,1} = 0 = \mathcal{I}(1)_1$.

4.3.4. The induction: The dimension invariant operations

- $\mathsf{E} = k\mathsf{F}$ then $\mathcal{T}_{\mathsf{E}} = k\mathcal{T}_{\mathsf{F}} = \left\langle \begin{pmatrix}1 & \boxed{\;0\;}\end{pmatrix}, \begin{pmatrix}0 & \boxed{kJ} \\ \boxed{0} & \boxed{F}\end{pmatrix}, \begin{pmatrix}kx \\ \boxed{U}\end{pmatrix} \right\rangle$.

Claim 1 holds by (20). Since Claim 2 and Claim 3 hold for $\mathcal{T}_{\mathsf{F}}$, they also hold for $\mathcal{T}_{\mathsf{E}}$ as $\mathrm{D}(\mathsf{E}) = \mathrm{D}(\mathsf{F})$, $\mathcal{U}(\mathsf{E}) = \mathcal{U}(\mathsf{F})$, and $\mathcal{F}(\mathsf{E}) = \mathcal{F}(\mathsf{F})$.

- $\mathsf{E} = \mathsf{F}k$ then $\mathcal{T}_{\mathsf{E}} = \mathcal{T}_{\mathsf{F}}\,k = \left\langle \begin{pmatrix}1 & \boxed{\;0\;}\end{pmatrix}, \begin{pmatrix}0 & \boxed{J} \\ \boxed{0} & \boxed{F}\end{pmatrix}, \begin{pmatrix}xk \\ \boxed{Uk}\end{pmatrix} \right\rangle$.

Claim 1 holds by (19). Claim 2 follows from

$$\forall \mathsf{K} \in \mathrm{D}(\mathsf{E}) \quad \mathsf{K} = \mathsf{H}k \quad \text{with} \quad \mathsf{H} \in \mathrm{D}(\mathsf{F}) \qquad \mathcal{U}(\mathsf{E})_K = \mathcal{U}(\mathsf{F})_H\, k = \mathsf{c}(\mathsf{H})\, k = \mathsf{c}(\mathsf{K}).$$

Claim 3 follows from the fact that by (20) $\mathcal{I}(\mathsf{H}k) = \mathcal{I}(\mathsf{H})$ and by (4), and the adequate renaming of row- and column-indices, $\mathcal{F}(\mathsf{F}k) = \mathcal{F}(\mathsf{F})$.

- $\mathsf{E} = \mathsf{F}^*$ then

$$\mathcal{T}_{\mathsf{E}} = (\mathcal{T}_{\mathsf{F}})^* = \left\langle \begin{pmatrix}1 & \boxed{\;0\;}\end{pmatrix}, \begin{pmatrix}0 & \boxed{x^*J} \\ \boxed{0} & \boxed{H}\end{pmatrix}, \begin{pmatrix}x^* \\ \boxed{Ux^*}\end{pmatrix} \right\rangle$$

with $H = U \cdot x^*J + F$.

Claim 1 holds by (21). Claim 2 follows from

$$\forall \mathsf{K} \in \mathrm{D}(\mathsf{E}) \quad \mathsf{K} = \mathsf{H} \cdot \mathsf{F}^* \quad \text{with} \quad \mathsf{H} \in \mathrm{D}(\mathsf{F}) \quad \text{and} \quad \mathcal{U}(\mathsf{E})_{\mathsf{K}} = \mathcal{U}(\mathsf{F})_{\mathsf{H}}\, x^* = \mathsf{c}(\mathsf{H})\,\mathsf{c}(\mathsf{F}^*) = \mathsf{c}(\mathsf{K}).$$

In order to prove Claim 3, let K in $\mathrm{D}(\mathsf{E})$, hence $\mathsf{K} = \mathsf{H} \cdot \mathsf{F}^*$ with H in $\mathrm{D}(\mathsf{F})$. By (23),

$$\mathcal{I}(\mathsf{K}) = \mathcal{I}(\mathsf{H}) + \mathsf{c}(\mathsf{H})\,\mathcal{I}(\mathsf{F}^*) = \mathcal{I}(\mathsf{H}) + \mathsf{c}(\mathsf{H})\,x^*\,\mathcal{I}(\mathsf{F})\,.$$

By Lemma 25, $\mathrm{D}(\mathsf{K}) \subseteq \mathrm{D}(\mathsf{F}^*)$, that is the dimension of $\mathcal{I}(\mathsf{K})$ is contained in $\mathrm{D}(\mathsf{E})$. By induction the claims imply that $\mathcal{I}(\mathsf{H})$ is the row of index H of $\mathcal{F}(\mathsf{F}) = F$ and $\mathsf{c}(\mathsf{H}) = \mathcal{U}(\mathsf{F})_{\mathsf{H}} = U_{\mathsf{H}}$. Hence (25) tells that $\mathcal{I}(\mathsf{K})$ is equal to the row of index K of $\mathcal{F}(\mathsf{F}^*) = H$.

4.3.5. The operations addition and product

In addition to the notation taken in (15), let G be another $\mathbb{K}$-expression and

$$\mathcal{T}_{\mathsf{G}} = \left\langle \begin{pmatrix} 1 & \boxed{0} \end{pmatrix}, \begin{pmatrix} 0 & \boxed{K} \\ \boxed{0} & \boxed{G} \end{pmatrix}, \begin{pmatrix} y \\ \boxed{V} \end{pmatrix} \right\rangle$$

its standard term automaton, which fulfil the running claims.

- $\mathsf{E} = \mathsf{F} + \mathsf{G}$. We first form

$$\mathcal{T}_{\mathsf{F}} + \mathcal{T}_{\mathsf{G}} = \left\langle \begin{pmatrix} 1 & \boxed{0} & \boxed{0} \end{pmatrix}, \begin{pmatrix} 0 & \boxed{J} & \boxed{K} \\ \boxed{0} & \boxed{F} & \boxed{0} \\ \boxed{0} & \boxed{0} & \boxed{G} \end{pmatrix}, \begin{pmatrix} x+y \\ \boxed{U} \\ \boxed{V} \end{pmatrix} \right\rangle,$$

a standard automaton of dimension $\mathrm{D}(\mathsf{F}) \sqcup \mathrm{D}(\mathsf{G})$. Let φ be the map

$$\varphi \colon \mathrm{D}(\mathsf{F}) \sqcup \mathrm{D}(\mathsf{G}) \to \mathrm{D}(\mathsf{F}) \cup \mathrm{D}(\mathsf{G}) = \mathrm{D}(\mathsf{E})\,,$$

that maps two terms K and K' of $\mathrm{D}(\mathsf{F}) \sqcup \mathrm{D}(\mathsf{G})$ onto one if they are *equal*, hence if $\mathsf{K} \in \mathrm{D}(\mathsf{F})$, $\mathsf{K}' \in \mathrm{D}(\mathsf{G})$ and $\mathsf{K} = \mathsf{K}'$, or, to state it otherwise, if $\mathsf{K} \in \mathrm{D}(\mathsf{F}) \cap \mathrm{D}(\mathsf{G})$.

By Claim 2 and Claim 3 for $\mathcal{T}_{\mathsf{F}}$ and $\mathcal{T}_{\mathsf{G}}$, if $\mathsf{K} \in \mathrm{D}(\mathsf{F}) \cap \mathrm{D}(\mathsf{G})$ then $U_{\mathsf{K}} = V_{\mathsf{K}}$ and $F_{\mathsf{K},.} = G_{\mathsf{K},.} = \mathcal{I}(\mathsf{K})$. This is sufficient for φ to be a morphism of automata and establishes Proposition 21.

With our convention, both J and K can be considered as vectors of dimension $\mathrm{D}(\mathsf{F}) \cup \mathrm{D}(\mathsf{G}) = \mathrm{D}(\mathsf{E})$. The image $\varphi(\mathcal{T}_{\mathsf{F}} + \mathcal{T}_{\mathsf{G}})$ is $\mathcal{T}_{\mathsf{E}}$ and can be written

$$\mathcal{T}_{\mathsf{E}} = \left\langle \begin{pmatrix} 1 & \boxed{0} \end{pmatrix}, \begin{pmatrix} 0 & \boxed{J+K} \\ \boxed{0} & \boxed{H} \end{pmatrix}, \begin{pmatrix} x+y \\ \boxed{W} \end{pmatrix} \right\rangle$$

where H is the 'fusion' of F and G and W is the 'fusion' of U and V.

Claim 1 then holds by (22). Claim 2 and Claim 3 are directly inherited from the corresponding properties for $\mathcal{T}_{\mathsf{F}}$ and $\mathcal{T}_{\mathsf{G}}$ (and the convention).

- $\mathsf{E} = \mathsf{F} \cdot \mathsf{G}$. We first form

$$\mathcal{T}_{\mathsf{F}} \cdot \mathcal{T}_{\mathsf{G}} = \left\langle \begin{pmatrix} 1 & \boxed{0} & \boxed{0} \end{pmatrix}, \begin{pmatrix} 0 & \boxed{J} & \boxed{xK} \\ \boxed{0} & \boxed{F} & \boxed{U \cdot K} \\ \boxed{0} & \boxed{0} & \boxed{G} \end{pmatrix}, \begin{pmatrix} xy \\ \boxed{Uy} \\ \boxed{V} \end{pmatrix} \right\rangle,$$

a standard automaton *a priori* of dimension $\mathrm{D}(\mathsf{F}) \sqcup \mathrm{D}(\mathsf{G})$, but which we consider as a standard automaton of dimension $\mathrm{D}(\mathsf{F}) \cdot \mathsf{G} \sqcup \mathrm{D}(\mathsf{G})$, that is, we multiply all indices from $\mathrm{D}(\mathsf{F})$ by G on the right.

Let ψ be the 'natural' map

$$\psi \colon \mathrm{D}(\mathsf{F}) \cdot \mathsf{G} \sqcup \mathrm{D}(\mathsf{G}) \to \mathrm{D}(\mathsf{F}) \cdot \mathsf{G} \cup \mathrm{D}(\mathsf{G}) = \mathrm{D}(\mathsf{E})\,,$$

that is, ψ maps two terms K and K' of $\mathrm{D}(\mathsf{F}) \cdot \mathsf{G} \sqcup \mathrm{D}(\mathsf{G})$ onto one if they are *equal*, hence if $\mathsf{K} \in \mathrm{D}(\mathsf{F}) \cdot \mathsf{G}$, $\mathsf{K}' \in \mathrm{D}(\mathsf{G})$ and $\mathsf{K} = \mathsf{K}'$, or, to state it otherwise, if $\mathsf{K} \in \mathrm{D}(\mathsf{F}) \cdot \mathsf{G} \cap \mathrm{D}(\mathsf{G})$.

Let K be such an expression, that is, $\mathsf{K} = \mathsf{H} \cdot \mathsf{G}$ with H in $\mathrm{D}(\mathsf{F})$ and K belongs to $\mathrm{D}(\mathsf{G})$. We consider first the final vector of $\mathcal{T}_{\mathsf{E}}$.

By Claim 2, we have on one hand $\mathcal{U}(\mathsf{G})_{\mathsf{K}} = V_{\mathsf{K}} = \mathsf{c}(\mathsf{K})$ and on the other hand $\mathsf{c}(\mathsf{K}) = \mathsf{c}(\mathsf{H})\,\mathsf{c}(\mathsf{G}) = U_{\mathsf{H}}\, y$. Hence, the two entries of index K of $U y$ and V are equal.

We then consider $\mathcal{F}(\mathsf{E})$. By Claim 3, the row of index K in G is $\mathcal{I}(\mathsf{K})$ which, by (23), is written as $\mathcal{I}(\mathsf{K}) = \mathcal{I}(\mathsf{H}) + \mathsf{c}(\mathsf{H})\, \mathcal{I}(\mathsf{G})$, that is,

$$\forall \mathsf{L} \in \mathrm{D}(\mathsf{G}) \qquad \mathcal{I}(\mathsf{K})_{\mathsf{L}} = \mathcal{I}(\mathsf{H})_{\mathsf{L}} + \mathsf{c}(\mathsf{H})\, \mathcal{I}(\mathsf{G})_{\mathsf{L}}\,,$$

which implies in particular that the non-zero entries of $\mathcal{I}(\mathsf{H})$ all correspond to derived terms of G.

The same Claim 3 on the other hand implies that the row of index H of F (of index $\mathsf{H} \cdot \mathsf{G}$ in $\mathcal{T}_{\mathsf{E}}$) is $\mathcal{I}(\mathsf{H})$. The row of index H of the matrix $U \cdot K$ is $\mathsf{c}(\mathsf{H})\, \mathcal{I}(\mathsf{G})$. If we sum all entries of equal index in $\mathrm{D}(\mathsf{F}) \cdot \mathsf{G}$ on one hand and in $\mathrm{D}(\mathsf{G})$ on the other hand, we obtain a row-vector Z such that

$$\forall \mathsf{L} \in \mathrm{D}(\mathsf{F}) \cdot \mathsf{G} \cup \mathrm{D}(\mathsf{G}) \qquad Z_{\mathsf{L}} = \mathcal{I}(\mathsf{H})_{\mathsf{L}} + \mathsf{c}(\mathsf{H})\, \mathcal{I}(\mathsf{G})_{\mathsf{L}}\,,$$

and, with our convention, $Z = \mathcal{I}(\mathsf{K})$.

Together with the property shown above for V and $U y$ this proves that ψ is a morphism of automata and Proposition 22 is established.

Moreover, the same computations establish Claims 1 to 3 for $\psi(\mathcal{T}_{\mathsf{F}\cdot\mathsf{G}})$ and by this fact, complete the definition of the standard derived-term automaton.

4.4. The derived-term automaton

Finally, let us define yet another automaton associated with an expression E which is indeed the one we are ultimately aiming at. By convention, we consider that the initial state of $\mathcal{T}_{\mathsf{E}}$ is indexed by E. If E belongs to $\mathrm{D}(\mathsf{E})$, let ω be the 'natural' map

$$\omega \colon \mathsf{E} \sqcup \mathrm{D}(\mathsf{E}) \to \mathrm{D}(\mathsf{E})\,.$$

Proposition 26. *The map ω is a morphism of automata.*

Proof. Let

$$\mathcal{T}_{\mathsf{E}} = \left\langle \begin{pmatrix} 1 & \boxed{\;0\;} \end{pmatrix}, \begin{pmatrix} 0 & \boxed{\;J\;} \\ \boxed{0} & \boxed{\;F\;} \end{pmatrix}, \begin{pmatrix} x \\ \boxed{U} \end{pmatrix} \right\rangle.$$

By Claim 1, $J = \mathcal{I}(\mathsf{E})$. If E is in $\mathrm{D}(\mathsf{E})$, then, by Claim 2, $U_{\mathsf{E}} = \mathsf{c}(\mathsf{E}) = x$ and, by Claim 3, $F_{\mathsf{E},.} = \mathcal{I}(\mathsf{E})$. These equalities tell that ω is a morphism of automata. ■

Definition 27. For every valid $\mathbb{K}$-rational expression E, *the derived-term automaton* $\mathcal{D}_{\mathsf{E}}$ of E is defined by $\mathcal{D}_{\mathsf{E}} = \omega(\mathcal{T}_{\mathsf{E}})$ if E is in $\mathrm{D}(\mathsf{E})$ and $\mathcal{D}_{\mathsf{E}} = \mathcal{T}_{\mathsf{E}}$ otherwise.

We then finally can state:

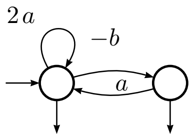
Theorem 28. *For every valid $\mathbb{K}$-rational expression E, the derived-term automaton $\mathcal{D}_{\mathsf{E}}$ is a quotient of the standard automaton of E, $\mathcal{S}_{\mathsf{E}}$ (and hence realises the series denoted by E).*

Example 7 (Continued)**.** Let $\mathsf{E}_1 = a^* \cdot (a^* + (-1)b^*)^*$.
We have seen that $\mathrm{D}(\mathsf{E}_1) = \{a^* \cdot (a^* + (-1)b^*)^*,\ b^* \cdot (a^* + (-1)b^*)^*\}$ and

$$\mathcal{T}_{\mathsf{E}_1} = \left\langle \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2a & -b \\ 0 & 2a & -b \\ 0 & a & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle.$$

It holds that E_1 is in $\mathrm{D}(\mathsf{E}_1)$ and we observe that the first and second lines of the matrix, as well as the first and second entries of the final vector, both indexed by instances of the derived term E_1, are equal. The quotient of $\mathcal{T}_{\mathsf{E}_1}$ by the morphism ω is

$$\mathcal{D}_{\mathsf{E}_1} = \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 2a & -b \\ a & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle \qquad \text{drawn as}$$



The expression E_1 has also the property that the 'Thompson construction' (when generalised to weighted automata) applied to it yields a *non-valid* automaton (see [14]).

5. BACK TO DERIVATION

Finally, we reconnect this work with the previous ones and show that the derived-term automaton we have just described coincides — in the case where M is a free monoid — with the automaton defined by the derivation of expressions process introduced in [1] for Boolean automata and in [13] for weighted automata (see also [18, 19, 20]).

5.1. Preparation: The differential of an expression

We begin with a definition and a property that are valid in the case of general (graded) monoids. The specialisation to the case of free monoids allows a particular writing that will be used in the sequel.

Definition 29. Let E be a $\mathbb{K}$-expression over M. The *differential of* E, denoted by $\mathrm{d}\,\mathsf{E}$, is the expression

$$\mathrm{d}\,\mathsf{E} = \sum_{\mathsf{H} \in \mathrm{D}(\mathsf{E})} \mathcal{I}(\mathsf{E})_{\mathsf{H}} \cdot \mathsf{H}. \tag{25}$$

Equation (25) allows to write a 'first-order development' of the expression *via* the following statement.

Proposition 30. $|\mathsf{E}| = \mathsf{c}(\mathsf{E}) + |\mathrm{d}\,\mathsf{E}|$.

Proof. By induction on the formation of E. Proposition 30, which we rather write under the form $|\mathsf{E}| = \mathsf{c}(\mathsf{E}) + \sum_{\mathsf{H}\in\mathrm{D}(\mathsf{E})} \mathcal{I}(\mathsf{E})_{\mathsf{H}} \,|\mathsf{H}|$, is based on Equations (17) to (23) which have been established with the construction of the standard derived-term automaton.

Base cases

- $\mathsf{E} = 0$ and $\mathsf{E} = 0$ obvious by the emptyness of $\mathrm{D}(\mathsf{E})$.
- $\mathsf{E} = m \in M$ as obvious since $|m| = m$, $\mathsf{c}(m) = 0_{\mathbb{K}}$, $\mathrm{D}(m) = 1$ and $\mathcal{I}(m)_1 = m$.

Induction

- $\mathsf{E} = k\,\mathsf{F}$ $\quad \mathsf{c}(k\,\mathsf{F}) = k\,\mathsf{c}(\mathsf{F})$, $\mathrm{D}(k\,\mathsf{F}) = \mathrm{D}(\mathsf{F})$ and $\mathcal{I}(k\,\mathsf{F}) = k\,\mathcal{I}(\mathsf{F})$,
  hence $\quad \mathsf{c}(\mathsf{E}) + |\mathrm{d}\,\mathsf{E}| = k\,\mathsf{c}(\mathsf{F}) + k\,\sum_{\mathsf{H}\in\mathrm{D}(\mathsf{F})} \mathcal{I}(\mathsf{F})_{\mathsf{H}}\,|\mathsf{H}| = k\,|\mathsf{F}| = |\mathsf{E}|$.

- $\mathsf{E} = \mathsf{F}\,k$ $\quad \mathsf{c}(\mathsf{F}\,k) = \mathsf{c}(\mathsf{F})\,k$, $\mathrm{D}(\mathsf{F}\,k) = \mathrm{D}(\mathsf{F})\;k$ and $\mathcal{I}(\mathsf{F}\,k) = \mathcal{I}(\mathsf{F})$,
  hence $\quad \mathsf{c}(\mathsf{E}) + |\mathrm{d}\,\mathsf{E}| = \mathsf{c}(\mathsf{F})\,k + \sum_{\mathsf{H}\in\mathrm{D}(\mathsf{F})} \mathcal{I}(\mathsf{F})_{\mathsf{H}}\,|\mathsf{H}\,k| = |\mathsf{F}|\,k = |\mathsf{E}|$.

- $\mathsf{E} = \mathsf{F}+\mathsf{G}$ $\quad \mathsf{c}(\mathsf{F}+\mathsf{G}) = \mathsf{c}(\mathsf{F}) + \mathsf{c}(\mathsf{G})$, $\mathrm{D}(\mathsf{F}+\mathsf{G}) = \mathrm{D}(\mathsf{F}) \cup \mathrm{D}(\mathsf{G})$ and $\mathcal{I}(\mathsf{F}+\mathsf{G}) = \mathcal{I}(\mathsf{F}) + \mathcal{I}(\mathsf{G})$,
  hence $\quad \mathsf{c}(\mathsf{E}) + |\mathrm{d}\,\mathsf{E}| = \mathsf{c}(\mathsf{F}) + \mathsf{c}(\mathsf{G}) + \sum_{\mathsf{H}\in\mathrm{D}(\mathsf{F})\cup\mathrm{D}(\mathsf{F})} (\mathcal{I}(\mathsf{F}) + \mathcal{I}(\mathsf{G}))\,|\mathsf{H}| = |\mathsf{F}| + |\mathsf{F}| = |\mathsf{E}|$.

- $\mathsf{E} = \mathsf{F}\cdot\mathsf{G}$ $\quad \mathsf{c}(\mathsf{F}\cdot\mathsf{G}) = \mathsf{c}(\mathsf{F})\,\mathsf{c}(\mathsf{G})$, $\mathrm{D}(\mathsf{F}\cdot\mathsf{G}) = \mathrm{D}(\mathsf{F})\cdot\mathsf{G} \cup \mathrm{D}(\mathsf{G})$ and $\mathcal{I}(\mathsf{F}\cdot\mathsf{G}) = \mathcal{I}(\mathsf{F}) + \mathsf{c}(\mathsf{F})\,\mathcal{I}(\mathsf{G})$,
  more precisely:
  $\forall \mathsf{H} \in \mathrm{D}(\mathsf{F}) \quad \mathcal{I}(\mathsf{F}\cdot\mathsf{G})_{\mathsf{H}\cdot\mathsf{G}} = \mathcal{I}(\mathsf{F})_{\mathsf{H}} + \mathsf{c}(\mathsf{F})\,\mathcal{I}(\mathsf{G})_{\mathsf{H}\cdot\mathsf{G}}$, and
  $\forall \mathsf{K} \in \mathrm{D}(\mathsf{G}) \setminus \mathrm{D}(\mathsf{F})\cdot\mathsf{G} \quad \mathcal{I}(\mathsf{F}\cdot\mathsf{G})_{\mathsf{K}} = \mathsf{c}(\mathsf{F})\,\mathcal{I}(\mathsf{G})_{\mathsf{K}}$.

  It then comes

$$\begin{aligned}
\mathsf{c}(\mathsf{E}) + |\mathrm{d}\,\mathsf{E}| &= \mathsf{c}(\mathsf{F})\,\mathsf{c}(\mathsf{G}) + \sum_{\mathsf{K}\in\mathrm{D}(\mathsf{E})} \mathcal{I}(\mathsf{E})_{\mathsf{K}}\,|\mathsf{K}| \\
&= \mathsf{c}(\mathsf{F})\,\mathsf{c}(\mathsf{G}) + \sum_{\mathsf{H}\in\mathrm{D}(\mathsf{F})} \mathcal{I}(\mathsf{F})_{\mathsf{H}}\,|\mathsf{H}\cdot\mathsf{G}| + \mathsf{c}(\mathsf{F}) \sum_{\mathsf{H}\in\mathrm{D}(\mathsf{F})} \mathcal{I}(\mathsf{G})_{\mathsf{H}\cdot\mathsf{G}}\,|\mathsf{H}\cdot\mathsf{G}| \\
&\qquad + \mathsf{c}(\mathsf{F}) \sum_{\mathsf{K}\in\mathrm{D}(\mathsf{G})\setminus\mathrm{D}(\mathsf{F})\cdot\mathsf{G}} \mathcal{I}(\mathsf{G})_{\mathsf{K}}\,|\mathsf{K}| \\
&= \mathsf{c}(\mathsf{F})\,\mathsf{c}(\mathsf{G}) + \left(\sum_{\mathsf{H}\in\mathrm{D}(\mathsf{F})} \mathcal{I}(\mathsf{F})_{\mathsf{H}}\,|\mathsf{H}|\right) |\mathsf{G}| + \mathsf{c}(\mathsf{F}) \sum_{\mathsf{K}\in\mathrm{D}(\mathsf{G})} \mathcal{I}(\mathsf{G})_{\mathsf{K}}\,|\mathsf{K}| \\
&= \mathsf{c}(\mathsf{F}) \left(\mathsf{c}(\mathsf{G}) + \sum_{\mathsf{K}\in\mathrm{D}(\mathsf{G})} \mathcal{I}(\mathsf{G})_{\mathsf{K}}\,|\mathsf{K}|\right) + \left(\sum_{\mathsf{H}\in\mathrm{D}(\mathsf{F})} \mathcal{I}(\mathsf{F})_{\mathsf{H}}\,|\mathsf{H}|\right) |\mathsf{G}| \\
&= |\mathsf{F}|\,|\mathsf{G}| = |\mathsf{E}|.
\end{aligned}$$

- $\mathsf{E} = \mathsf{F}^*$ $\quad \mathsf{c}(\mathsf{F}^*) = (\mathsf{c}(\mathsf{F}))^*$, $\mathrm{D}(\mathsf{F}^*) = \mathrm{D}(\mathsf{F})\cdot\mathsf{F}^*$ and $\mathcal{I}(\mathsf{F}^*) = (\mathsf{c}(\mathsf{F}))^*\,\mathcal{I}(\mathsf{F})$, hence

$$\begin{aligned}
\mathsf{c}(\mathsf{E}) + |\mathrm{d}\,\mathsf{E}| &= (\mathsf{c}(\mathsf{F}))^* + (\mathsf{c}(\mathsf{F}))^* \sum_{\mathsf{K}\in\mathrm{D}(\mathsf{F})} \mathcal{I}(\mathsf{F})_{\mathsf{K}}\,|\mathsf{K}\cdot\mathsf{F}^*| \\
&= (\mathsf{c}(\mathsf{F}))^* + (\mathsf{c}(\mathsf{F}))^* \left(\sum_{\mathsf{K}\in\mathrm{D}(\mathsf{F})} \mathcal{I}(\mathsf{F})_{\mathsf{K}}\,|\mathsf{K}|\right) |\mathsf{F}^*|.
\end{aligned}$$

The term $\sum_{\mathsf{K}\in\mathrm{D}(\mathsf{F})} \mathcal{I}(\mathsf{F})_{\mathsf{K}}\,|\mathsf{K}|$ is the *proper part* $|\mathsf{F}|_{\mathsf{p}}$ of $|\mathsf{F}|$. Let us write $x = \mathsf{c}(\mathsf{F})$. It then comes

$$\begin{aligned}
\mathsf{c}(\mathsf{E}) + |\mathrm{d}\,\mathsf{E}| &= x^* + x^*|\mathsf{F}|_{\mathsf{p}}\,(|\mathsf{F}|)^* = x^* + x^*|\mathsf{F}|_{\mathsf{p}}\,x^*\,(|\mathsf{F}|_{\mathsf{p}}\,x^*)^* = x^*\,(1_{\mathbb{K}} + |\mathsf{F}|_{\mathsf{p}}\,x^*\,(|\mathsf{F}|_{\mathsf{p}}\,x^*)^*) \\
&= x^*\,(|\mathsf{F}|_{\mathsf{p}}\,x^*)^* = (|\mathsf{F}|)^* = |\mathsf{E}|.
\end{aligned}$$

■

If $M = A^*$ is a free monoid, every entry of $\mathcal{I}(\mathsf{E})$ is a linear combination of letters in A. In (25), we can reorder the terms and see the vector $\mathcal{I}(\mathsf{E})$ as the sum $\mathbb{K}$-vectors of dimension $\mathrm{D}(\mathsf{E})$ multiplied by the letters of A

$$\mathcal{I}(\mathsf{E}) = \sum_{a \in A} \langle \mathcal{I}(\mathsf{E}) \,, a \rangle \cdot a,$$

and the differential becomes

$$\mathrm{d}\,\mathsf{E} = \sum_{a \in A} a \cdot \sum_{\mathsf{H} \in \mathrm{D}(\mathsf{E})} \langle \mathcal{I}(\mathsf{E}) \,, a \rangle_{\mathsf{H}} \, \mathsf{H}. \tag{26}$$

As recalled in the introduction, the *quotient operation* may be defined on languages

$$\forall L \in \mathfrak{P}\left(A^*\right), \ \forall u \in A^* \qquad u^{-1}L = \{v \in A^* \mid u\,v \in L\},$$

and on series over a free monoid

$$\forall s \in \mathbb{K}\langle\langle A^* \rangle\rangle \,, \ \forall u \in A^* \quad u^{-1}s \quad \text{is defined by} \quad \forall v \in A^* \quad \langle u^{-1}s, v \rangle = \langle s, u\,v \rangle.$$

From Proposition 30 and (26), directly follows then

$$a^{-1}|\mathsf{E}| = \sum_{\mathsf{H} \in \mathrm{D}(\mathsf{E})} \langle \mathcal{I}(\mathsf{E}) \,, a \rangle_{\mathsf{H}} \, \mathsf{H}.$$

5.2. The derivation of an expression

The result of the *derivation of a (Boolean) expression*, as defined by Antimirov in [1] after modification of the definition of *derivatives* by Brzozowski [4], is a *set of expressions*. The result of the *derivation of a weighted expression*, which we have defined in [13] as a direct generalisation of the former, is a *linear combination of (weighted) expressions*.

Definition 31. Let E be a $\mathbb{K}$-expression over A^* and a in A. The *derivation* of E with respect to a, denoted by $\frac{\partial}{\partial a}\mathsf{E}$, is a linear combination of expressions in $\mathbb{K}\mathsf{RatE}\,A^*$, inductively defined by the following formulas.

Base cases

- $$\frac{\partial}{\partial a} 0 = \frac{\partial}{\partial a} 1 = 0_{\mathbb{K}}. \tag{27}$$
- $$\frac{\partial}{\partial a} b = \begin{cases} 1_{\mathbb{K}} & \text{if} \quad b = a, \\ 0_{\mathbb{K}} & \text{otherwise.} \end{cases} \tag{28}$$

Induction

- $$\frac{\partial}{\partial a}(k\,\mathsf{F}) = k\,\frac{\partial}{\partial a}\mathsf{F}. \tag{29}$$
- $$\frac{\partial}{\partial a}(\mathsf{F}\,k) = \left(\left[\frac{\partial}{\partial a}\mathsf{F}\right]\,k\right). \tag{30}$$
- $$\frac{\partial}{\partial a}(\mathsf{F}+\mathsf{G}) = \frac{\partial}{\partial a}\mathsf{F} \oplus \frac{\partial}{\partial a}\mathsf{G}. \tag{31}$$
- $$\frac{\partial}{\partial a}(\mathsf{F}\cdot\mathsf{G}) = \left(\left[\frac{\partial}{\partial a}\mathsf{F}\right]\cdot\mathsf{G}\right) \oplus \mathsf{c}(\mathsf{F})\,\frac{\partial}{\partial a}\mathsf{G}. \tag{32}$$
- $$\frac{\partial}{\partial a}(\mathsf{F}^*) = \mathsf{c}(\mathsf{F})^*\left(\left[\frac{\partial}{\partial a}\mathsf{F}\right]\cdot\mathsf{F}^*\right). \tag{33}$$

5.3. The reconciliation

Theorem 32. *Let* E *be a* $\mathbb{K}$*-expression over* A^* *and* a *in* A*. The derivation of* E *with respect to* a *is the coefficient of* a *in* $\mathrm{d}\,\mathsf{E}$

$$\frac{\partial}{\partial a}\,\mathsf{E} = \sum_{\mathsf{H}\in\mathrm{D}(\mathsf{E})} \langle \mathcal{I}(\mathsf{E})\,, a\rangle_{\mathsf{H}}\,\mathsf{H}. \tag{34}$$

A direct consequence of this statement is the fact that derivation is the lifting of the quotient of series at the level of expressions.

Corollary 33. $\left|\dfrac{\partial}{\partial a}\,\mathsf{E}\right| = a^{-1}|\mathsf{E}|$.

Proof. [Proof of Theorem 32] It is less a proof than a mere verification without mystery, by induction on the formation of E, and based on Equations (17) to (23).

Base cases

- $\mathsf{E}=\mathsf{0}$ and $\mathsf{E}=\mathsf{0}$ obvious by the emptyness of $\mathrm{D}(\mathsf{E})$.
- $\mathsf{E}=a\in A$ as obvious since $\mathrm{D}(a)=1$ and $\mathcal{I}(a)_{\mathsf{1}}=a$.

Induction

- $\mathsf{E}=k\,\mathsf{F}$ $\quad \dfrac{\partial}{\partial a}(k\,\mathsf{F}) = k\,\dfrac{\partial}{\partial a}\,\mathsf{F}$ on one hand-side,
  $\mathrm{D}(k\,\mathsf{F})=\mathrm{D}(\mathsf{F})$ and $\mathcal{I}(k\,\mathsf{F})=k\,\mathcal{I}(\mathsf{F})$ on the other;
  if (34) holds for F, it holds for $k\,\mathsf{F}$.
- $\mathsf{E}=\mathsf{F}\,k$ $\quad \dfrac{\partial}{\partial a}(\mathsf{F}\,k) = \dfrac{\partial}{\partial a}\,\mathsf{F}\,k$ on one hand-side,
  $\mathrm{D}(\mathsf{F}\,k)=\mathrm{D}(\mathsf{F})\;k$ and $\mathcal{I}(\mathsf{F}\,k)=\mathcal{I}(\mathsf{F})$ on the other;
  if (34) holds for F, it holds for $\mathsf{F}\,k$.
- $\mathsf{E}=\mathsf{F}+\mathsf{G}$ $\quad \dfrac{\partial}{\partial a}(\mathsf{F}+\mathsf{G}) = \dfrac{\partial}{\partial a}\,\mathsf{F}\oplus\dfrac{\partial}{\partial a}\,\mathsf{G}$ on one hand-side,
  $\mathrm{D}(\mathsf{F}+\mathsf{G})=\mathrm{D}(\mathsf{F})\cup\mathrm{D}(\mathsf{G})$ and $\mathcal{I}(\mathsf{F}+\mathsf{G})=\mathcal{I}(\mathsf{F})+\mathcal{I}(\mathsf{G})$ on the other;
  if (34) holds for F and G, it holds for $\mathsf{F}+\mathsf{G}$.
- $\mathsf{E}=\mathsf{F}\cdot\mathsf{G}$ $\quad \dfrac{\partial}{\partial a}(\mathsf{F}\cdot\mathsf{G}) = \left(\left[\dfrac{\partial}{\partial a}\,\mathsf{F}\right]\cdot\mathsf{G}\right)\oplus\mathsf{c}(\mathsf{F})\,\dfrac{\partial}{\partial a}\,\mathsf{G}$ on one hand-side,
  $\mathrm{D}(\mathsf{F}\cdot\mathsf{G})=\mathrm{D}(\mathsf{F})\cdot\mathsf{G}\cup\mathrm{D}(\mathsf{G})$ and $\mathcal{I}(\mathsf{F}\cdot\mathsf{G})=\mathcal{I}(\mathsf{F})+\mathsf{c}(\mathsf{F})\,\mathcal{I}(\mathsf{G})$ on the other;
  more precisely:
  $\forall\mathsf{H}\in\mathrm{D}(\mathsf{F})\quad \mathcal{I}(\mathsf{F}\cdot\mathsf{G})_{\mathsf{H}\cdot\mathsf{G}}=\mathcal{I}(\mathsf{F})_{\mathsf{H}}+\mathsf{c}(\mathsf{F})\,\mathcal{I}(\mathsf{G})_{\mathsf{H}\cdot\mathsf{G}}$ and
  $\forall\mathsf{K}\in\mathrm{D}(\mathsf{G})\setminus\mathrm{D}(\mathsf{F})\cdot\mathsf{G}\quad \mathcal{I}(\mathsf{F}\cdot\mathsf{G})_{\mathsf{K}}=\mathsf{c}(\mathsf{F})\,\mathcal{I}(\mathsf{G})_{\mathsf{K}}$;
  if (34) holds for F and G, it holds for $\mathsf{F}\cdot\mathsf{G}$.
- $\mathsf{E}=\mathsf{F}^*$ $\quad \dfrac{\partial}{\partial a}(\mathsf{F}^*) = \mathsf{c}(\mathsf{F})^*\left(\left[\dfrac{\partial}{\partial a}\,\mathsf{F}\right]\cdot\mathsf{F}^*\right)$ on one hand-side,
  $\mathrm{D}(\mathsf{F}^*)=\mathrm{D}(\mathsf{F})\cdot\mathsf{F}^*$ and $\mathcal{I}(\mathsf{F}^*)=(\mathsf{c}(\mathsf{F}))^*\,\mathcal{I}(\mathsf{F})$, on the other;
  if (34) holds for F, it holds for F^*. ∎

This conclude the proof that the derived-term automaton we have defined in this paper coincide with the one that was defined in the previous work dealing with expressions over the free monoids.

It is noteworthy that other works that dealt with the derivation of expressions outside from the scope of the free monoid [8, 12] have considered entities which are closed to ours. In particular, the differential of an expression is called the *linear form* in [12], and the sum of the differential and the constant term is the *expansion* in [8].

Nevertheless, we have taken here the formalism to its logical conclusion and designed a construction of the derived-term automaton that gets rid of the derivation, derivatives or their analogues.

REFERENCES

[1] V. Antimirov, "Partial derivatives of regular expressions and finite automaton constructions," *Theoret. Computer Sci.*, vol. 155, pp. 291–319, 1996.

[2] J. Berstel and C. Reutenauer, *Les séries rationnelles et leurs langages*, Masson, 1984. Translation: *Rational Series and Their Languages.* Springer, 1988.

[3] J. Berstel and C. Reutenauer, *Noncommutative Rational Series with Applications*, Cambridge University Press, 2011. New version of *Rational Series and Their Languages.* Springer, 1988.

[4] J. A. Brzozowski, "Derivatives of regular expressions," *J. Assoc. Comput. Mach.*, vol. 11, pp. 481–494, 1964.

[5] P. Caron and M. Flouret, "Glushkov construction for series: The non commutative case," *Int. J. Comput. Math.*, vol. 80, no. 4, pp. 457–472, 2003.

[6] J.-M. Champarnaud, F. Ouardi and D. Ziadi, "An efficient computation of the equation K-automaton of a regular K-expression," *Fundam. Inform.*, vol. 90, no. 1-2, pp. 1–16, 2009.

[7] J.-M. Champarnaud and D. Ziadi, "Canonical derivatives, partial derivatives and finite automaton constructions," *Theoret. Computer Sci.*, vol. 289, pp. 137–163, 2002.

[8] A. Demaille, "Derived-term automata of multitape expressions with composition," *Sci. Ann. Comput. Sci.*, vol. 27, no. 2, pp. 137–176, 2017.

[9] V. Diekert and G. Rozenberg (ed.), *The Book of Traces*, World Scientific, 1995.

[10] M. Droste, W. Kuich, and H. Vogler (Ed.), *Handbook of Weighted Automata*, Springer, 2009.

[11] V. M. Glushkov, "The abstract theory of automata," *Russian Math. Surveys*, vol. 16, pp. 1–53, 1961.

[12] S. Konstantinidis, N. Moreira, and R. Reis, "Partial derivatives of regular expressions over alphabet-invariant and user-defined labels," *Theoret. Computer Sci.*, vol. 870, pp. 103–120, 2021.

[13] S. Lombardy and J. Sakarovitch, "Derivatives of rational expressions with multiplicity," *Theoret. Computer Sci.*, vol. 332, pp. 141–177, 2005.

[14] S. Lombardy and J. Sakarovitch, "The validity of weighted automata," *Int. J. of Algebra and Computation*, vol. 23, no.4, pp. 863–914, 2013.

[15] D. Madore and J. Sakarovitch, An example of a non strong Banach algebra, to appear.

[16] J.-É. Pin (Ed.), *Handbook of Automata Theory*, European Mathematical Society Press, to appear. Vol. I and II.

[17] J. M. Rutten, "Behavioural differential equations: a coinductive calculus of streams, automata, and power series," *Theoret. Computer Sci.*, vol. 308, pp. 1–53, 2003.

[18] J. Sakarovitch, *Elements of Automata Theory*, Cambridge University Press, 2009. (Corrected English translation of *Éléments de théorie des automates*, Vuibert, 2003.)

[19] J. Sakarovitch, "Rational and recognisable power series," in: *Handbook of Weighted Automata*, M. Droste, W. Kuich and H. Vogler (ed.), Springer, 2009, pp. 105–174.

[20] J. Sakarovitch, Automata and expressions, in: *Handbook of Automata Theory, Vol. I*, J.-É. Pin (ed.), European Mathematical Society Press, 2021, pp. 39–78.

[21] A. Salomaa and M. Soittola, *Automata-Theoretic Aspects of Formal Power Series*, Springer, 1977.

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