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SCALABLE HUMAN KNOWLEDGE ABOUT NUMERIC TIME SERIES VARIATION AND ITS ROLE IN IMPROVING FORECASTING RESULTS

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Abstract. Instead of handling fuzzy sets associated with linguistic (L-) labels based on the developers' intuition immediately, the study follows the hedge algebras (HA-) approach to the time series forecasting problems, in which the linguistic time series forecasting model was, for the first time, proposed and examined in 2020. It can handle the declared forecasting L-variable word-set directly and, hence, the terminology linguistic time-series (LTS) is used instead of the fuzzy time-series (FTS). Instead of utilizing a limited number of fuzzy sets, this study views the L-variable under consideration as to the numeric forecasting variable's human linguistic counterpart. Hence, its word-domain becomes potentially infinite to positively utilize the HA-approach formalism for increasing the LTS forecasting result exactness. Because the forecasting model proposed in this study can directly handle L-words, the LTS, constructed from the numeric time series and its L-relationship groups, considered human knowledges of the given time-series variation helpful for the human-machine interface. The study shows that the proposed formalism can more easily handle the LTS forecasting models and increase their performance compared to the FTS forecasting models when the words' number grows.

Keywords. Linguistic time series; Linguistic logical relationship; Hedge algebras; Quantitative words semantics.

1. INTRODUCTION

Human activity achievements are characterized by their effective decision-making processes, in which forecasting problems play an essential role. However, in their daily lives,

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they usually observe, analyze their real-world counterparts, and make their own natural L-words decisions. In this context, the statistical approaches that deal with only the numerical time series (NTS) cannot efficiently solve such decision-making problems. The fuzzy time series (FTS) was first introduced by Song and Chissom [1–3], based on the fuzzy set theory introduced by Zadeh [4], in which the fuzzy sets are considered the meaning of their L-words. Their study originated from the observations that the weather of a particular place in North America can be described by L-words like good, very good, *quite good*, *very very good*, *cool*, *very cool*, *quite cool*, *hot*, *very hot*, *cold*, *very cold*, *very very cold*. Further, a person's mood during periods can be described as *good*, *very good*, *very very good*, *really good*, *bad*, *very bad*, *not too bad*, and so forth.

Their studies opened a new intensive research field, attracting a significant number of methodological and application-oriented studies. Ones can find in the literature the methodological studies including, e.g., model improvement by simplifying computational method [2,3], applying Fibonacci sequence [5]; high-order FTS [6–9]; multi-factor [10–12]; type-2 fuzzy set [13–15]; intuitionistic fuzzy set [16–21]; picture fuzzy set [22]; suitably determining 'fuzzy' intervals of the universe of discourse assigned to the pre-specified L-words and their numeric or defuzzification values [8, 23, 24]; optimizing these 'fuzzy' length [8, 25, 26]. In the first enrollment forecasting models of Song [2, 3] and Chen [5, 7], the number of 'fuzzy' intervals of the universe of discourse is 7, and their length is equal. In Huarng's study [45], he has stated that the length of intervals affects forecasting results in fuzzy time series. Hence, to improve the forecasting results for the enrollment and other forecasting problems, some optimization methods such as genetic algorithm [36–39], particle swarm optimization [11, 37] or clustering techniques [25, 26, 33] are used to adjust the interval length. The other computational techniques are also applied to the different steps of the forecasting model such as artificial neural network [27–32], machine learning [33, 34]. That study of Huarng has also indirectly stated that reducing the length of the intervals will increase the number of intervals. Because of that in recent studies of the enrollment forecasting, the greater number of intervals of the universe of discourse is used such as 14, 16 instead of 7 to improve the forecasting results.

The application-oriented studies are, for instance, those in temperature forecasting [26, 35–37]; tourism demand [38–40]; stock price forecasting [36, 41, 42] dry bulk shipping index forecasting [43], and so on.

In 2005, Yu proposed a weighted FTS forecasting model to resolve recurrence issues and weighting in the FTS forecasting procedure [41]. Interestingly, the repeat of the historical observations can be considered as a local regression model. Therefore, some studies improved the weighted fuzzy time series forecasting models by considering recurrent fuzzy logical relationships [19, 44–48].

Though there are significant achievements, the FTS involves two main noticeable methodological drawbacks. *The first and essential one* is the so-called word-semantics problem: though the L-data present in the FTS's examples above-mentioned are human natural language's words with their inherent semantics well-familiar with the human users, they are considered only as labels of the fuzzy sets appearing in the FTS. There is no formal connection of the L-labels with their inherent semantics with their FTS developer's associated fuzzy sets based on his intuition. *The second* is The determination of the designed fuzzy sets. While the inherent L-labels' semantics is objective, these constructed fuzzy sets are entirely subjective.

Thus, a natural novel question emerging is: Is there a formalism to develop LTS forecasting models (LTS-FM), whose L-words can be considered human words with their inherent semantics, that can immediately handle L-words to solve a given NTS forecasting problem, though the models calculate their quantitative semantics?

The hedge algebras (HA) were introduced in an axiomatic way to model L-variables? inherent semantics structures [49, 50]. It establishes another formalism to handle uncertain data in terms of L-words. HAs have been effectively applied in many fields, e.g., fuzzy control [51–54], data classification and regression using fuzzy rule-based systems [55, 56], multichannel image contrast enhancement [57]. In particular, they were, for the first time, applied in the linguistic time series (LTS) forecasting problems [58, 59]. Methodologically, though the LTS-FM proposed in these two studies deal immediately with L-variable words and their inherent qualitative semantics, it was restricted more or less in the FTS-FM methodological frame. Therefore, the number of word-data is limited by 7, very small compared to the L-variable word-domain cardinality, which is potentially infinite.

This study will stand on a novel viewpoint: the real-world variable, X_{RW}, described by the NTS's variable, X_N, can also be described by an L-variable denoted X_L, from the human users' standpoint. One can observe the X_{RW}'s states on the X_N's or X_L's standpoints, which are correlated with each other. In reality, it implies that the X_L's word-domain may be infinite, though, in applications, ones usually use only finite declared X_L's word-set. Therefore, in this study, the LTS' L-variable word-domain, consisting of all words in the natural language, which human domain experts use to describe the X_{RW}'s states in practice, is potentially infinite in principle. The study shows that the HA-approach mentioned above offers sufficient and reliable formalism to ensure the compatible semantic relationships between the L-words' semantics and their numeric ones.

Another new feature, compared to the two studies, also in the HA-approach, abovementioned is that the study follows the so-called repeated LTS forecasting model, in which words can repeatedly occur in the right side of a linguistic, logical relationship group.

To show the approach's performance, we apply the LTS-FM to forecast the NTS data of the well-known Alabama enrollments and the daily average temperature from June 1996 to September 1996 in Taipei. We perform two experiments over the first dataset with the aims: The first one is to show the performance of the LTS forecasting model in comparison with the FTS forecasting model proposed by Song , and Chissom [2] and Chen [60], so the number of used words is limited to seven which is the same as seven fuzzy sets in the counterparts. The second one is to show the increase of LTS forecasting model performance when the currently used words grow. More generally, this study would like to establish a more comprehensive formalized methodology for handling words directly, to simulate how human exerts make a forecasting process in terms of their words in the natural language. Therefore, this study ignores external factors that influence the forecasting process, such as applying optimization techniques, high-order or multi-factor time series, etc. Those matters would be considered in future studies.

The rest of this paper is organized as follows. In Section 2, we recall some background knowledge, including fuzzy time series and hedge algebras. Section 3 will introduce the linguistic time series forecasting method and its performance, using linguistic words' inherent semantics. The experimental simulation and discussions will be in Section 4. Finally, the conclusion and future work are covered in Section 5.

2. BACKGROUND

2.1. Fuzzy time series

Because of the high uncertainty of NTSs, in 1993, Song and Chissom proposed the FTS model to forecast such uncertain NTS data. Thus, we first recall the following basic definitions. Let $U = \{u_1, u_2, \ldots, u_n\}$ be a discourse universe of a FTS variable. Every fuzzy set A defined on U is considered an $A = \{(\mu_A(u_1))/u_1, (\mu_A(u_2))/u_2, \ldots, (\mu_A(u_n))/u_n\}$, where μ_A is called the membership function of A, $\mu_A : U \to [0,1]$. The notation $\mu_A(u_i)$ indicates the degree of membership of u_i belonging to the fuzzy set A, $f_A(u_i) \in [0,1]$, where $1 \le i \le n$.

Definition 1.(Fuzzy time series) [2] Let $Y(t), (t = 0, 1, 2, \ldots)$, a subset of R, be the universe of discourse on which fuzzy sets $f_i(t),\ i = 1, 2, \ldots$ are defined, and if $F(t)$ is a collection of $f_1(t), f_2(t), \ldots$, then $F(t)$ is called an FTS defined on $Y(t)\,(t = 0, 1, 2, \ldots)$.

Definition 2. (Fuzzy logical relationship) [60] The relationship between $F(t)$ and $F(t-1)$ is expressed as $F(t-1) \to F(t)$. Let $A_j = F(t)$ and $A_i = F(t-1)$, then the relationship between $F(t)$ and $F(t-1)$ can be expressed by fuzzy logical relationship (FLR) $A_i \to A_j$, where A_i and A_j refer to the left-hand side and the right-hand side of FLR, respectively.

Definition 3. (Fuzzy logical relationship group) [60] The FLRs having the same left-hand side can be further grouped into a fuzzy logical relationship group (FLRG). Assume that there are FLRs $A_i \to A_{j1}, A_i \to A_{j2}, \ldots, A_i \to A_{jk}$, Chen suggested that these FLRs with the same left-hand side can be grouped into FLRG as $A_i \to A_{j1}, A_{j2}, \ldots, A_{jk}$.

Studies on FTS achieve many significant results with high accuracy, but it is also crucial that FTS forecasting models handle very well historical data with high variation, short or uncertain data. However, as discussed at the beginning of this study, though the LTS inspired by the FTS forecasting model, there is no formalized formalism to link its fuzzy sets with L-words. It suggests that if we could formalize L-variable's word-domains and develop their quantification theory to handle the L-data of the LTS, it would be beneficial for human experts to solve NTSs' forecasting problems. Thus, in the next section, we present the necessary concepts and properties of the HA as mathematical models of the word-domains and their quantification to form a formal basis to study the LTS forecasting model (LTS-FM).

2.2. Some hedge algebras knowledge

2.2.1. The L-variable word-domain as an algebraic order-based structure and its properties

1) Description of the algebraic order-based structure – hedge algebras

For every L-variable, say the temperature, $\mathcal{X}_{tmp}$, its word-domain, denoted by X_{tmp}, is the set of all the words in the natural language, which indicate the daily temperature. It can be observed that the words in X_{tmp} can be generated from two atomic (or primary) words, '*low*' and '*high*', by applying L-hedges, such as '*very*', '*little*', and linearly ordered based on their inherent qualitative semantics. It is also interesting that for every hedge h and every word x of $\mathcal{X}_{tmp}$, x and hx are comparable. These suggest Nguyen and Wechler introduced the concept of hedge algebras (HA) [49, 50] for every L-variable $\mathcal{X}$, the hedge algebras of $\mathcal{X}$ is an algebraic order-based structure $\mathcal{AX} = (X, G, C, H, \leq)$, where X is an L-word set of $\mathcal{X}$ and $X = \mathrm{Dom}(\mathcal{X})$;

G is a set of two generator words c^- and c^+, where $c^- \leq c^+$; they are called the negative and positive atomic words, respectively;

$\leq$ is the order relation induced by the inherent word semantics of $\mathcal{X}$;

$H = H^- \cup H^+$ is a set of L-hedges of $\mathcal{X}$, considered as *unary* operations, where H^- and H^+ are two sets of, respectively, negative and positive hedges;

$C = \{0, W, 1\}$ is a set of word constants being fixed points, which are called, in turn, the least, the medium, and the greatest words and satisfy the conditions, $0 \leq c^- \leq W \leq c^+ \leq 1$.

The terminology 'hedge algebras' comes from the hedges' specific syntactic and semantic functionalities: they syntactically generate the words and modify or intensify their inherent semantics of a given L-variable $\mathcal{X}$. By their inherent semantic functionalities, for every hedge h, and every word x of $\mathcal{X}$, x and hx are always comparable. Further, by modifying functionality, the set $H(x)$ is called the fuzziness model of the word x [61].

By HA's definition, every word x in X can be represented as the string representation, $x = \sigma c = h_m \ldots h_1 c,\ h_j \in H,\ j = 1, \ldots, m,\ c \in \{c^-, c^+\}$. I.e., $\sigma \in H$, the set of all strings of hedges, including the empty string ε, $\varepsilon = \emptyset$. Then, the x's length, denoted by $|x|$, is just the string's length σx. Note that $\varepsilon x = x$ and $\sigma\alpha = \alpha$ for all the constants $\alpha \in C$. For every $x \in X$, put $H(x) = \{\sigma x : \sigma \in H\}$. It can be easily verified that $X = H(c^-) \cup H(c^+) \cup C$.

2) *Primary properties of the hedge algebras*

Consider an HA of $\mathcal{X}$, $\mathcal{AX} = (X, G, C, H, \leq)$, where $H = \{L\,(Little), V\,(Very)\}$ being of only two hedges for simplicity. Then, one can discover the 'algebraic' sign of its words and its hedges as follows:

- First, c^- and c^+ have contrary semantic trends observed by $Vc^- \leq c^-$, and $c^+ \leq Vc^+$. From these, we define $\mathrm{sign}(c^-) = -1$ and $\mathrm{sign}(c^+) = +1$.

  From $Lc^+ \leq c^+ \leq Vc^+$ (or $Vc^- \leq c^- \leq Lc^-$), set $\mathrm{sign}(L) = 1$ and $\mathrm{sign}(V) = +1$.

- Furthermore, the inequalities $VLc^+ \leq Lc^+ \leq c^+$ imply that when V acts on Lc^+, it modifies the word Lc^+ in the *same* direction as L modifies c^+. Then, we write, $\mathrm{sign}(V, L) = +1$, and call it the *relative* sign of V to L. Similarly, we observe that $c^+ \leq LVc^+ \leq Vc^+$, i.e., L modifies Vc^+ in the direction *contrary* to the one V modifies c^+. Hence, we write $\mathrm{sign}(L, V) = -1$. Generally, one can always define the relative sign of any two L-hedges.

Now, for every $x = h_m \ldots h_1 c,\ c \in \{c^-, c^+\}$, we can write

$$\mathrm{sign}(x) = \mathrm{sign}(h_m, h_{m-1}) \times \ldots \times \mathrm{sign}(h_2, h_1) \times \mathrm{sign}(h_1) \times \mathrm{sign}(c).$$

Thus, every word x its algebraic sign, $\mathrm{sign}(x)$, and its meaning can be formulated as follows.

Proposition 1. $\mathrm{sign}(hx) = +1$ *implies* $x \leq hx$, *and* $\mathrm{sign}(hx) = -1$ *implies* $hx \leq x$.

Also, assuming $H^- = \{L, R\,(Rather)\}$ and $H^+ = \{M\,(More), V\}$, i.e., H has four hedges, to capture a more generality, we have the following properties of the words and their fuzziness models:

$$\{0\} \leq H\left(c^{-}\right) \leq \{W\} \leq H\left(c^{+}\right) \leq \{1\}, \text{ and } H\left(c^{-}\right) \cup H\left(c^{+}\right) = X \backslash C, \tag{1}$$

$$H(x) = H(Lx) \cup H(Rx) \cup \{x\} \cup H(Mx) \cup H(Vx), \tag{2}$$

$$\operatorname{sign}(Vx) = +1 \Rightarrow H(Lx) \leq H(Rx) \leq \{x\} \leq H(Mx) \leq H(Vx), \tag{3}$$

$$\operatorname{sign}(Vx) = -1 \Rightarrow H(Vx) \leq H(Mx) \leq \{x\} \leq H(Rx) \leq H(Lx). \tag{4}$$

Thus, it is essential that every L-variable now involves a (semantic) algebraic structure compatible with its numeric counterpart variable having a math-structure.

2.2.2. The quantification of hedge algebras

For a given real-world variable, considering its L-variable, denoted by $\mathcal{X}_L$, as the counterpart of its respective numeric variable, denoted by $\mathcal{X}_N$, naturally, genuine semantic relationships exist between the words of $\mathcal{X}_L$ and their $\mathcal{X}_N$'values.

1) *The numeric semantics of* $\mathcal{X}_L$

We begin with the numeric semantics of the $\mathcal{X}_L$'s words, X_L, defined by an assignment mapping $f_{\mathcal{X}}$ from X_L to $[0,1]$, the normalized numeric universe of $\mathcal{X}_N$, whose constraint conditions are as minimal as possible [61].

Definition 4. A one-to-one mapping, $f : X_L \to [0,1]$, of an L-attribute $\mathcal{X}_L$ is said to be a semantically quantifying mapping (SQM) of $\mathcal{X}_L$, provided that

(SQM1) $f_{\mathcal{X}}$ preserves the structure $(X_L, \leq)$, i.e. $(\forall x, y \in X_{\mathcal{X}})\,(x \leq y \Rightarrow f_{\mathcal{X}}(x) \leq f_{\mathcal{X}}(y))$.

(SQM2) The image of the word set X_L, $f_{\mathcal{X}}(X_L)$ is *dense* in $[0,1]$; i.e. $C(f_{\mathcal{X}}(X_L)) = [0,1]$, where C is the ordinary topological closure operator on $[0,1]$.

2) *An axiomatization method of the HA' quantification*

The HA' quantification axiomatization method is also essential for establishing a formal basis to guarantee that the proposed LTS-FM deals directly with L-words, though it handles their numeric semantics instead of themselves. The words' fuzziness, characterized by their fuzziness interval examined above, is an essential characteristic of fuzzy data. Point 1) has shown that once the SQM $f_{\mathcal{X}}$ of $\mathcal{X}$ is given, the fuzziness intervals of the words are entirely determined. Thus, as our expectation, a strictly closed relationship exists between the variable's semantic structure and its numeric structure universe: the variable word-domain semantic structure and a given semantic mapping $f_{\mathcal{X}}$ mentioned above determine its words' fuzziness intervals on its universe.

We will now show that for a given specific words' fuzziness intervals' information, say their fuzziness intervals' lengths, called the words' *fuzziness measure*, denoted by m, inversely, one can determine the semantic mapping $f_{\mathcal{X}}$. In the HA-approach, the fuzziness measure m of an L-variable $\mathcal{X}$ can be defined by the following axioms [61].

Definition 5. Let $\mathcal{AX} = (X, G, C, H, \leq)$ be an HA. A function $\mathrm{m} : X \to [0,1]$ is said to be a fuzziness measure of words in X provided that the following conditions are satisfied:

F1) $\mathrm{m}(c^{-}) + \mathrm{m}(c^{+}) = 1$ and $\sum_{h \in H} \mathrm{m}(hu) = \mathrm{m}(u)$, for $\forall u \in X$;

F2) For the constants 0, W, and 1: $\mathrm{m}(0) = \mathrm{m}(W) = \mathrm{m}(1) = 0$;

F3) For $\forall x, y \in X$ and $x, y \notin C$, $\forall h \in H$, we have $\dfrac{\mathrm{m}(hx)}{\mathrm{m}(x)} = \dfrac{\mathrm{m}(hy)}{\mathrm{m}(y)}$ and it does not depend on particular words in X. Hence, it is called the hedge h's fuzziness measure, denoted by $\mu(h)$.

From conditions F3) and F1) it follows that
m1) $\mathrm{m}(hx) = \mu(h)\,\mathrm{m}(x)$, for $\forall x \in X$;
m2) $\mathrm{m}(c^-) + \mathrm{m}(c^+) = 1$;
m3) $\sum_{h \in H} \mu(h) = 1$.

From m1) – m3), it can be verified that, when the values of $\mathrm{m}(c^-)$, and $\mu(h), h \in H \setminus \{h'\}$, for some $h' \in H$, are known, the fuzziness measure m is wholly defined. Therefore, they are called the *independent* fuzziness parameters of the L-variable $\mathcal{X}$.

It is essential that conversely to Point 1), for every L-variable, the fuzziness parameters of $\mathcal{X}$ do define all quantitative semantics of $\mathcal{X}$: the numeric semantics and the fuzziness intervals of its words. From the discussion in Point 1), it is required to show that the numeric, semantic mapping (SQM) $f_{\mathcal{X}}$ of $\mathcal{X}$ is completely defined for the given independent fuzziness parameters' values. The study by Nguyen and Nguyen [61], proved that the following defined numeric mapping v is just a numeric, semantic mapping (SQM) $f_{\mathcal{X}}$ defined in Definition 4, Point 1):

Definition 6. For the given fuzziness parameters' values of an L-variable, the SQM of $\mathcal{X}$, $\nu : X \to [0, 1]$ is recursively calculated as follows:

1) $\nu(W) = \theta = \mathrm{m}(c^-)$, (5)

2) $\nu(c^-) = \theta - \alpha \mathrm{m}(c^-)$ and $\nu(c^+) = \theta + \alpha \mathrm{m}(c^+)$, (6)

3) $\nu(h_j x) = \nu(x) + \mathrm{sign}(h_j x) \left\{ \sum_{i=\mathrm{sign}(j)}^{j} \mathrm{m}(h_i x) - \omega(h_j x)\,\mathrm{m}(h_j x) \right\}$, (7)

where $\omega(h_j x) = \frac{1}{2}\left[1 + \mathrm{sign}(h_j x)\,\mathrm{sign}(h_p h_j x)(\beta - \alpha)\right] \in \{\alpha, \beta\}$ and $\alpha + \beta = 1$.

Thus, one can see that Section 3 provides a formalized formalism to ensure that when the proposed LTS forecasting method (LTS-FMd) handling the L-words' quantitative semantics based on this formalism, it does, indeed, deal with the L-words themselves.

3. THE LTS-FMD USING L-WORDS' INHERENT SEMANTICS AND ITS PERFORMANCE AND ADVANTAGES

The LTS-FMd can be understood as follows. Because human dataset users are well-familiar with their natural language, they observe a given NTS variation diagram using their NTS's L-variable words. Thus, a problem arising is that how one can develop a forecasting method to transform a given NTS forecasting problem into its respective LTS forecasting problem with, of course, a formal human L-words' semantics authenticity guarantee. It is guaranteed that there is a reliable formalized formalism to ensure that the interaction between the L-words and their respective numeric values in the desired LTS forecasting model is the same as such a similar interaction in human dataset user's activity.

3.1. The LTS and their forecasting models

The concept of FTS and their forecasting model are indeed fascinating because it is the first time L-words, represented by fuzzy sets, have been used to solve NTS forecasting problems, which, in nature, must be solved in an environment of very high uncertainty. However, though the FTS forecasting models are originated from the time series with L-data, no studies in this field can deal immediately with L-data with their inherent semantics in the natural language. It motivated Nguyen et al. to introduce the LTS-forecasting model (LTS-FM) [58], which can be shortly described as follows:

1) *The linguistic time series concept*

Because human beings are incredibly well-familiar with their natural language and do their reasoning using their language, LTS exist naturally and seem very useful. Thus, it can naturally be formalized as follows.

Definition 7. Let X be a set of L-words in the natural language of a variable $\mathcal{X}$ defined on the universe of discourse $U_{\mathcal{X}}$ to describe its numeric quantities. Then, any series $L(t)$, $t = 0, 1, 2, \ldots$, where $L(t)$ is a collection of words of X, is called an LTS.

The LTS are very natural, as mentioned by Chen [60] but they were still not considered his study subject and in the framework of the fuzzy sets.

2) *Linguistic, logical relationships of a given LTS*

In general, $L(t)$ is a finite subset, e.g., a few given L-words may describe the weather of a day. In this study, we deal only with the LTS, whose $L(t)$'s are singleton. Like the (fuzzy) logical relationship concept introduced by that paper, one can also define linguistic, logical relationships (LLRs) from a given $\mathcal{X}$'s LTS $\mathcal{T}_L$. For example, if the L-value at every time $t = k$ of the LTS is x_i, and at its time $t = k + 1$ is x_j, then it defines an LLR of the form $x_i \to x_j$. Then, from the established LLRs of the given LTS $\mathcal{T}_L$ with the same left-hand side, say x_i and $x_i \to x_{j_l}$, $l = 1, \ldots, n$, one can group into an LLRs' group (LLRG) written in the form, $x_i \to x_{j_1}, x_{j_2}, \ldots, x_{j_n}$.

Thus, in this way, every given LTS defines a unique collection of LLRG.

However, to handle LTS computationally, one must establish a formalism to deal with the LTS' words' inherent qualitative and quantitative semantics. We will show that the formalism presented in Section 2 is sufficient to develop an LTS-FMd, formally ensuring an LTS's words' semantics authenticity, to solve NTS forecasting problems.

3.2. Linguistic time series forecasting method

Let us consider an NTS $\mathcal{T}$ of a numeric variable $\mathcal{X}$ and its forecasting problem. For convenience, its respective forecasting L-variables is denoted by $\mathcal{X}_L$. Then, the LTS-FMd to develop LTS-FMs for solving the NTS's forecasting problem of $\mathcal{T}$ comprises two phases:

1) Constructing an LTS from $\mathcal{T}$, denoted by $\mathcal{T}_L$, and establish its LLRGs;

2) Developing a reasoning method working on the constructed LTS $\mathcal{T}_L$ to solve the given $\mathcal{T}$'s forecasting problem.

The efficiency of the developed LTS-FM depends on both phases.

1) Constructing an LTS from a given NTS $\mathcal{T}$ and a reasoning method on its

Let be given a NTS $\mathcal{T}$ of a variable $\mathcal{X}$, $\mathcal{T} = \{d_i : i = 0\ldots, n\}$. From the human users' standpoint, can also be considered a forecasting L-variable, denoted by $\mathcal{X}_L$, whose different aspect semantics, in an application, should be defined. Assume, for instance, that they are defined as follows:

Declaring its syntactical semantics, including two atomic words, c^- and c^+, a set of hedges, H, simply say $H^= \{L\,(little)\}$ and $H^+ = \{V\,(very)\}$, and the highest words' specificity, say κ; Then, one can determine the currently declared word-set of κ-specificity, denoted by $X_{(\kappa)}$. With this declared syntactical semantics, $X_{(1)}$ have five words, $X_{(2)}-$ nine words, and $X_{(3)}-$ seventeen words;

Its qualitative semantics specifies the atomic words' names, the hedges (algebraic) signs, and their relative signs.

Its quantitative semantics is specified by providing the forecasting variable fuzziness parameters' values.

$\circ$ *A method* M *to* construct *the LTS* $\mathcal{T}_L$ *from* $\mathcal{T}$. This construction comprises the following tasks:

Task 1. Compute the numeric semantics of the declared $\mathcal{X}'s$ *words*. For a given the currently declared $\mathcal{X}_L$'s word-set of κ-specificity, $X_{(\kappa)}$, arrange its words in an increasing chain following their order, denoted by $C\left(X_{(\kappa)}\right)$, where $C\left(X_{(\kappa)}\right) = \{x_j : 0 \leq j \leq p = |X\,(\kappa)|$, and $v\,(x_i) < v\,(x_j)\, x_i < x_j\}$.

Applying the SQM v, defined in Definition 6, assigning to every word x its numeric semantics, $v\,(x) \in [0, 1]$, the normalized $\mathcal{X}_L$'s universe, we obtain the numeric chain $v\left(C\left(X_{(\kappa)}\right)\right) = \{v\,(x_j) : 0 \leq j \leq p$, and $v\,(x_i) < v\,(x_j) \Leftrightarrow x_i < x_j\}$. Clearly, we have $v\,(x_0) = v\,(0) = 0; v\,(x_p) = v\,(1) = 1$.

Task 2. Denote by $\mathrm{Cent}\,(v\,(x_l)\,, v\,(x_{l+1}))$ the center of the interval $[v\,(x_l)\,, v\,(x_{l+1})]$. Consider the following intervals:

For $l = 0$, put $I\,(x_0) = I\,(0) = [v\,(0)\,, \mathrm{Cent}\,(v\,(x_0)\,, v\,(x_1))]$;

For $l = 1$ to $p - 1$, put $I\,(x_l) = [\mathrm{Cent}\,(v\,(x_{l--1})\,, v\,(x_l))\,, \mathrm{Cent}\,(v\,(x_l)\,, v\,(x_{l+1}))]$;

And for $l = p$, put $I\,(x_p) = I\,(1) = [\mathrm{Cent}\,(v\,(x_{p-1})\,, v\,(x_p))\,, v\,(1)]$.

It can be verified that $I\,(x_k)$ contains the numeric semantics of x_k, $v\,(x_k)$, and the set I, $I = \{I\,(x_k) : 0 \leq k \leq p\}$ forms a partition of the normalized L-variable universe, $[0, 1]$.

Then, the LTS constructed from the given NTS $\mathcal{T}$ is generated as follows:

Denote by $N\,(d_j)$ where its value computed by $N(d_j) = (d_j\text{–}\min(\mathcal{T}))/(\max(\mathcal{T})\text{–}\min(\mathcal{T}))$, the normalized value of d_j. Then, we determine the desired LTS's words in the following manner:

For $i = 0$, if d_0 falls in the interval $I(x_{k_0})$, put $u_0 = x_{k_0}$.

For $i = 1, \ldots, n$, if d_i falls in $I(x_{k_i})$, and $x_{k_i} \neq u_i - -1$, then d_i is replaced by the word x_{k_i}, otherwise consider $i = i + 1$.

In this way, we obtain an LTS generated from $\mathcal{T}$, denoted by $\mathcal{T}_L$.

$\circ$*Forecasting using the obtained LTS*: There several fuzzy forecasting methods examined in the literature, including the high-order FTS forecasting method. However, since the LTS forecasting model, whose forecasting L-variable domain is potentially infinite, is examined for the first time in this study, we restrict ourselves to examine the first order LTS forecasting method.

Let be given an LTS, $\mathcal{T}_L$, formed from the declared word-set $X_{(\kappa)}$ of an L-variable $\mathcal{X}_L$ of the forecasting variable $\mathcal{X}$, whose declared multi-aspect semantics determines its numeric semantics mapping $v_{\mathcal{X}}$. As discussed previously, $\mathcal{T}_L$ defines a set of LLRGs of the forms

$$x_i \rightarrow x_{ij_1},\ x_{ij_2}, \ldots,\ x_{ij_l},\ i = 1, \ldots, q, \tag{8}$$

where q is the number of words in the right-hand side.

Then, the LTS forecasting method computes the forecasting value simply as follows. For a given numeric $\mathcal{X}$'s value, d considered as its numeric input at the time $t = t_s$, it must determine a word x_d in $\mathcal{T}_L$ so that $I(x_d)$ contains d or its numeric semantics $v(x_d)$ is nearest d. The word x_d must appear on the left-hand side of an expression in the form (8), say, $x_d \rightarrow x_{d1}, x_{d2}, \ldots, x_{dl}$. Then, it computes the forecasting value $F(d)$, at the time $t = t_s + 1$, by the following formula

$$F(d) = \left(v\left(x_{d1}\right) + v\left(x_{d2}\right) + \ldots + v\left(x_{dl}\right)\right)/l. \tag{9}$$

2) An LTS forecasting procedure

From the above discussion, the procedure to solve a given NTS forecasting problem of a forecasting variable $\mathcal{X}$, using LTS, comprises four steps, shortly described as follows:

Step 1. Determine the multi-semantic aspects of $\mathcal{X}$. Because the multi-aspect words' semantics is essential for the proposed approach, this step is crucial for guaranteeing the method performance. It includes declaring the $\mathcal{X}$'s syntactical and qualitative semantics and selecting $\mathcal{X}$'s fuzziness parameters' values by trial and error.

Step 2. Compute the numerical semantic of the declared $\mathcal{X}$*'s word-set.* Quantify the semantics of the declared words by applying the quantitative semantics assignment v, defined by the selected $\mathcal{X}$'s fuzziness parameters' values (Definition 6, Section 2).

Step 3. Generate the LTS for the given NTS. Transform the given NTS $\mathcal{T}$ into the LTS $\mathcal{T}_L$, and generate LLRs and LLRGs from $\mathcal{T}_L$;

Step 4. Compute the forecasting value. For a given input datum at the time t, compute the forecasting value at time $t + 1$, utilizing the generated LLRGs. Specifically, in case the right-hand side of LLRGs is not empty, the forecasting value is computed by formula (9), otherwise the forecasting value is $v(x_d)$.

3) Outstanding features of the proposed HA-approach

Usually, in the fuzzy time series forecasting, human words of a forecasting L-variable, as a linguistic counterpart of the given numeric forecasting one, describing a given numeric time-series play a motivation factor for developing FTS forecasting models because ones consider them as only L-labels of the designed fuzzy sets. Though they represent the L-labels' semantics, they are constructed only based on the developers' intuition but not on a mathematical formalization of the L-variables' word domains. It implies that one cannot consider L-labels associated with the constructed fuzzy sets as application users' words.

The proposed HA-approach has the following specific features to overcome these drawbacks:

Utilizing the hedge algebras' formalism, similar to the case of the fuzzy set-based words' semantics interpretability and scalability examined in [62], it ensures that the numeric words' semantics (the SQM v's values) is interpretable and scalable by Definition 4 and 6. Its scalability is ensured because v depends on only the whole forecasting variable's fuzziness parameters' values despite how many words are currently used. Thus, the study has a sound formalism to guarantee that, instead of handling words directly, one can handle their numeric semantics.

It implies that the words appearing in the LTS constructed from the given NTS $\mathcal{T}$ presented above are human words, i.e., it is indeed a human linguistic time series. This feature is essential and useful for establishing interactions with the application users. Consequently, the constructed LTS can simulate how human users describe $\mathcal{T}$ in terms of their words.

Since the larger number of the words used to describe the given NTS $\mathcal{T}$, the precise the constructed LTS can describe the given NTS $\mathcal{T}$ when the forecasting word-set grows. Then, the proposed LTS-FM can increase prediction results' precision, as shown next.

4. THE EXPERIMENTAL SIMULATION AND DISCUSSIONS

There are two standard literature methods to group the fuzzy logical relationships (FLRs) into the fuzzy logical relationship groups (FLRGs). In Chen's method [60], a fuzzy set cannot be repeated in the right-hand side of the FLRGs, whereas it can be repeated in Yu's method [41]. Inspired by these two kinds of FLRGs, this experiment study applies both these FLRGs' kinds. Thus, we denote the respective two LTS-FMs in our approach, for short, by Unrepeated LTS (ULTS) and Repeated LTS (RLTS).

In order to justify the proposed approach performance, the study designs a comparative study to compare the proposed LTS forecasting model performance with those examined by Song , and Chissom [2] and Chen [60], using the University of Alabama historical data observed from 1971 to 1992, and Chen and Hwang [35], using the daily average temperature data from June 1996 to September 1996 in Taipei.

The experiment simulation performs two experiments aiming to show the following statements:

Statement 1: The developed LTS-FMs outperform the fuzzy counterpart models, restricted its L-words' number to the same counterpart models' fuzzy sets' number;

Statement 2: Since a specific new feature of the proposed LTS-FMs is considering the forecasting L-variable $\mathcal{X}_L$ as the L-counterpart of the numerical forecasting one, we assume no restriction on its declared word set's cardinality. Then, the simulation study will show that when the declared $\mathcal{X}_L$'s word-set grows, the proposed LTS-FM can considerably increase the forecasting precision.

4.1. The proposed LTS-FM performance

A) Comparison with the two counterparts examined in Song and Chissom 1993a and Chen 1996

To show the efficiency and robustness of the proposed approach, we apply the ULTS, the RLTS, and the two counterpart methods examined by Song and Chissom [2] and Chen [60], using the University of Alabama historical data, as mentioned above, shown in Table 2. To expose the approach features more obviously, the experiment follows the proposed procedure, step by step, as follows:

Step 1. Determine the multi-semantic aspects of the forecasting L-variable $\mathcal{X}$.

One determines its syntactical semantics by declaring $c^- = small\,(s)$, $c^+ = large\,(l)$ and two hedges sets, $H^- = \{Little\,(L)\}$, $H^+ = \{Very\,(V)\}$. For the comparative study, the declared word set is the 2-specificity word-set $X_{(2)}$, (the number of declared words is equal to the number of intervals in [45]):

$U_{\mathcal{X},L} = \{$***0***, *V_small, small, L_small, medium, L_large, large, V_large,* ***1***$\}$, ***0*** and ***1*** stand for *Absolutely small (A_small)* and *Absolutely large (A_large).*

Its $\mathcal{X}$'s *qualitative* semantics is determined by specifying the hedges' signs and the relative signs between the hedges

$$\text{sign}\,(V) = \text{sign}\,(V, V) = \text{sign}\,(V, L) = +1.$$

Its quantitative semantics (or the numerical semantics defined by v (Definition 6)) is defined by specifying the $\mathcal{X}$'s fuzziness parameters' values $\text{m}\,(c^-) = 0.48$ and $\mu\,(L) = 0.29$.

Step 2. Compute the numerical semantic of the declared $\mathcal{E}$.'*s word-set.* Using the mapping v defined by Definition 6 and transforming their normalized values into the $\mathcal{E}$'s universe values (or unnormalized values) in [13000, 2000], we obtain the numerical semantics of the declared words:

Table 1: The actual semantics of seven words in HA

The declared $\mathcal{X}$'s words	*A_small*	*V_small*	*small*	*L_small*	*medium*	*L_large*	*large*	*V_large*	*A_large*
Their quantitative semantics	13000	14693.8	15385.6	15668.2	16360	17109.5	17415.6	18165.1	20000

Step 3. Generate the LTS for the given NTS. When the numerical semantics are computed, the proposed method, $\mathcal{M}$, generates an LTS $\mathcal{T}_L$, represented in the 3rd-column of Table 2, from the NTS $\mathcal{T}$ given by the University of Alabama historical data. From $\mathcal{T}$, one can quickly determine the LLRs and, then, the LLRGs exposed in the fourth and fifth columns in Table 2, respectively.

Step 4. Compute the forecasting value. For $t = 1971$ to 1991, from the given input enrolment datum at year t, compute the forecasting value at time $t+1$, using the normalized input data, the generated LLRGs (Repeated/Unrepeated) and formula (10); denormalize the computed forecasting value to generate its respective output data. The obtained forecasting results and those made by Song and Chissom's model, Chen's model, and Cheng's model are shown in Table 3 and visualized in Figure 1.

The evaluation of forecasting models bases on mean square error (MSE), $MSE = (1/N)\sum_i (F_i - A_i)^2$, where F_i and A_i, respectively, are the forecasting and actual values, and N is the number of historical data taken for forecasting.

The University of Alabama enrollments' forecasting results from 1971 to 1992 in terms of 7 words are shown in Table 3. We apply the same fuzziness parameter values, $m\,(c^-) = 0.48$ and $\mu\,(V) = 0.29$, for both ULTS and RLTS models. It shows that the MSE values $162,754.0$ and $161,331.0$ made, respectively, by the ULTS and RLTS models are much smaller than the ones $423,027$, $407,507$, and $191,844.0$ caused, respectively, by the three counterpart models.

It allows us to conclude that both proposed forecasting models can significantly outperform their counterparts, and the RLTS model's performance is better than that of the ULTS (about 0.88%.) By statistical analysis, we observe that the forecasting errors of the RLTS model range from 0.11% to 5.17%, and the average error is 1.93%, while the forecasting

Table 2: The enrollments of the University of Alabama from 1971 to 1992 and LLRs in the case of 9 declared words

Year	**NTS (Enrol. values)**	**The constructed LTS**	**Linguistic, Logical Relationships**	**Repeated Linguistic, Logical Relationship Groups**
1971	13055	A_small		∘A_small → A_small, V_small ∘V_small → small ∘small → small, L_small, small, small, small, L_small, ∘L_small → L_small, L_large, L_large ∘L_large → L_large, medium, V_large ∘medium → small, ∘V_large → V_large, large ∘A_large → A_large, large
1972	13563	A_small	A_small → A_small	
1973	13867	V_small	A_small → V_small	
1974	14696	V_small	V_small → V_small	
1975	15460	small	V_small → small	
1976	15311	small	small → small	
1977	15603	L_small	small → L_small	
1978	15861	L_small	L_small → L_small	
1979	16807	L_large	L_small → L_large	
1980	16919	L_large	L_large → L_large	
1981	16388	medium	L_large → medium	
1982	15433	small	medium → small	
1983	15497	small	small → small	
1984	15145	small	small → small	
1985	15163	small	small → small	
1986	15984	L_small	small → L_small	
1987	16859	L_large	L_small → L_large	
1988	18150	V_large	L_large → V_large	
1989	18970	V_large	V_large → V_large	
1990	19328	A_large	V_large → A_large	
1991	19337	A_large	A_large → A_large	
1992	18876	V_large	A_large → V_large	

errors of the ULTS model range from 0.15% to 5.17%, but the average error is 2.01%. So, the forecasted values of the ULTS model are closer to the actual values than the ones of the RLTS model.

B) Comparison with Chen and Hwang's method using the daily average temperature data

In order to enhance our Statement 1 more reliably, the study compares the proposed model, ULTS, with the model examined by Chen and Hwang [35], using a more sophisticated time series, the daily average temperature from June 1996 to September 1996 in Taipei, shown in the column AV (Actual Values) of Table 5. The highest and the lowest values of the observed temperatures from June to September are 31.6 and 23.3, respectively. However, the lowest and the highest temperatures of each months are different. Hence, assume that $U_{\mathcal{E}}$ of the months from June to September are $[25.5, 31.5]$, $[27.0, 32.0]$, $[25.5, 31.0]$ and $[23.0, 31.0]$, respectively.

We can perform this experiment step by step, similar to above, except the multi-aspect semantics is defined as follows.

We assume that $c^- = cool$, $c^+ = hot$ and, as above, also two hedges $H^- = \{Little\,(L)\}$, $H^+ = \{Very\,(V)\}$. Hence, as previously, after eliminating the constants **0** and **1**, we have

$$U_{\mathcal{E},L} = \{Verycool, cool, Littlecool, normal\,(W)\,, Littlehot, hot, Veryhot\}\,.$$

For determining the quantitative word semantics, we adopt $m\,(c^-) = 0.52, \mu\,(L) = 0.528$. Then, the numerical semantics of the $U_{\mathcal{E},R}$'s words of June, for example, can be calculated

Table 3: A comparison of the forecasting models of Enrollments of Alabama from 1971 to 1992 in case of 9 declared words

Year	**NTS (Enrollment values)**	**Song and Chissom [2] 1993a**	**Chen 1996 [60]**	**Cheng 2008 [45]**	**ULTS model**	**RLTS model**
1971	13055					
1972	13563	14000	14000	13680.5	13847	13847
1973	13867	14000	14000	13731.3	13847	13847
1974	14696	14000	14000	13761.7	15040	15040
1975	15460	15500	15500	15194.6	15040	15040
1976	15311	16000	16000	15374.8	15527	15480
1977	15603	16000	16000	15359.9	15527	15480
1978	15861	16000	16000	16410.3	16389	16629
1979	16807	16000	16000	16436.1	16389	16629
1980	16919	16813	16833	17130.7	17212	17212
1981	16388	16813	16833	17141.9	17212	17212
1982	15433	16789	16833	15363.8	15386	15386
1983	15497	16000	16000	15372.1	15527	15480
1984	15145	16000	16000	15378.5	15527	15480
1985	15163	16000	16000	15343.3	15527	15480
1986	15984	16000	16000	15345.1	15527	15480
1987	16859	16000	16000	16448.4	16389	16629
1988	18150	16813	16833	17135.9	17212	17212
1989	18970	19000	19000	18915	19083	19083
1990	19328	19000	19000	18997	19083	19083
1991	19337	19000	19000	19032.8	19083	19083
1992	18876	-	19000	19033.7	19083	19083
MSE		**423,027.0**	**407,507.0**	**191,844.0**	**162,754.0**	**161,331.0**
RMSE		**650.4**	**638.36**	**438.0**	**403.43**	**401.66**

as exposed as follows

$$U_{\mathcal{E},R} = \{26.195, 26.973, 27.842, 28.62, 29.338, 30.14, 30.858\}.$$

The same as previously, once the numerical words' semantics are defined, one can easily construct the temperature LTS and produce the LLRs, and their unrepeated or repeated LLRGs for the ULTS or RLTS models, respectively. Because the RLTS's LLRGs are too long, only the ULTS's LLRGs for June are exhibited in Table 4 for simplifying the presentation.

Table 4: Linguistic, logical relationship groups of the historical temperature of June

Group	**LLRGs of ULTS model**
Group 1	$V\_cool \;\rightarrow\; L\_cool$
Group 2	$L\_cool \rightarrow L\_hot, , cool$
Group 3	$L\_hot \rightarrow V\_hot, L\_hot, normal, hot$
Group 4	$V\_hot \rightarrow hot, V\_hot, normal$
Group 5	$hot \rightarrow L\_hot, hot, V\_hot$
Group 6	$normal \rightarrow L\_hot, normal, L\_cool$
Group 7	$cool \rightarrow L\_cool, normal$

Following Chen and Hwang (2000), both proposed LTS-FMs, ULTS (UFM) and RLTS (RFM) are also applied to predict the daily forecasted temperature simulation results of

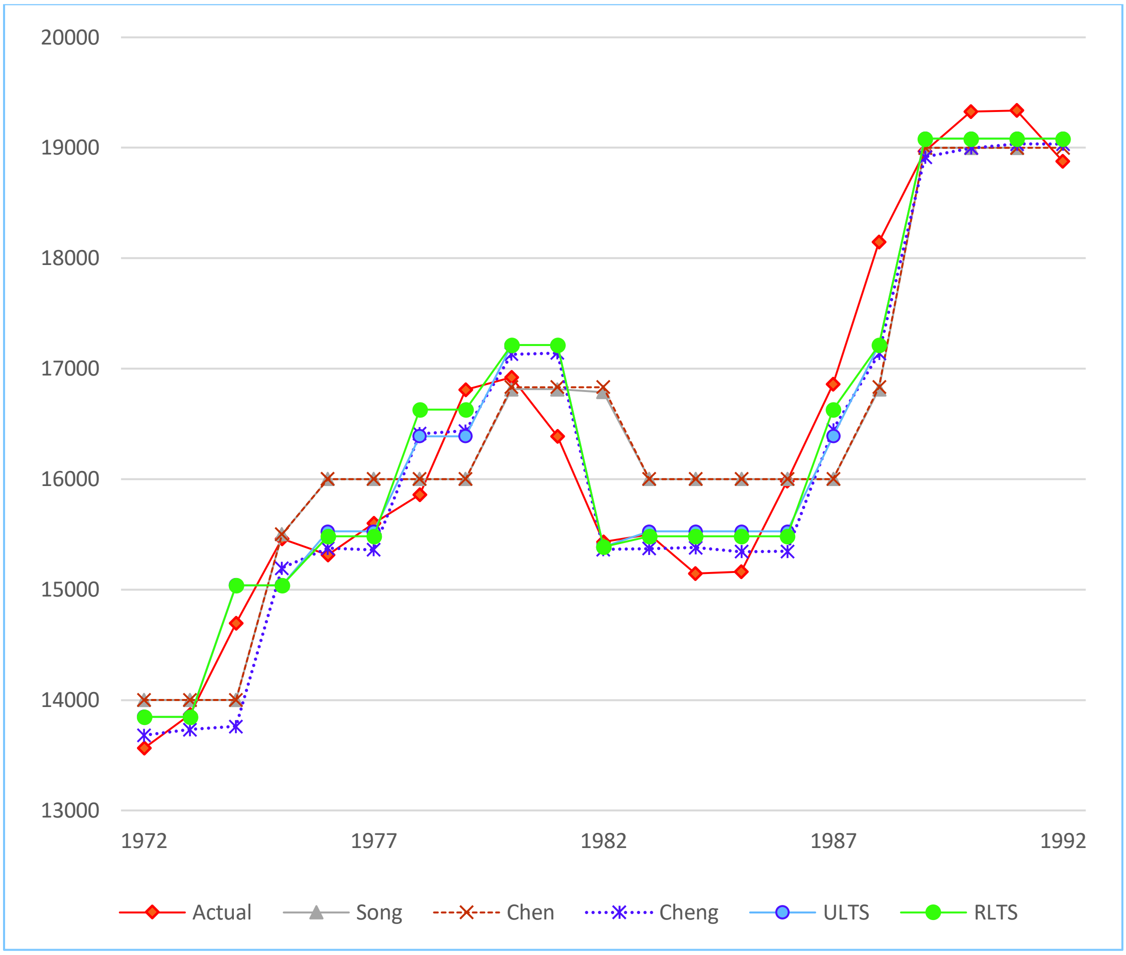

Figure 1: The comparison of enrollment forecasted results in case of 9 words

each month exhibited in Table 5 for the compatible comparison. Similar to that study, the mean absolute percentage error (MAPE) is applied and defined by the following formula to make the time series forecasting models' performance comparison, where F_i denotes the forecasting value and A_i is its actual value at the time i

$$MAPE = \frac{100\%}{N} \sum_i \left| \frac{F_i - A_i}{A_i} \right|. \tag{10}$$

The month simulation MAPE values of the proposed models, RFM and UFM, and the best Chen and Hwang's study are exposed in Figure 4.1. justify that the RFM outperforms the UFM in all four months, and the UFM outperforms the best results of Chen and Hwang's methods in all four months, June (Algorithm-B* and window size is 2), July (Algorithm-B and window size is 2), August (Algorithm-B* and window size is 2), and September (Algorithm-B* and window size is 3), in turn of 20.13%, 9.38%, 15.3% and 18.7%.

Table 5: The temperature forecasted results of ULTS and RLTS models

Day	June			July			August			September		
	AV	UFM	RFM	AV	UFM	RFM	AV	UFM	RFM	AV	UFM	RFM
1	26.1	-	-	29.9	-	-	27.1	-	-	27.5	-	-
2	27.6	27.84	27.84	28.4	29.56	29.57	28.9	28.34	28.34	26.8	26.64	26.38
3	29.0	28.16	28.39	29.2	29.28	29.17	28.9	28.69	28.74	26.4	26.64	26.38
4	30.5	29.74	29.45	29.4	28.90	28.68	29.3	28.69	28.74	27.5	26.06	26.08
5	30.0	29.87	29.87	29.9	28.67	28.76	28.8	28.69	28.74	26.6	26.64	26.38
6	29.5	30.11	30.11	29.6	29.56	29.57	28.7	28.69	28.74	28.2	26.06	26.08
7	29.7	29.74	29.45	30.1	28.67	28.76	29.0	28.69	28.74	29.2	27.81	27.96
8	29.4	29.74	29.45	29.3	29.56	29.57	28.2	28.69	28.74	29.0	29.15	29.15
9	28.8	29.74	29.45	28.1	28.67	28.76	27.0	28.50	28.6	30.3	29.15	29.15
10	29.4	28.60	28.47	28.9	29.28	29.17	28.3	28.34	28.34	29.9	29.67	29.99
11	29.3	29.74	29.45	28.4	28.90	28.68	28.9	28.50	28.6	29.9	29.67	29.99
12	28.5	29.74	29.45	29.6	29.28	29.17	28.1	28.69	28.74	30.5	29.67	29.99
13	28.7	28.60	28.47	27.8	28.67	28.76	29.9	28.50	28.6	30.2	29.67	29.99
14	27.5	28.60	28.47	29.1	28.25	28.25	27.6	28.81	28.81	30.3	29.67	29.99
15	29.5	28.16	28.39	27.7	28.90	28.68	26.8	27.60	27.60	29.5	29.67	29.99
16	28.8	29.74	29.45	28.1	28.25	28.25	27.6	28.34	28.34	28.3	29.15	29.15
17	29.0	28.60	28.47	28.7	29.28	29.17	27.9	27.60	27.60	28.6	27.81	27.96
18	30.3	29.74	29.45	29.9	28.90	28.68	29.0	27.60	27.60	28.1	27.81	27.96
19	30.2	30.11	30.11	30.8	29.56	29.57	29.2	28.69	28.74	28.4	27.81	27.96
20	30.9	30.11	30.11	31.6	31.47	31.47	29.8	28.69	28.74	28.3	27.81	27.96
21	30.8	29.87	29.87	31.4	30.21	30.96	29.6	28.81	28.81	26.4	27.81	27.96
22	28.7	29.87	29.87	31.3	30.21	30.96	29.3	28.81	28.81	25.7	26.06	26.08
23	27.8	28.60	28.47	31.3	30.21	30.96	28.0	28.69	28.74	25.0	26.06	26.08
24	27.4	28.16	28.39	31.3	30.21	30.96	28.3	27.60	27.60	27.0	27.16	27.16
25	27.7	28.23	28.23	28.9	30.21	30.96	28.6	28.50	28.6	25.8	26.64	26.38
26	27.1	28.16	28.39	28.0	28.9	28.68	28.7	28.50	28.6	26.4	26.06	26.08
27	28.4	28.23	28.23	28.6	29.28	29.17	29.0	28.69	28.74	25.6	26.06	26.08
28	27.8	28.60	28.47	28.0	28.90	28.68	27.7	28.69	28.74	24.2	26.06	26.08
29	29.0	28.16	28.39	29.3	29.28	29.17	26.2	27.60	27.60	23.3	23.93	23.93
30	30.2	29.74	29.45	27.9	28.67	28.76	26.0	26.89	26.89	23.5	23.93	23.93
31				26.9	28.25	28.25	27.7	26.89	26.89			

4.2. The proposed LTS-FM performance in increasing the forecasting precision by allowing the existing used forecasting L-variable word-set to grow

In this section, the study experiments to justify a specific feature mentioned at the end of Section 3, the proposed LTS-FM can increase the forecasting results by allowing the existing used forecasting L-variable word-set to grow. Recall that, as the linguistic counterpart of the numeric forecasting variable, its word-domain is potentially infinite.

The experimental scenario is described as follows:

- The selected fuzziness parameters' values for illustration: $\mathrm{m}(c^-) = 0.46$ and $\mu(L) = 0.52$.

- Let us assume that the time series application admin applied the proposed LTS-FM, using a forecasting L-variable 2-specificity word-set, $X_{(2)}$, of nine words to solve the forecasting time series problem. Now, the application users and admin require to increase the forecasting precision by permitting the word-set $X_{(2)}$ to grow to the 3-specificity one $X_{(3)}$ of 17 words.

- At the two latter application's life-cycle moments, assume that the application users

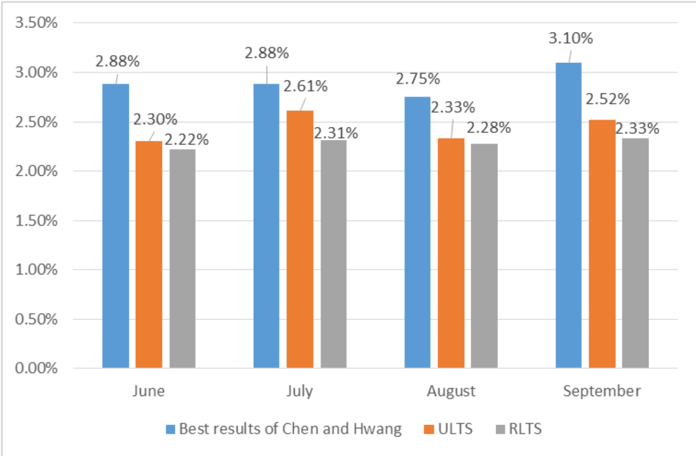

Figure 2: The MAPE values of temperature forecasted results by month (the lower, the better)

and admin require again two times increases the forecasting precision by permitting the word-set to grow, in turn, to the 4-specificity word-set, $X_{(4)}$, consisting of 33 words, and the 5-specificity one, $X_{(5)}$, consisting of 65 words. Where,

$X_{(2)}$ = {0, *V_small, small, L_small, medium, L_large, large, V_large*, 1},

$X_{(3)}$ ={0, *VV_small, V_small, LV_small, small, LL_small, L_small, VL_small, medium, VL_large, L_large, LL_large, large, LV_large, V_large, VV_large*, 1},

$X_{(4)}$ ={0, *VVV_small, VV_small, LVV_small, V_small, LLV_small, LV_small, VLV_small, small, VLL_small, LL_small, LLL_small, L_small, LVL_ small, VL_small, VVL_small, medium, VVL_large, VL_large, LVL_large, L_large, LLL_large, LL_large, VLL_large, large, VLV_large, LV_large, LLV_large, V_large, LVV_large, VV_large, VVV_large*, 1}.

Similarly, $X_{(5)}$ consists of 65 ordered words, which cannot be exposed here because of the limited space.

The experiments will justify that the proposed LTS-FM can considerably increase the forecasting results' precision when the word-set grows. Indeed, the study first constructs four LTS using in turn 9, 17, 33, 65 words, exposed in Figure 3, including the enrolment chart represented by the red line. One can observe that the richer in the words of the currently used word-sets, the more approximately approach the constructed LTS, recalling that, though the word-sets under consideration grow, their qualitative and quantitative words semantics are defined in the total forecasting L-variable context and, hence, they are consistent.

The experimental results in each case are represented as follows for economizing the space:

1) The LTS constructed from the numeric time series when the forecasting L-variable word-set growing.

The LTS constructed from the numeric time series using the 9-elements word-set when the forecasting L-variable word-set grows to, in turn, 17, 33, and 65 words, are represented in Table 6. Using their words' numeric values (or numeric semantics), i.e., their SQM v's values, one can represent these four constructed LTS in Figure 3. The red line represents the enrolment time series. For comparative analyzing the experimental results' examples, we assign words of only two constructed LTS, using the minimal number of words (9 words) and the maximal one (65 words), to the graph points determined by their numeric semantics. For example, the 65-words-graph point, at the year-value 1977, is represented by a boded marina color point associated with the word "*LVL_small*," while a rhombus point represents the 9-word-graph point associated with "*L_small*".

Table 6: The linguistic time series of enrollment data in case of 9, 17, 33 and 65 words

Year	**Enroll.**	**LTS with 9 words**	**Their numeric values**	**LTS with 17 words**	**Their numeric values**	**LTS with 33 words**	**Their numeric values**	**LTS with 65 words**	**Their numeric values**
1971	13055	*A_small*	13000	*A_small*	13000	*A_small*	13000	*VVVV_small*	13082
1972	13563	*V_small*	13742	*V_small*	13742	*LVV_small*	13557	*LVV_small*	13557
1973	13867	*V_small*	13742	*V_small*	13742	*LLV_small*	13943	*VLLV_small*	13838
1974	14696	*small*	14546	*small*	14546	*VLL_small*	14746	*VLL_small*	14746
1975	15460	*L_small*	15416	*L_small*	15416	*L_small*	15416	*L_small*	15416
1976	15311	*L_small*	15416	*L_small*	15416	*L_small*	15416	*VLLL_small*	15312
1977	15603	*L_small*	15416	*L_small*	15416	*LVL_small*	15616	*LVL_small*	15616
1978	15861	*medium*	16220	*VL_small*	15834	*VL_small*	15834	*VL_small*	15834
1979	16807	*L_large*	17163	*VL_large*	16672	*LVL_large*	16927	*LLVL_large*	16795
1980	16919	*L_large*	17163	*L_large*	17163	*LVL_large*	16927	*LVL_large*	16927
1981	16388	*medium*	16220	*medium*	16220	*VVL_large*	16437	*VVL_large*	16437
1982	15433	*L_small*	15416	*L_small*	15416	*L_small*	15416	*L_small*	15416
1983	15497	*L_small*	15416	*L_small*	15416	*L_small*	15416	*VLVL_small*	15513
1984	15145	*L_small*	15416	*LL_small*	14964	*LLL_small*	15199	*LLL_small*	15199
1985	15163	*L-small*	15416	*LL_small*	14964	*L LL_small*	15199	*LLL_small*	15199
1986	15984	*medium*	16220	*VL_small*	15834	*VVL_small*	16035	*VVL_small*	16035
1987	16859	*L_large*	17163	*VL_large*	16673	*LVL_large*	16928	*LLVL_large*	16795
1988	18150	*large*	18186	*large*	18186	*large*	18186	*large*	18186
1989	18970	*V_large*	19129	*V_large*	19129	*LLV_large*	18893	*VLLV_large*	19016
1990	19328	*V_large*	19129	*V_large*	19129	*LVV_large*	19346	*LVV_large*	19346
1991	19337	*V_large*	19129	*V_large*	19129	*L VV_large*	19346	*LVV_large*	19346
1992	18876	*V_large*	19129	*LV_large*	18638	*LLV_large*	18893	*LLV_large*	18893

2) The linguistic, logical groups (LLGs) of the constructed LTS and the forecasting simulation.

From the LTS constructed above, one can quickly determine their LLRs, from which LLGs can quickly be established and represented in Table 7 and Table 8, noting that for the constructed LTS utilizing 65 words, the LLRs of both LTS-FMs ULTS and RLTS are identical. Similar to above, utilizing the established individual constructed LTS' LLGs and applying the proposed LTS-FM to each of these LTS, one can obtain the forecasting simulation results of the LTS-FMs under examination represented in Table 9.

3) The advantages and performance of the proposed LTS-FMd and its developed LTS-FMs

This point aims to analyze the above simulation results to explain some proposed LTS-FMd's superiorities and advantages and justify the proposed LTS-FMs in increasing the forecasting results.

∘*Methodological advantages*: When a time series forecasting application is provided with the developed LTS-FM, which can deal with natural human words directly, it brings many outstanding advantages:

Usually, human users observe numeric time series (NTS) charts in terms of their words present in their natural language. Thus, natural and genuine relationships exist between the

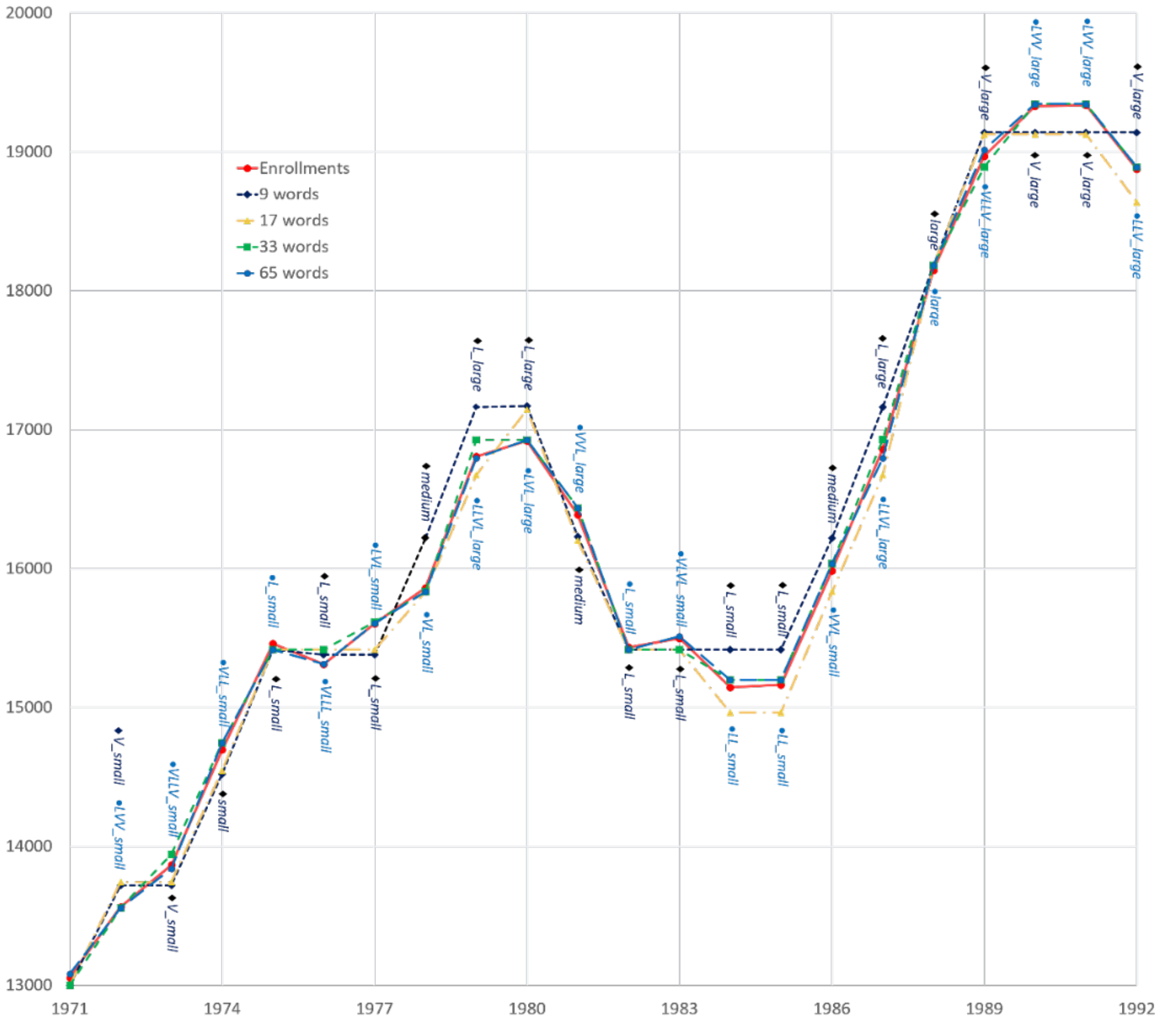

Figure 3: Four constructed LTS' graphs using the numeric semantics of, in turn, 9, 17, 33, and 65 words, and one graph (the red line) of the enrolment time series

inherent words' semantics and their numeric semantics that human users have established based on a long-term historical learning process. In this context, the proposed LTS-FMd dealing immediately with words is very natural to simulate how human users forecast based on a given NTS. Therefore, its LTS, constructed from the given NTS gave in Table 7, and their established LLGs are very useful for human-machine interaction.

In the established methodology's formalism, similar to the numeric domain, the semantics of every word of its L-counterpart variable, $\mathcal{X}_L$, should be defined in the context of the total $\mathcal{X}_L$. Consequently, the currently used word set words' semantics are scalable, i.e., their qualitative and quantitative semantics are maintained when the currently used word set grows. It is an outstanding feature because the constructed fuzzy sets and their associated L-labels of the existing fuzzy time series forecasting models do not have this feature.

This feature is necessary to simulate how human users use their words to make forecasting and essential in applications when human users desire to increase the forecasting results

Table 7: Linguistic, logical relationship groups of the enrollments in case of 9 and 17 words

Group	**ULTS with 9 words**	**RLTS with 9 words**	**ULTS with 17 words**	**RLTS with 17 words**
Group 1	*A_small* → *V_small*	*A_small* → *V_small*	*A_small* → *V_small*	*A_small* → *V_small*
Group 2	*V_small* → *V_small, Small*	*V_small* → *V_small, small*	*V_small* → *V_small, small*	*V_small* → *V_small, small*
Group 3	*small* → *L_small*	*small* → *L_small*	*small* → *L_small*	*Small* → *L_small*
Group 4	*L_small* → *L_small, medium*	*L_small* → *L_small, L_small, medium, L_small, L_small, L_small, medium*	*L_small* → *L_small, VL_small, LL_small*	*L_small* → *L_small, L_small, VL_small, L_small, LL_small*
Group 5	*medium* → *L_large, L_small*	*medium* → *L_large, L_small, L_large*	*VL_small* → *VL_large*	*VL_small* → *VL_large, VL_large*
Group 6	*L_large* → *L_large, medium, large*	*L_large* → *L_large, medium, large*	*VL_large* → *L_large, large*	*VL_large* → *L_large, large*
Group 7	*large* → *V_large*	*large* → *V_large*	*L_large* → *medium*	*L_large* → *medium*
Group 8	*V_large* → *V_large*	*V_large* → *V_large, V_large, V_large*	*medium* → *L_small*	*medium* → *L_small*
Group 9			*LL_small* → *LL_small, VL_small*	*LL_small* → *LL_small, VL_small*
Group 10			*large* → *V_large*	*large* → *V_large*
Group 11			*V_large* → *V_large, LV_large*	*V_large* → *V_large, V_large, LV_large*

precision. For instance, *Figure 3* exposes four LTS' charts using four word-sets of, in turn, 9, 17, 33, and 65 words. The proposed LTS-FMd guarantees that the qualitative and the quantitative semantics of four currently used word-sets, in different application life-cycle moments, are unique and do not vary when the currently used word-sets grew. It is *compatible* with how human experts use their words in reality.

For example, in *Figure 3*, at point (year) 1982, two words, "*L_small*" with marina color and "*L_small*" with black color, are identical though their semantics are defined in two different contexts of, respectively, nine and sixty-five words. The fuzzy time series formalism cannot ensure this property because the fuzzy sets, constructed in a singularity structure, are context-dependent.

∘*Forecasting performances*: The experiments were designed to justify that two proposed LTS-FMs performance can increase the forecasting precision significantly when the forecasting L-variable word-set size grows from 9 to, in turn, 17, 33, and 65 words. Based on the criteria MSE's and RMSE's values given in Table 9, it follows that.

When word-set size grows as above, the MSE's value of the proposed ULTS (RLTS) decreased, in turn, 49.85% (47.84%), 54.02% (49.55%), and 74.49% (72.11%), compared to the case of 9 words.

Similarly, the *RMSE*'s value of the proposed ULTS (RLTS) decreased, in turn, 29.05% (27.78%), 32.08% (28.98%), and 49.49% (47.19%).

The reason may be that the more affluent the forecasting L-variable word-set, the more

Table 8: Linguistic, logical relationship groups of the enrollments in case of 33 and 65 words

Group	**ULTS with 33 words**	**RLTS with 33 words**	**ULTS with 65 words & RLTS with 65 words**
Group 1	*A_small* → *LVV_small*	*A_small* → *LVV_small*	*VVVV_small* → *LLVV_small*
Group 2	*LVV_small* → *LLV_small*	*LVV_small* → *LLV_small*	*LLVV_small* → *V_small*
Group 3	*LLV_small* → *VLL_small*	*LLV_small* → *VLL_small*	*V_small* → *small*
Group 4	*VLL_small* → *L_small*	*VLL_small* → *L_small*	*small* → *LLL_small*
Group 5	*L_small* → *L_small, LVL_small, LLL_small*	*L_small* → *L_small, LVL_small, L_small, LLL_small*	*LLL_small* → *LLLL_small, VLLL_small*
Group 6	*LVL_small* → *VL_small*	*LVL_small* → *VL_small*	*LLLL_small* → *L_small*
Group 7	*VL_small* → *LVL_large*	*VL_small* → *LVL_large*	*L_small* → *LVL_small*
Group 8	*LVL_large* → *LVL_large, VVL_large, large*	*LVL_large* → *LVL_large, VVL_large, large*	*LVL_small* → *LVVL_large*
Group 9	*VVL_large* → *L_small*	*VVL_large* → *L_small*	*LVVL_large* → *VL_large, VVLL_large*
Group 10	*L_small* → *L_small, LVL_small, LLL_small*	*L_small* → *L_small, LVL_small, LLL_small*	*VL_large* → *VVVL_small*
Group 11	*LLL_small* → *LLL_small, VVL_small*	*LLL_small* → *LLL_small, VVL_small*	*VVVL_small* → *LLL_small*
Group 12	*VVL_small* → *LVL_large*	*VVL_small* → *LVL_large*	*VLLL_small* → *LL_small*
Group 13	*large* → *LLV_large*	*large* → *LLV_large*	*LL_small* → *LL_small, LLVL_small*
Group 14	*LLV_large* → *LVV_large*	*LLV_large* → *LVV_large*	*LLVL_small* → *LVVL_large*
Group 15	*LVV_large* → *LVV_large, LLV_large*	*LVV_large* → *LVV_large, LLV_large*	*VVLL_large* → *VLLV_large*
Group 16			*VLLV_large* → *LVV_large*
Group 17			*LVV_large* → *LVV_large, LLV_large*

highly approximates the given NTS the constructed LTS. Further, considering the set of the LLRGs generated from such a constructed LTS as human knowledge, it is evident that the richer such a human knowledge, the higher precision the forecasting results the proposed LTS-FM can offer. For the word-sets of 33 and 65 words, one can see that the constructed LTS' charts in *Figure 3* well approximate the given enrollments time series and, hence, their LLRGs are much richer and of higher quality than the LLRGs generated in the case of 9 words. It is a methodological basis to ensure that the proposed LTS-FMd can develop linguistic time series forecasting models with expected high performance.

5. CONCLUSION AND FUTURE WORKS

The study aims to establish a more comprehensive formalized methodology, dealing with words directly, to simulate how human exerts make a forecasting process in terms of their words in the natural language when observing a given numeric time series (NTS). Then, the study ignores external political and economic factors that influence the forecasting process, focuses on the variations of the given NTS themselves, and tries to simulate the human

Table 9: Simulation results of the forecasting models under examination of Alabama Enrollments from 1971 to 1992 with different used words.

Year	**Real values**	**ULTS.9**	**RLTS.9**	**ULTS.17**	**RLTS.17**	**ULTS.33**	**RLTS.33**	**ULTS.65 & RLTS.65**	
1971	13055								
1972	13563	13742	13742	13742	13742	13557	13557	13557	
1973	13867	14144	14144	14144	14144	13943	13943	13838	
1974	14696	14144	14144	14144	14144	14746	14746	14746	
1975	15460	15416	15416	15416	15416	15416	15416	15416	
1976	15311	15818	15646	15405	15409	15411	15412	15412	
1977	15603	15818	15646	15405	15409	15411	15412	15617	
1978	15861	15818	15646	15405	15409	15834	15834	15834	
1979	16807	1690	16581	16673	16673	16928	16928	16795	
1980	16919	17190	17190	17675	17675	17184	17184	17557	
1981	16388	17190	17190	16220	16220	17184	17184	16437	
1982	15433	16290	16581	15416	15416	15416	15416	15416	
1983	15497	15818	15646	15405	15409	15411	15412	15412	
1984	15145	15818	15646	15405	15409	15411	15412	15199	
1985	15163	15818	15646	15399	15399	15617	15617	15617	
1986	15984	15818	15646	15399	15399	15617	15617	15617	
1987	16859	16290	16581	16673	16673	16928	16928	16795	
1988	18150	17190	17190	17675	17675	17184	17184	17557	
1989	18970	19129	19129	19129	19129	18894	18894	19016	
1990	19328	19129	19129	18884	18966	19346	19346	19346	
1991	19337	19129	19129	18884	18966	19120	19120	19120	
1992	18876	19129	19129	18884	18966	19120	19120	19120	
MSE		**231,856**	**212,053**	**116,278 (49.85%)**	**110,600 (47.84)**	**106,962 (54.02%)**	**106,971 (49.55%)**	**59,146 (74.49%)**	**59,146 (72.11%)**
RMSE		**481.51**	**460,49**	**341.65 (29.05%)**	**332.57 (27.78%)**	**327.04 (32.08%)**	**327.06 (28.98%)**	**243.2 (49.49%)**	**243.2 (47.19%)**

experts' capability in forecasting, based on the given NTS, through their natural language. One can imagine that by observing the given NTS's variations, human experts can quickly transform its variation into words' variations, though not exact, called a linguistic time series (LTS). Then, based on the words' relationships of the LTS, they do reasoning in some way, in their mind, to estimate the forecast value. For achieving this aim, this study establishes a more comprehensive and complete methodological viewpoint than the one given in Nguyen et al. (2019, 2020), justify its outstanding performance advantages, and resolve the following main questions. Human beings uses their natural language to interact with the reality around them and do reasoning for effectively making decisions to struggle for their existence and development successfully. The ways human experts resolve their decision-making problems, including NTS forecasting ones, must realize, of course, based on specific particular inherently existing nature of the natural language, including its structure and formalism, that one should reveal or simulate in some formalized ways. Restricted to the NTSs' forecasting problems, this study tried to establish a formalized methodology to simulate the way just mentioned that human experts apply "their language-based formalism" to solve NTSs' forecasting problems in daily lives.

1) Establishing a formal basis to directly handle the forecasting L-variable words' semantics in relations to its numeric universe. In order to computationally simulate how to forecast the human experts based on a given numeric time series (NTS) in terms of their own forecasting L-variable's words present in the natural language, one should formalize the following problems underlined the human linguistic forecasting process:

(i) The forecasting L-variable word-domain, which consist of all possible L-variable words existing in the natural language and, hence, potentially infinite, must first be soundly formalized, called a hedge algebra, to model the inherent semantic structure of any L-variables;

(ii) A method for simulating the human expert capability in transforming the given NTS into an LTS based on a "similarity" of its numeric values to the numeric semantics of the currently used forecasting L-variable words. However, in this context, it can make sense only when the developed method is equipped with a formalism to handle L-words together with their numeric semantics consistently. This study's formalism was ensured by Tarski et al. interpretability concept, which requires that the forecasting L-variable's inherent semantic word-domain structure must be interpretable in its numeric semantic structure. Alternatively, it means that the word-domain numeric semantics must be an isomorphic image of its inherent semantic word-domain structure.

Moreover, this formalism ensures that the inherent words' semantics, both qualitative and quantitative, are defined in the full L-variable context. Therefore, the declared currently used forecasting L-variable word-set is scalable, i.e., when human experts permit it to grow, all distinct semantic types of the currently used words cannot be changed. It is an essential, natural, and practical feature to guarantee the consistency of the NTS's variation knowledge growing in time. As examples, one can easily find out many LLRGs present in Table 's 7 and 8, e.g., Groups 1 and 2 in Table 7, that were already established in the former phases, whose semantics were still thoroughly maintained in the next phases;

(iii) The proposed methodology permits developing a reasoning method based on the LLRGs' human experts' knowledge generated from the constructed LTS. Similar to those utilized by the counterpart methods, in this study, we used the simple reasoning method developed in the paper by Nguyen et al. (2019, 2020) mentioned above, i.e., reasoning by taking the average of the numeric semantics of the words appearing on the right side of the LLRG under consideration.

2) Developing an LTS-FM that can soundly and directly handle the potentially infinite forecasting L-variable words and their numeric semantics for increasing the forecasting results precision. This formalism developed motivated by the following practical scenario: Assume that human experts already applied the developed LTS-FM to an NTS forecasting application at its life-cycle moment in the past. At the present moment, the application experts may, naturally, desire to obtain more forecasting precision. Since the richer the experts' knowledge describing the NTS variation, the higher exact forecasting decisions the developed LTS-FM can achieve, this study offers a formalism for permitting the proposed LTS-FM to enlarge the knowledge LLRGs generated from the constructed LTS, by allowing the currently used word-set to grow, to rich a more plentiful knowledge about the given NTS variation. As discussed next, one can see that the proposed LTS-FM can increase the forecasting results significantly.

3) Considering the LLRGs as human users' linguistic knowledge about the NTS variation, an outstanding feature of the proposed LTS-FM is its ability to simulate the scalable users' knowledge growth to increase their prediction. One can observe that human domain (e.g., the medical) knowledge always grow in practice to increase the prediction precision, based on the scalability feature of this domain knowledge, i.e., the existing knowledge meaning is, in general, still maintained when it is required to grow. It is an essential property for increasing the forecasting precision by allowing the existing NTS's variation knowledge to grow. For the first time, the study simulated this situation by permitting the existing used word-set under consideration to grow from 9 words to, in turn, 17, 33, and 65 words, which implies the growth of their respective LLRGs' sets, considering the given NTS's variation knowledge. The obtained experimental results justified that the forecasting results' precision was also, in turn, increased significantly for both criteria MSE and RMSE, see Table 10, and, therefore, show the effectiveness of the proposed approach.

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