INFORMATION AND MATHEMATICAL STRUCTURES CONTAINED IN THE NATURAL LANGUAGE WORD DOMAINS AND THEIR APPLICATIONS

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Abstract. The study stands on the standpoint that there exist close and essential relationships between real-world structures and their information provided to human beings in reality. Because the natural language plays a specifically vital role in daily human activities, e.g., in capturing, conveying, and communicating information and accumulating knowledge containing useful high-level information, consequently, it must contain certain semantics structures, including linguistic (L-) variables' semantic structures, which are fundamental, similar to the math variables' structures. In this context, the fact that the (L-) variables' word domains can be formalized as algebraic semantics-based structures in an axiomatic manner, called hedge algebras (HAs,) is still a novel event and essential for developing computational methods to simulate the human capabilities in problem-solving based on the so-called natural language-based formalism. Hedge algebras were founded in 1990. Since then, HA-formalism has been significantly developed and applied to solve several application problems in many distinct fields, such as fuzzy control, data classification and regression, robotics, L-time series forecasting, and L-data summarization. The study gives a survey to summarize specific distinguishing fundamental features of HA-formalism, its applicability in problem-solving, and its performance and advantages.

Keywords. Fuzzy sets; Hedge algebras; Linguistic rule bases; Linguistic variable semantics; Qualitative word semantics; Words' fuzziness.

1. INTRODUCTION

The natural language plays an essential role in human beings' daily activities in the struggles for their existence and the development of their societies. Especially, their different domain knowledge has been accumulative and grown along with such struggles and developments. These facts assert that the natural language can communicate necessary information between people, represent and store specific human domain knowledge, essential components for human reasoning and decision-making. Accepting the strictly close relationship between

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Dedicated to Professor Phan Dinh Dieu on the occasion of his 85th birth anniversary.

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the structure and the information, which states that the higher structure of a real-world entity, the more information it can provide us, one may conclude that the natural language itself does involve different structural-semantic relationships (between L-variables/attributes,) or, in other words, different L-semantic structures.

Moreover, as shown in this study, the L-variables' word-domains contain inherent math structures that can be established in an axiomatization manner and, then, one can discover L-structures defined on the Cartesian product of L-variable word-domains. Notice that the natural language accompanies human beings since the very moment humans appeared on earth, but the math structures contained in the L-variables' word domains had not been discovered until 1990 and 1992 in the studies [1, 2]. These L-variables' word-domains math-structures are called hedge algebras (HAs), in which linguistic hedges, i.e., those adverbs, like *very*, *extremely*, *little*, *rather*, . . . , that have a functionality to modify or intensify words play their unary operators. Thus, it should still be emphasized more strongly the structure-information relation. It can be said that the closed relationships between *Structure, Information, and Information processing* play a fundamental role in motivating the authors of that studies to discover the L-variables' word domains math-structures, though necessary fundamental knowledge of the universal algebras and math-logics is also very important.

Since then, many fundamental theoretical studies, such as the studies on the refined HAs to make the word domains partially ordered semantic structures to become lattices [4], complete and linear HAs [7], and their quantification [8]. Interestingly, the quantification of an L-variable word domain could be established based on only the L-words fuzziness model concept, their fuzziness intervals, and an isomorphic order mapping for every L-variable. These achieved theoretical results sufficiently form a necessary formalized basis for solving application problems in different fields. One can find its applications in cybernetics [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30], in classification and regression [31, 32, 33], decision making [38, 39], multi-channel image processing [37], robotics [45, 46, 47, 48, 49], networking [43, 44], fuzzy linguistic time series [40, 41, 42], linguistic data summaries [35, 36], databases and object-oriented databases based on hedge algebraic semantic [58, 59, 60], and it is not the end.

It should also emphasize that the establishment and the development of the HAs theory and its initial application results were inherited from the fuzzy sets theory's ideas and achievements. However, since both theories aim to simulate how human experts do reasoning and make decisions in their daily lives, in terms of their L-sentences, an essential fact makes an HA-approach different from a fuzzy set-based approach as follows. The HA approach can directly handle L-words and simulate how human experts utilize L-words and their knowledge to solve the problems encountered in their daily lives. In this formalism, the semantic order and generality-specificity relationships between every L-variable's words play a crucial role, noting that the individual L-variables' words are nothing but their semantic relationships, i.e., is their *structures* essential. Recall that, in mathematics, one focuses not on individual universe elements but their relationships, i.e., their *structure.*

On the other hand, fuzzy sets are ideal numerically computational objects to represent the L-variables' words' fuzziness. Thus, in an HA-approach, when necessary, it is necessary to utilize fuzzy sets to represent the words' semantics soundly, where 'soundness' here means that one must have a reliable formalism to ensure that his designed fuzzy sets can *appropriately* represent their associated words' semantics. Therefore, because the designed

words' fuzzy sets *structures* are essential, the HA approach requires that the fuzzy sets *structures* designed to represent the L-variables' word domains must be the *isomorphic* images of the inherent semantic L-variables' word domains structures, following Tarski et al.'s interpretability concept in the math-logics field. It implies that to simulate how human experts solve a problem in their natural language-based formalism, the declared L-variables' semantic word *structures* under consideration must be *interpretable in* their respective constructed fuzzy sets' *structures*. In other words, there exists an isomorphism translating the declared L-variables' semantic word *structures* into their respective constructed fuzzy sets' *structures*.

The study's primary purpose is to give a survey on the HA-approach to simulate the human experts' capability to handle their natural language's words to solve their daily problems and emphasize specific features distinguished from the fuzzy set-based approaches. Especially, it emphasizes that the constructed individual fuzzy sets representing their L-labels are nothing, but their defined *structures* are essential to solving a given application problem. With this outstanding specific feature, the terminology '*structure*' is written in italic, if necessary, to emphasize the essential role of the objects' structure under consideration.

The study organization is as follows. Sect. 2 offers to present how one can formalize the L-variables' word domains to become HAs in an axiomatization manner, based on some essential inherent semantic words' properties, and axiomatize their quantification, or the inherent relationships between the words' qualitative semantics and their numeric semantics in reality. It forms a formalized basis to construct fuzzy sets' structures to represent their L-variables' word domain semantic structure, whose fuzzy sets' cores are singleton.

In practice and application, the fuzzy set cores may not be a singleton, say intervals. They numerically represent the so-called words semantics cores. Thus, Sect. 3 devotes to axiomatize the word semantics cores' structure in the context of the total word domains' semantic structures. It establishes a formalized math basis to construct the fuzzy set structures representing the L-variable words' semantics, whose fuzzy set cores are intervals.

Applications of the HA-methodology to justify the HAs theory's applicability and its performance can be found in Sect. 4. Restricted to the study presentation space, we present only a few typical applications, comprising an HA's application in the fuzzy control, the ones in data classification, and the ones in the linguistic data summarization, which exposes a specific feature that characterizes the HA-methodology advantages and benefits.

Conclusions and some research discussions will be discussed in Sect. 5.

2. HEDGE ALGEBRAS - THE DISCOVERED QUALITATIVE SEMANTIC STRUCTURES OF THE L-VARIABLES' WORD DOMAINS

2.1. Inherent abstract algebras hidden in every L-variable word-domain in the natural language

The word-domain of an L-variable $\mathfrak{X}$ in the natural language (NL), like the human AGE or the cars' SPEED, in reality, is the set of its words present in the NL, denoted by $D(\mathfrak{X})$. For example, $D(\text{AGE})$ contains the words like '*young*', '*old*', '*Very (V_-) young*', '*V_-old*', '*Rather (R_-) young*', '*R_-old*', ... As discussed in Sect. 1, the NL can describe and convey extremely rich amounts of information about the real world (RW) and, therefore, from the closed relationship between *structure* and *information*, one can infer that its word-domain $D(\mathfrak{X})$ must contain a *complex semantic structure*.

This section explains how this structure can be discovered mathematically.

2.1.1. The syntactical semantics of the given L-variable $\mathfrak{X}$ and its abstract algebra

Similar as shown just above, for every $\mathfrak{X}$, $Dom(\mathfrak{X})$ contains two *atomic* (or *primary*) words, denoted by c^- and c^+, e.g., '*young*' and '*old*' (or '*slow*' and '*fast*,') respectively, of AGE (or, of SPEED.) Let us denote by H the set of all possible hedges, i.e., the adverbs that have the functionality to modify or intensify an adverb or an adjective. For example, we may list here the adverbs such as *very, extremely, little, rather, more-or-less*, and it is not the last ones. However, the study is restricted to those hedges so that the $\mathfrak{X}$'s word domain will become a *lineally* semantics-based ordered structure. Moreover, for a given application, in practice, its human users require to use an $\mathfrak{X}$'s 'words dictionary' with a restricted number of its words, and not all hedges at each moment of the application life-cycle. Thus, for an application, the $\mathfrak{X}$'s syntactical semantics determination is essential, and its users require a careful $\mathfrak{X}$'s syntactical semantics declaration. Consequently, which set H of hedges is declared to determine the $\mathfrak{X}$'s syntactical semantics is crucial.

2.1.2. The $\mathfrak{X}$'s syntactical semantics or its word-domain viewed as an abstract algebra

If we search on Google those words containing the string of five the adverbs '*very*', we may obtain many, many words, one of which is '*very, very, very, very, very, very, very, very long* time. Thus, one can observe that for a given declared $\mathfrak{X}$'s syntactical semantics or, equivalently, a given set H, every $\mathfrak{X}$'s word-domain word, x, is of the string form, say $x = h_n \ldots h_1 c$, $c \in G = \{c^-, c^+\}$, and $h_j \in H$, $\forall j \in \{1, \ldots, n\}$. As a string, the length of x is its string length, denoted by $|x|$, i.e., $|x| = n + 1$.

Hence, at this moment, such an $\mathfrak{X}$'s syntactical semantics can be represented by an abstract algebra, $\mathcal{A}^{\mathfrak{X}} = (X^{\mathfrak{X}}, G, C, H)$, where $C = \{\boldsymbol{0}, W, \boldsymbol{1}\}$ is the set of $\mathfrak{X}$'s L-constants, e.g., for the L-variable TRUTH, whose constants, in turn, are '*absolutely false*,' '*neutral*,' and '*absolutely true*,' for instance, and every hedge h, $h \in H$, is considered as a unary operation of $\mathcal{A}^{\mathfrak{X}}$. Therefore, it is called *hedge algebra* (HA).

Fuzziness models of the $\mathfrak{X}$'s words: An essential feature characterizing the fuzzy information, including the fuzzy sets, is their fuzziness. Thus, a question arising is how one can describe or model the words' fuzziness. In the HA-approach to the fuzzy linguistic information, we can define this concept mathematically, i.e., the words' fuzziness can be defined in the algebraic structure of $\mathcal{A}^{\mathfrak{X}}$.

In fact, from the intensifying functionality of hedges, for every $\mathfrak{X}$'s word x, the set $H(x)$ consists of all $\mathfrak{X}$'s words intensified by the hedges in H, i.e., $\forall y \in H(x)$, y is different, but intensified/modified from x by hedges in H. In addition, for every linguistic constant $c \in C$, we have $hc = c$, which implies that $H(c) = \{c\}$, which is singleton. Hence, the constants are not uncertain, or they are exact information. Thus, $H(x)$ can be considered a fuzziness model of x. The larger the set $H(x)$, the more fuzziness of x.

The $\mathfrak{X}$'s words fuzziness models have the following properties offering for establishing HAs in an axiomatization manner: For $\forall x, y \in X^{\mathfrak{X}}$,

a) $y \in H(x)$ implies that $H(y) \subseteq H(x)$. That is, y (or, x) is less (or, more) fuzziness than x (or, y).

b) For $\forall x, y \in X^{\mathfrak{X}}$, x and y are said to be *independent* if we have $x \notin H(y)$ & $y \notin H(x)$.

   Then, if x and y are *independent*, we have $H(y) \cap H(x) = \emptyset$. Thus, c^- and c^+ are independent.

   For all different h, $h' \in H$, and every $x \in X^{\mathfrak{X}}$, different from the constants, $H(hx) \cap H(h'x) = \emptyset$.

c) For every $x \in X^{\mathfrak{X}}$, $H(x) = \bigcup_{h \in H} H(hx)$.

By the intensifying or modifying functionality of the given hedges, we semantically have that the order of the set of $H(hx)$ follows the order of their words, hx's, $h \in H$, i.e., $hx \le h'x \Rightarrow H(hx) \le H(h'x)$. Moreover, we also have $H(G) = H(c^-) \cup H(c^+)$, $H(c^-) \cap H(c^+) = \emptyset$, and $X^{\mathfrak{X}} = H(G) \cup C$. Similar as above, the order of the sets, $H(c^-)$ and $H(c^+)$, follows the semantic order of c^- and c^+.

2.1.3. The multi-aspect qualitative semantics of L-variables

To describe, convey, and represent the information, including structural information (human domain linguistic knowledge, for instance) sufficiently, how rich the qualitative semantics of the L-variables in the natural language is essential. This point shows that the qualitative semantics of every L-variable, say $\mathfrak{X}$, involves two relations, the semantic order-based relation and the generality vs. specificity relation, which play essential roles for human domain experts accumulating their domain knowledge, doing reasoning, and making their decisions to solve their application problems encountered in practice.

$\circ$ *The semantic order-based relation on $X^{\mathfrak{X}}$*: Because, in order to make decisions, human beings need comparison criteria to evaluate their encountered alternatives, it is natural that there exists an inherent semantics order-based relation on $X^{\mathfrak{X}}$ in the natural language. Consequently, a semantic order-based relation on $X^{\mathfrak{X}}$ exists, denoted by $\le$ and called the semantic order relation. It was shown in the studies [1, 2, 4, 7, 8, 32] that this order relation can be determined based on the following observations, referring to the syntactical semantics discussed in Subsections 2.1.1 and 2.1.2:

The semantic order of the atomic words and hedges, and their 'algebraic' signs: Let the syntactical semantics of $\mathfrak{X}$ is declared by giving the names of the atomic words in G and the set of hedges, $H = \{R \ (rather), L \ (Little), M \ (More), V \ (Very), E \ (Extremely)\}$. First, it is observed that $c^- \le c^+$, e.g., *young* $\le$ *old*, and they have the reverse semantic tendency directions, called their (algebraic) sign, deduced from the fact that, considering the hedge V and fix it, while $Vc^- \le c^-$, one sees that $c^+ \le Vc^+$. Conventionally, the semantic tendency direction of c^- (or, c^+) is called *negative* (or, *positive*) and denoted as $sign(c^-) = -1$ ($sign(c^+) = +1$.) One can check that the hedges M and E have the same direction action effect as V. For example, for M, we also have $Mc^- \le c^-$ and $c^+ \le Mc^+$. Since these hedges' direction effect, which intensifies the atomic word semantics, is compatible with their respective atomic tendency direction, they are called *positive* hedges. Therefore, we define $sign(M) = sign(V) = sign(E) = +1$.

It can be verified that such hedges' direction effect of the remaining hedges, R and L, is reverse to the one of the positive hedges and, hence, they are called *negative,* and we define $sign(R) = sign(L) = $ -1. Thus, we may set $H^- = \{R, L\}$ and $H^+ = \{M, V, E\}$.

Accordingly, at this moment, we begin to see that the L-variable $\mathfrak{X}$ inherently contains many interesting qualitative semantic properties.

The relative signs between hedges and the sign of the words different from the constants: As discussed previously, for every $\mathfrak{X}$'s word x different from the constants, it is syntactically represented as a string, $x = h_n \ldots h_1 c$, $c \in G$, h_j's $\in H$, e.g., $x = VL\_young$, $y = RL\_young$, and $y = VE\_young$. From the *inherent* words' semantics, we can verify that $V\_young \leq young \leq RL\_young \leq L\_young \leq VL\_young$.

Table 1. The relative sign of a hedge in a row w.r.t. a hedge in a column

$sign(k,h)$	E	V	M	R	L
E	+	+	+	−	+
V	+	+	+	−	+
M	+	+	+	−	+
R	−	−	−	+	−
L	−	−	−	+	−

From the inequalities, $young \leq RL\_young \leq L\_young$, it follows that, while L modifies '*young*' to becomes a *greater* word, '*L_young*,' R modifies the word L_young to become a *smaller* word, '*RL_young*.' That is, the two modification directions caused by the hedge L, applying to '*young*,' and R applying to '*L_young*' are reverse. In this case, we say that the *relative* sign of R applying to L is *negative* and write $sign(R, L) = $ -1. By a similar argument, from the inequalities, $young \leq L\_young \leq VL\_young$, we infer that the two modification directions caused by L, applying to '*young*,' and V applying to '*L_young*' are the same, and we say that the *relative* sign of V applying to L is *positive* and write $sign(V, L) = +1$.

Consequently, for every x given as above, i.e., $x = h_n \ldots h_1 c$, $c \in G$, x has also now a sign defined as follows

$$sign\,(x) = sign\,(h_n, h_{n-1}) \ldots sign\,(h_2, h_1)\,sign\,(c) \in \{-1, +1\}.$$

The meaning of algebraic sign of a word x different from the constants is: $sign(hx) = +1 \Leftrightarrow hx \geq x$. Since $hx \neq x$ and, we can semantically observe that $x \notin H(hx)$, for all $h \in H$.

$\circ$ *The generality-specificity relation on* $X^{\mathfrak{X}}$: An especially specific feature of the L-variables' word-semantics is the so-called generality-specificity (GS-) relation. It is well-known that the GS-relation is essential for human beings to formulate their knowledge pieces when they accumulate their domain knowledge. It can be seen that this concept can be formally defined in a very general way, as follows:

By the hedges' functionality, every hedge h intensifies a word x to obtain a more specific one, the hx. Then, if $|y| = k > 0$, y is said to be $k-$specific or of the $k-$specificity. The larger the length of the words, the higher their specificity. Thus, in the HA-approach, a word x is said to be more general than another word y if the inclusion, $H(x) \supseteq H(y)$, is valid and denoted by $g(x, y)$, called the generality-specificity relation.

It can be formally verified that $g(x, y)$ a binary *partial* relation $g \subseteq X^{\mathfrak{X}} \times X^{\mathfrak{X}}$ that satisfies the following conditions (refer to [36]):

(i) For $x \in X$ and $h \in H$, $G(x, hx)$ holds.

(ii) *Reflexiveness*: For $x \in X$, $G(x, x)$ is valid.

(iii) *Antisymmetry*: For $x, y \in X$, $G(x, y)$ and $G(y, x) \Rightarrow x = y$.

(iv) *Transitivity*: For $x, y, z \in X$, $G(x, y)$ and $G(y, z) \Rightarrow G(x, z)$.

2.1.4. The L-variables' Hedge algebras, developed in an axiomatic way, as algebraic math-models of their word domains semantic structures

It should emphasize that all syntactical and qualitative semantic properties are objectively discovered from the natural language, i.e., they inherently exist in the natural language, and one can observe them. Some of these words' semantic properties are utilized to establish the axiom systems in studies [1, 2, 4, 7, 8, 32] to develop their specific hedge algebras (HAs). Hence, mathematically, the HAs can model the L-variables' word domains in the natural language soundly.

Since, as discussed above, for every L-variable $\mathfrak{X}$, there exists an inherent *partial* generality-specificity relation on $X^{\mathfrak{X}}$, i.e., $g(x, y)$ is valid, for certain $x, y \in X^{\mathfrak{X}}$, if and only if $H(x) \supseteq H(y)$ Thus, be restricted to the underlying set $X^{\mathfrak{X}}$ of the HA $\mathcal{A}^{\mathfrak{X}}$ and focus to only the relations on $X^{\mathfrak{X}}$, $\leq$ and g, we obtain the *structure* $\mathcal{A}^{\mathfrak{X}} = (X^{\mathfrak{X}}, \leq, g)$ called the *multi-aspect semantic structure* of the L-variable $\mathfrak{X}$'s semantics to emphasize its semantic structural complexity. The natural language capability in describing, conveying, and representing real-world information results in a complex semantic structure that exists and is contained in the L-variable word domains. It justifies again an essential close relationship between the *structure* and the *information* for the natural language.

2.2. An axiomatic quantification of the L-variables multi-aspect semantic structures

The *multi-aspect semantic* structure examined in Subsection 2.1 will be utilized to construct interpretable fuzzy set-based semantics of L-variables, whose cores are singleton, e.g., the triangular fuzzy sets. In the HA approach, one can also establish, in an axiomatic way, a formalism to construct interpretable fuzzy set-based semantics, whose cores are intervals, e.g., the trapezoid fuzzy sets, as will be discussed in the next section.

The readers, now, know that the L-variables in the natural language contain their rich and complex math structures that can be defined in an axiomatic way. However, they still have no numerically computational characteristics essential for developing software products to simulate the expert capability, for example, in drawing L-information or human knowledge from a given large numeric dataset. Thus, how can one quantify the L-variable word-domains multi-aspect semantic structures in an axiomatic manner? Based on a long-term trial-error learning process, human experts now can establish conventional uncertain relationships between the L-variables words semantics and their respective numerical semantics, acceptable for the daily activities of a human community. For example, knowing the high of a person is 174 cm, an adult can linguistically inform another person that that person is '*high.*' That is, inherent relationships exist between the word-semantics and their numeric one.

Another type of the words' numerically computational semantics is the words' interval ($i-$) semantics. For example, one may accept that the Vietnamese persons' middle age is approximately from 43 to 49 years. Thus, the $i-$semantics of '*middle age*' is the interval [43, 49].

In this point, we show that one can discover the *inherent* relationships between the *qualitative*, the *numerical*, and the *interval* semantics of the L-variable words in an axiomatic way. From the study standpoint, this task is theoretically essential, even though there were no formal relationships between the designed fuzzy sets and their associated L-labels in the fuzzy sets' framework (hence, in which experimental study is essential to justify the applicability of a proposed method.) However, recall Zadeh's idea that a developer may lose some information if he uses an exact computational object to model uncertain/fuzzy information. Despite this, the study will justify that the hedge algebras theory, including their quantification one, can establish a math-formalism to connect the qualitative words' semantics with their desired exact numerically computational semantics, and it can bring many advantages for practical problem-solving.

This formalism is still also flexible by tuning the L-variables' fuzziness parameters only.

The axiomatic quantification of an HA, $\mathcal{A}^{\mathfrak{X}}$, can be shortly described as follows. It starts with the determination of the words' numerical semantics. Because the linearly semantic order-based structure is now a math model of $\mathfrak{X}$'s word-domain, its words' numerical semantics must be the order-isomorphic images of the $\mathfrak{X}$'s word-domain order-based structure. Thus, the desired quantification of $\mathcal{A}^{\mathfrak{X}}$ should start with a given order isomorphism $f^{\mathfrak{X}}$: $X^{\mathfrak{X}} \to [0, 1]$, where $[0, 1]$ is the normalized universe of $\mathfrak{X}$.

$f^{\mathfrak{X}}$ is one-to-one mapping satisfying:

(i) It preserves the order of the words, i.e., $x \leq y \Rightarrow f^{\mathfrak{X}}(x) \leq f^{\mathfrak{X}}(y)$;

(ii) $f^{\mathfrak{X}}(X^{\mathfrak{X}})$, which is countable, is dense in $[0, 1]$.

This isomorphic mapping $f^{\mathfrak{X}}$ is called in [8] a semantically quantifying mapping (SQM). It was proved that such a given SQM $f^{\mathfrak{X}}$ could determine, for every word x, an x's fuzziness interval, a subset of $[0, 1]$, denoted by $\Im^{\mathfrak{X}}(x)$ which is the topological closure of the countable set $f(H(x))$. One can interpret $\Im^{\mathfrak{X}}(x)$ as the set of all the $\mathfrak{X}$'s numerical universe values semantically compatible with the x's semantics. Hence, it is called the *fuzziness interval* of x, and its length is called its *fuzziness measure*

It was shown in [36] that the fuzziness intervals' structure of every L-variable is also rich and complex, and it can be represented by a bush-graph similar as shown in Fig. 1 below, when imagining that the oval figures, associated with the strings of the form $h_0 x$, is reduced to single points, in this case. However, the limited study space does not allow to present here, but the readers can refer to the axiomatic quantification of the enlarged hedge algebras described in the next section.

2.3. The enlarged hedge algebra of a given ordinary hedge algebra – a formal math basis for constructing the words fuzzy set-based semantics, whose cores are intervals

The section aims to show that the HA-approach follows a principle that one should always establish a math-formalism, established in an axiomatic way, to produce the L-variable words' *interpretable* fuzzy set-based semantics whose cores are intervals, from just its right word domain semantic structure. Here, 'interpretable' is in Tarski et al.'s sense mentioned in the introduction section. Thus, instead of using the numeric SQM mapping $f^{\mathfrak{X}}$ examined above, we must establish a math-formalism in an axiomatic way to define an *interval-valued* isomorphic mapping to construct the interval-cores of the words' fuzzy set-based semantics.

This section describes how one can develop such a formalism.

2.3.1. Enlarged hedge algebra – An enlargement of a given ordinary HA

Since the L-variables words are fuzzy, uncertain, or soft data, similar to fruits, they have their kernels inside, called the words' semantics cores. Imaginarily, in relation with the fuzziness interval of a words x, $\Im^{\mathfrak{X}}(x)$, one can accept that the numeric values of a proper subinterval of $\Im^{\mathfrak{X}}(x)$, located at about the center of $\Im^{\mathfrak{X}}(x)$, are most compatible with the semantics of x. A unique primary difference of the desired enlarged HA from its ordinary HA, $\mathcal{A}^{\mathfrak{X}} = (X^{\mathfrak{X}}, G, C, H, \leq)$, is that every $\mathfrak{X}$'s word has a core, and the question arising is that how one can model the words' semantics cores based on the given HA $\mathcal{A}^{\mathfrak{X}}$.

In this section, we show that one can establish a math-formalism to enlarge the given $\mathcal{A}^{\mathfrak{X}}$ to the so-called enlarged hedge algebra (EnHA), denoted by $\mathcal{A}_{en}^{\mathfrak{X}} = (X_{en}^{\mathfrak{X}}, G, C, H_{en}, \leq)$, where $H_{en} = H \cup \{h_0\}$ and h_0 is an artificial hedge used to produce the word-cores. Thus, $X_{en}^{\mathfrak{X}} = X^{\mathfrak{X}} \cup \{h_0 x : x \in X^{\mathfrak{X}}\}$. In this way, the words' semantics cores can be axiomatized using the following additional axiom [32]:

(A5^{en}) *Axioms for the s-core of words*: For $\forall x, y \in X_{en}$ and $x \neq y$,

(i) $h' h_0 x = h_0 x$, for $\forall h' \in H_{en}$. That is, $h_0 x$ is always a *fixed point*. In addition, for $x \in X$, $h_0 x = x \Leftrightarrow x \in C$, otherwise, $h_0 x$ and x are *incomparable* (recall that $h_0 x \notin X^{\mathfrak{X}}$).

(ii) $\forall, y \in X, x < y \Rightarrow h_0 x < y$ & $x < h_0 y$.

We see again that when the semantics cores of their L-words can convey useful information, which indicates the sets of the L-variable numerical universe values most compatible with their respective L-words' qualitative semantics, there is a math structure describing the L-variable words and their semantics cores. It contributes to turns out the importance of the *structure* and *information* relationship.

Similar to the case of the L-variables' HAs, the rich and complex semantic structure of every declared L-variable semantics, $\mathcal{S}_{en}^{\mathfrak{X}}$, can be described by *an infinite semantic bush-graph structure* given in Fig. 1, denoted by $\mathfrak{B}^{\mathfrak{X}}$, whose every node is associated with either a word x or its fuzziness model $H_{en}(x)$. One can read the semantic structure $\mathfrak{B}^{\mathfrak{X}}$ as follows:

By the hedges' functionality, every hedge intensifies the word x to be more specific. If $|x| = k > 0$, x is said to be $k-$specific or of the $k-$specificity. The larger the length of the words, the higher their specificity. Moreover, to our knowledge, when a human expert considering choosing a $\mathfrak{X}$'s word of the $k-$specificity, in the natural language, to describe an RW-event, he needs to choose it in the context of the all $\mathfrak{X}$'s words of the same specificity level, i.e., the words in $X_{en,k}^{\mathfrak{X}}$ carefully. For this reason, the words of the k-specificity are arranged on the same level, called the k-level. If there is an arrow from a note associated with a word x, called simply the node x, to another node y of the bush-graph, it represents the validity of $g(x, y)$, i.e., $H_{en}(x) \supseteq H_{en}(y)$, where the fuzziness model $H_{en}(\cdot)$ of a word '$\cdot$' represented by an oval containing the word '$\cdot$'. Thus, the vertical paths from top to bottom represent the words' generality-specificity relationships.

If one imagines that, for $\forall x \in X_{en}^{\mathfrak{X}}$, node x could have its horizontal coordinate, the words horizontal coordinate relationships represent their semantics order. For example, it can be verified, in Fig. 2, that $VVc^- \leq LVc^- \leq hc^-$. It results in the fact that, on each k-level, the k-specificity words and their fuzziness models, $H_{en}(\cdot)$s, are linearly arranged from left to right. Moreover,

$$(\bigcup_{x \in X^{\mathfrak{X}}_{en,k}} H_{en}(x)) \cup C = X^{\mathfrak{X}}_{en}.$$

2.3.2. The quantification of EnHAs

1) A description of how to quantify $X^{\mathfrak{X}}_{en}$ from the semantic structure $\mathcal{S}^{\mathfrak{X}}_{en}$. Similar to Subsection 2.2, for a given L-variable $\mathfrak{X}$ with its declared syntactical semantics, its quantification of EnHA starts with the *interval $\mathfrak{X}$-words' semantics defined by giving an interval-valued order-isomorphism, called interval*-valued semantically quantifying mapping (I-SQM) $f^{\mathfrak{X}}_{en}$: $X^{\mathfrak{X}}_{en} \to \mathrm{P}([0, 1])$, where $\mathrm{P}([0, 1])$ denotes the set of all subintervals of the normalized $\mathfrak{X}$-universe, $[0, 1]$, i.e., for $\forall x, y \in X^{\mathfrak{X}}_{en}$, $x \le y \Leftrightarrow f^{\mathfrak{X}}_{en}(x) \le f^{\mathfrak{X}}_{en}(y)$, and the set $f^{\mathfrak{X}}_{en}(X^{\mathfrak{X}}_{en})$ is dense in $[0, 1]$ The quantitative value $f^{\mathfrak{X}}_{en}(x)$ is called the interval-valued semantics, or $i-$valued semantics, of the word x.

The study [32] shown that for $\forall x \in X^{\mathfrak{X}}_{en}$, $f^{\mathfrak{X}}_{en}(x) = \Im(h_0 x)$ and the set, $f^{\mathfrak{X}}_{en}(H_{en}(x)$, is dense in an interval, denoted by $\Im(x) \subseteq [0, 1]$, and called the fuzziness interval of the word x Since $h_0 x$ is a fixed point, we have $f^{\mathfrak{X}}_{en}(H_{en}(h_0 x) = f^{\mathfrak{X}}_{en}(h_0 x) = f^{\mathfrak{X}}_{en}(x)$. It implies that $\Im(h_0 x) = f^{\mathfrak{X}}_{en}(x)$.

It was shown in [36] that the discovered properties of the fuzziness intervals of the words of $X^{\mathfrak{X}}_{en}$ can be arranged in a structure of the bush, also represented in Fig. 1. In this figure, similar to the semantic bush structure, the $k-$specificity words' fuzziness intervals are linearly arranged on the $k-$level, and if $g(x, y)$ is valid, or $H_{en}(x) \supseteq H_{en}(y)$, then $\Im(x) \supseteq \Im(y)$. They also form a bush structure of the words' $i-$valued semantics and their fuzziness intervals, and it is called *the word domain's quantitative semantics bush.*

The length of $\Im(x)$ is the x's fuzziness measure, denoted also by $m(x)$, as previously.

Since $f^{\mathfrak{X}}_{en}$ is an isomorphism, it transforms the structure $\mathcal{S}^{\mathfrak{X}}_{en}$, represented by the word domain's semantics bush structure in Fig. 1 onto its quantitative semantics structure also in the form of the quantitative semantics bush-structure that is also represented in the same figure.

It was shown that the following properties, which can be described by the word domain's quantitative semantics bush graph exhibited in Fig. 1, can be proved in an axiomatic manner:

- On every k-level, we have

a1) The order of $\Im(x)$, $x \in X_{en,k}$, follows the order of their words;

a2) $\bigcup_{x \in X_{en,k}} \Im(x) = [0, 1]$ and, for $k = 1$, $\Im(\boldsymbol{0}) \cup \Im(c^-) \cup \Im(W) \cup \Im(c^+) \cup \Im(\boldsymbol{1}) = [0, 1]$;

a3) For $\forall x \in X_{en,k}$, $H(x) = \bigcup_{h \in H_{en}} H(hx)$, which implies $\Im(x) = \bigcup_{h \in H_{en}} \Im(hx)$.

- *The fuzziness measure words and hedges*: As a consequence, putting $m(x) = |\Im(x)|$, we have

b1) $\sum_{x \in X_{en,1}} m(x) = m(0) + m(c^-) + m(W) + m(c^+) + m(1) = 1$, and $\sum_{h \in H_{en}} m(hx) = m(x)$, which implies that $m(hx) \le m(x)$;

b2) Adopt the assumption that the quantity, $\frac{m(hx)}{m(x)}$ *does not depend* on x. Then, it is called the effect or the fuzziness measure of the hedge h, denoted by $\mu(h)$. Thus, $m(hx) = \mu(h)m(x)$.

As a consequence, we have $m(x) = \mu(h_n)\ldots\mu(h_1)m(x)$, for $x = h_n\ldots h_1 c$, $c \in G$;

b3) Moreover, from the second equality in b1), it follows that $\sum\limits_{h \in H_{en}} \mu(h) = 1$.

It can be check that giving the values of $m(\mathbf{0})$, $m(c^-)$, $m(W)$ and $m(c^+)$, and $\mu(h)$, for $h \in H_{en}\setminus\{h\}$, the fuzziness measure m of the L-variable $\mathfrak{X}$, is completely determined. Therefore, they are called the *independent* fuzziness parameters of $\mathfrak{X}$.

Now, we see that utilizing only an interval-valued order-isomorphism $f_{en}^{\mathfrak{X}}$, one can define many quantitative words' semantic concepts and their properties from the multi-aspect semantic structure, $\mathcal{S}_{en}^{\mathfrak{X}}$.

2) *The axiomatization of the above quantification of* $X_{en}^{\mathfrak{X}}$: Considering, now, properties b1) – b3) axioms of the fuzziness measure of the given $\mathcal{A}_{en}^{\mathfrak{X}}$ (a more economic quantification axiomatization system can be found in [32],) it was proved in that paper that the I-SQM f_{en} above can be entirely determined and, for every $\mathfrak{X}$'s word x of $k-$specificity, the left endpoint of the interval-value of $f_{en}(x)$ is computed as follows

$$Lft\left(f_{en,m}(x)\right) = \sum_{y \in X_{en,(k+1)}\&y<x} m(y). \tag{$*$}$$

Therefore, in turn, given the independent fuzziness parameters' values of $\mathfrak{X}$, one can compute all quantitative quantities presented in Point 1).
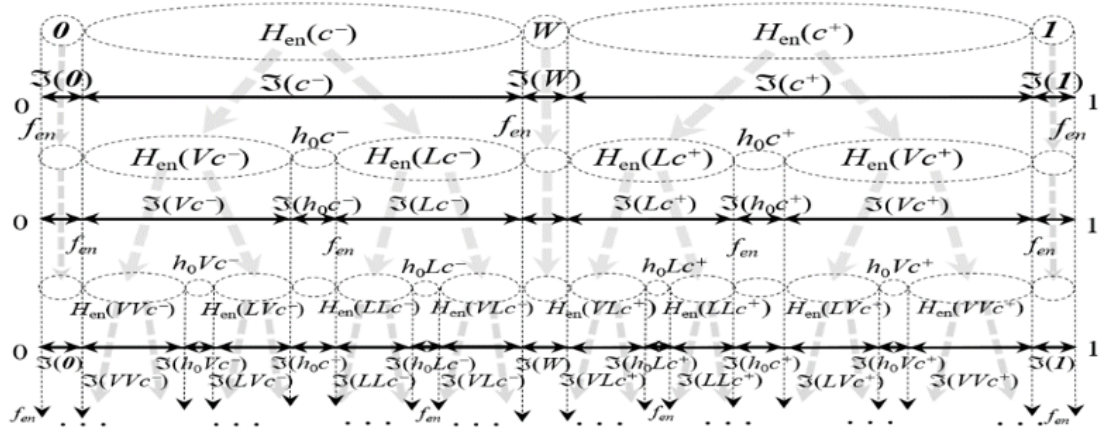

Figure 1. The semantic bush structures $\mathfrak{B}^A$ of the different semantic-aspects of an attribute A and their relationships: a) The semantic bush of the words; b) The semantic bush of the word-fuzziness models; c) The semantic bush of the word-fuzziness intervals.

3.   THE INTERPRETABLE AND SCALABLE CURRENTLY DECLARED $\mathfrak{X}$'S WORDS' FUZZY SET-BASED SEMANTICS

From Tarski et al.'s interpretability concept, it follows that to computationally simulate how human experts make their decisions, based on their accumulative knowledge, in terms of their words in the natural language, one must guarantee that the constructed words' fuzzy set-based semantics must soundly represent their qualitative semantics. That is, the L-variables' word domain semantic structures must be interpretable in their respective constructed words' fuzzy set-based semantics structures. On the other hand, the experts' accumulative knowledge implies that their knowledge can usually grow and, hence, so do their currently used L-variables' word domains. It results in the scalability of their knowledge and its currently used L-variables' word domains, i.e., their semantics are maintained when they grow. Thus, in this section, the interpretability and scalability of the fuzzy set-based semantics of the currently declared L-variables' words are taken into consideration.

3.1.   The currently declared L-variables' word-set semantic structures and their scalability

As mentioned above, one should establish a math-formalism to handle the conflict that when dealing with L-words directly, a developed method handles, at the current moment, the words of the currently declared L-variables' word-sets while their semantics are defined in the context of their total L-variables. Thus, in Section 2, the study assumes that the underlying set of the multi-aspect semantic structure of an L-variable $\mathfrak{X}$, $\mathcal{S}_{en}^{\mathfrak{X}} = (X_{en}^{\mathfrak{X}}, \leq, g)$, the word-domain of an L-variable $\mathfrak{X}$, $X_{en}^{\mathfrak{X}}$ is infinite. Meanwhile, at an application life-cycle moment, one requires to use only its finite word-set, called the $\mathfrak{X}$'s currently declared word-set, denoted by $F_{en,k}^{\mathfrak{X}}$ (F means a linguistic *Framework* of cognition,) where the integer k, $k > 0$, indicates that the word-set $F_{en,k}^{\mathfrak{X}}$ consists of all $\mathfrak{X}$'s words whose specificity degrees are not greater than k. That is, $F_{en,k}^{\mathfrak{X}} = X_{en,(k)}^{\mathfrak{X}}$, and called the $k-$specificity word-set of $\mathfrak{X}$.

3.1.1.   The scalability concept of the k-specificity word-set $F_{en,k}^{\mathfrak{X}}$ of $\mathfrak{X}$

The natural language and linguistic domains' knowledge are essential tools for human beings to interact with their RW counterparts to realize their daily activities. Along with the existence and development of human society, the natural human language and their *accumulative* domain knowledge grow in time. It implies that the L-variables' word-domains also grow and, then, they are potentially infinite in reality. It implies that human beings use words with their inherent semantics in the context of their total L-variables. However, they are assumed to be finite at each current application life-cycle moment for a given application, called the currently declared L-variables word sets.

The capability of human beings to handle their currently used word sets and their existing domain knowledge, when these word-sets and knowledge grow, is called its *scalability*, a concept in the system area. In terms of word-set structure formalism, it means that the way one handles the word-set structure's words is unchanged when the word-set structure grows. Thus, for convenience, in the HA-approach, this property of a current word-set structure is called its scalability.

Thus, aiming to develop a fuzzy set-based method to deal directly with L-words, one should establish a formalism to guarantee that this developed method can handle the scalability of the current L-variable word sets. Recall that in the fuzzy set framework, usually the number of words and their associated fuzzy sets are hardly limited and fixed and, therefore, the scalability concept of the L-labels' sets and their fuzzy representations have been ignored.

3.1.2. The multi-aspect semantic structure of $F_{en,k}^{\mathfrak{X}}$, as math-substructure of $\mathcal{S}_{en}^{\mathfrak{X}} = (X_{en}^{\mathfrak{X}}, \leq, g)$, represented by its semantic $k-$bush-graph, and its scalability

Let us consider the multi-aspect semantic structure of an L-variable $\mathfrak{X}$, $\mathcal{S}_{en}^{\mathfrak{X}} = (X_{en}^{\mathfrak{X}}, \leq, g)$. Restrict the $\mathfrak{X}$'s word-domain $X_{en}^{\mathfrak{X}}$ to a finite word-set $F_{en,k}^{\mathfrak{X}}$, it can be seen that the structure $\mathcal{S}_{en,k}^{\mathfrak{X}} = (F_{en,k}^{\mathfrak{X}}, \leq_k, g_k)$, in which $g_k(x, y)$ elegantly defined by $g_k(x, y) \Leftrightarrow H_{en}(x) \cap F_{en,k}^{\mathfrak{X}} \supseteq H_{en}(y) \cap F_{en,k}^{\mathfrak{X}}$ and $\leq_k$ is induced by the linear order-relation $\leq$ on $X_{en}^{\mathfrak{X}}$, is a semantic k-specificity ordered substructure of $\mathcal{S}_{en}^{\mathfrak{X}}$. Similarly, $F_{en,k}^{\mathfrak{X}}$ is closed to the relation g_k. Thus, $\mathcal{S}_{en,k}^{\mathfrak{X}}$ is a k-specificity substructure of $\mathcal{S}_{en,l}^{\mathfrak{X}}$, for $l > k$.

The math substructure means that the handling objects of $\mathcal{S}_{en,k}^{\mathfrak{X}}$ using its math-formalism is the same as handling them in the math-structure $\mathcal{S}_{en,l}^{\mathfrak{X}}$, with $l > k$. Its shows the *scalability* of $\mathcal{S}_{en,k}^{\mathfrak{X}}$. Moreover, referring to Fig. 1, we see that it is represented by the so-called k-portion of $\mathfrak{B}^{\mathfrak{X}}$, denoted by $\mathfrak{B}_{\kappa}^{\mathfrak{X}}$ Observing this k-portion $\mathfrak{B}_{\kappa}^{\mathfrak{X}}$ of $\mathfrak{B}^{\mathfrak{X}}$, one can also verify that the relationships between its words, using the semantic order relation or the generality-specificity relation, restricted to $\mathfrak{B}_{\kappa}^{\mathfrak{X}}$, are unchanged when $F_{en,k}^{\mathfrak{X}}$ grows. It contributes to justify the scalability of $F_{en,k}^{\mathfrak{X}}$ more clearly

3.2. The interpretable and scalable trapezoid fuzzy set-based words' semantics of a declared L-variable LFoC

Though L-variables' word domains are mathematically formalized, they have a very poor numerically computational structure for representing the words' fuzziness and uncertainty, limiting the interaction between word data and the numeric data in larger datasets. Fuzzy sets offer a tool to present L-words' fuzzy set-based semantics; however, there is no formal connection between the inherent qualitative words' semantics and their fuzzy sets constructed by an expert in the fuzzy set frameworks. It leads to a conflict that while the inherent qualitative words' semantics is objective (because it is commonly understood the same in a domain experts' community), their fuzzy sets constructed, based on the designer intuition, to represent the qualitative semantics of the words, called their fuzzy set-based semantics is entirely subjective. This drawback may result in the fact that if one develops a method to extract a knowledge piece of a given dataset, using the designed fuzzy set-based words' semantics, such an extracted knowledge piece may represent inappropriate content contained in the dataset.

Thus, a problem arising is how one can construct the words' fuzzy set-based semantics soundly? Restricted to the L-variables' word-domains semantic structure, Tarski et al.'s interpretability concept in the math-logics [56] tells us that to solve this problem requires that the word-domain semantic structure of a given L-variable should be *interpretable* in the constructed math structure of the fuzzy set-based semantics of its words. Mathematically, it

means that this word-domain semantic structure must be *isomorphic* onto the constructed fuzzy sets math structure.

Restricted to the trapezoid fuzzy sets and following the above interpretability concept, this section describes a method to construct a trapezoid fuzzy set-based words' semantics structure, which is an isomorphic image of the multi-aspect semantic word domain's math structure of a given L-variable semantics.

3.2.1. Constructing the interpretable trapezoid fuzzy set-based semantics of L-variables' words

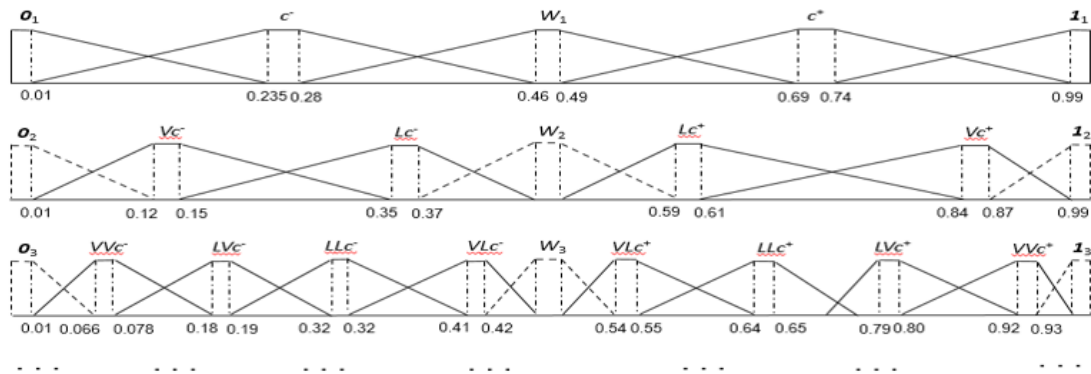1) *A method to construct trapezoid fuzzy set-based words' semantics*

Figure 2. The infinite trapezoids' bush-graph structure constructed to represent the multi-aspect semantic word-domain structure $\mathcal{S}_{en}^{\mathfrak{X}} = (X_{en}^{\mathfrak{X}}, \leq, g)$ of the attribute $\mathfrak{X}$.

Let us consider an L-variable $\mathfrak{X}$ with its given declared syntactic, qualitative, and quantitative semantics, and, hence, its interval-valued semantically quantifying mapping (I-SQM) $f_{en}^{\mathfrak{X}}$ is known. Theoretically, as previously, we assume that the word-domain of an L-variable $\mathfrak{X}$ is infinite and $\mathcal{S}_{en}^{\mathfrak{X}} = (X_{en}^{\mathfrak{X}}, \leq, g)$ is its multi-aspect semantic word-domain math structure represented in the bush-graphic form, $\mathfrak{B}^{\mathfrak{X}}$ examined in Sect. 2.3.2. Therefore, for ensuring the interpretability of the trapezoid fuzzy set-based $\mathfrak{X}$'s words semantics, their trapezoids should be constructed and arranged in the form of bush-graph, similar to $\mathfrak{B}^{\mathfrak{X}}$.

It can be constructed as follows:
- For $k = 1$, construct the small base of the trapezoid of every word x in $X_{en,1}^{\mathfrak{X}}$ so that its projection is just the interval-valued $f_{en}^{\mathfrak{X}}(x)$ Then, construct the trapezoids with their constructed small bases so that they form a strong fuzzy partition.
- For $k > 1$, the construction manner is the same as for $k = 1$, except that fact that the word-set of k−specificity, $X_{en,k}^{\mathfrak{X}}$, is added three L-constants, $\mathbf{0}$, W, and $\mathbf{1}$ to technically make the constructed trapezoids on k-level to form a fuzzy partition.

Demote by $\mathcal{T}(X_{en}^{\mathfrak{X}})$ the set of the constructed trapezoid fuzzy set-based semantics of the all $\mathfrak{X}$'s words.

2) *An interpretable math structure of the constructed fuzzy set-based semantics of $\mathfrak{X}$'s words*

On these constructed trapezoids, we define two relations, the order relation, $\leq_{tr}$, and the inclusion $\subseteq_{tr}$, corresponding to the semantic order-based relation, $\leq$, the generality-specificity relation, g, on $X_{en}^{\mathfrak{X}}$.

- The order relation on trapezoids: Assume that every trapezoid can be represented by a triple $(a, \boldsymbol{b}, c)$, where $[a, b]$ is the trapezoid large base and $\boldsymbol{b}$ denoted its small base. Then, for given any two triples, $(a, \boldsymbol{b}, c)$ and $(a', \boldsymbol{b}', c')$, we shall write $(a, \boldsymbol{b}, c) \leq_{tr} (a', \boldsymbol{b}', c') \Leftrightarrow \boldsymbol{b} \leq \boldsymbol{b}' \ \& \ (a \leq a' \text{ OR } b \leq b')$.

- The inclusion relation on trapezoids: We shall write $(a, \boldsymbol{b}, c) \subset_{tr} (a', \boldsymbol{b}', c') \Leftrightarrow [a, c] \subset [a', b']$. The inclusion means that the set of all numeric values compatible with the L-word, represented by the trapezoid $(a, \boldsymbol{b}, c)$, to some degree different from zero, must also be compatible with the L-word represented by $(a', \boldsymbol{b}', c')$.

- The math-structure of the constructed trapezoid fuzzy set-based semantics of the $\mathfrak{X}$'s words is denoted by $\mathbb{T}(X_{en}^{\mathfrak{X}}) = (\mathcal{T}(X_{en}^{\mathfrak{X}}), \leq_{tr}, \subset_{tr})$, called the fuzzy multigranularity *structure* (the fuzzy MGr*structure*). The order relation $\leq_{tr}$ is proved to be reflexive, antisymmetric, and transitive. Meanwhile, $\subset_{tr}$ is irreflexive, antisymmetric, and transitive.

Then, we have the following theorem states that $\mathbb{T}(X_{en}^{\mathfrak{X}})$ can *interpretably* represent $\mathcal{S}_{en}^{\mathfrak{X}}$. Its proof can be easily generated from the one in [36] for the finite set $X_{en,(k)}^{\mathfrak{X}}$ to the infinite $X_{en}^{\mathfrak{X}}$.

Theorem 1. *The fuzzy MGr-structure $\mathbb{T}(X_{en}^{\mathfrak{X}}) = (\mathcal{T}(X_{en}^{\mathfrak{X}}), \leq_{tr}, \subset_{tr})$ is the isomorphic image of the multi-semantic aspect word-domain structure $\mathcal{S}_{en}^{\mathfrak{X}} = (X_{en}^{\mathfrak{X}}, \leq, g)$, i.e., the set $\mathcal{T}(X_{en}^{\mathfrak{X}})$ representing the multi-semantic aspect word-domain structure of the word-domain $X_{en}^{\mathfrak{X}}$ is interpretable.*

Consequently, the fuzzy MGr-*structure* $\mathbb{T}(X_{en}^{\mathfrak{X}})$ is an isomorphic image of the bush-graph $\mathfrak{B}^{\mathfrak{X}}$ and, hence, it represents $\mathcal{S}_{en}^{\mathfrak{X}}$ soundly.

3.2.2. The interpretable and scalable trapezoid fuzzy set-based multi-granularity representation of a declared L-variable LFoC

As previously discussed, in practice, at a present moment of a given application life-cycle, it is necessary to declare only a finite word-set of a given L-attribute $\mathfrak{X}$, $X_{en,(k)}^{\mathfrak{X}}$, of the $k-$specificity, called the currently declare $\mathfrak{X}$'s word-set. Its $k-$multi-semantic aspect *structure*, $\mathcal{S}_{en,k}^{\mathfrak{X}} = (X_{en,k}^{\mathfrak{X}}, \leq, g)$, is a *substructure* of $\mathcal{S}_e^{\mathfrak{X}} = (X_{en}^{\mathfrak{X}}, \leq, g)$, represented by the $k-$portion $\mathfrak{B}_{\kappa}^{\mathfrak{X}}$ of the bush-graph $\mathfrak{B}^{\mathfrak{X}}$ of $\mathcal{S}_{en}^{\mathfrak{X}}$ and, hence *scalable*, because its specificity degree grows, the manipulation of its words is maintained. Let be given a fuzzy MGr-*structure* $\mathbb{T}(X_{en}^{\mathcal{X}})$ representing $\mathcal{S}_{en}^{\mathfrak{X}}$, whose trapezoids are just the trape-fuzzy set-based semantics of the $\mathfrak{X}$'s words. The $k-$portion of $\mathbb{T}(X_{en}^{\mathfrak{X}})$ is, then, just the *structure* $\mathbb{T}(X_{en,(k)}^{\mathfrak{X}})$. Since $\mathcal{S}_{en,k}^{\mathfrak{X}}$ is a *substructure* of $\mathcal{S}_{en}^{\mathfrak{X}}$ and $(X_{en}^{\mathfrak{X}})$ is an isomorphic image of $\mathcal{S}_{en}^{\mathfrak{X}}$, it follows that $\mathbb{T}(X_{en,(k)}^{\mathfrak{X}})$ is an isomorphic image of $\mathcal{S}_{en,k}^{\mathfrak{X}}$ and it is a *substructure* of $\mathbb{T}(X_{en}^{\mathfrak{X}})$. Its shows that $\mathbb{T}(X_{en,(k)}^{\mathfrak{X}})$ is interpretable for representing $\mathcal{S}_{en,k}^{\mathfrak{X}}$, the semantic *structure* of $X_{en,k}^{\mathfrak{X}}$, called the fuzzy $k-$MGr*structure*.

Moreover, since $\mathcal{S}_{en,k}^{\mathfrak{X}}$ is a *substructure* of $\mathcal{S}_{en}^{\mathfrak{X}}$, it follows that its isomorphic image, $\mathbb{T}(X_{en,(k)}^{\mathfrak{X}})$, is also a *substructure* of $\mathbb{T}(X_{en}^{\mathfrak{X}})$ and, hence, it is also scalable. Recall that since the natural language and human domain knowledge, accumulated by domain experts from their daily activities, are scalable, the scalability of the constructed trape-fuzzy set *substructure* is essential for simulating the human capability in accumulating the linguistic knowledge of a given application dataset.

4.   HEDGE ALGEBRAS' APPLICATIONS AND THEIR ADVANTAGES

This section justifies that when the inherent semantic L-variables' word domain structures of the natural language are mathematically discovered correctly, they offer a reliable formal basis for developing methods to simulate human experts' capabilities, based on their natural language formalism, to solve their daily life problems. Therefore, hopefully, the HA approach could achieve many specific application results in problem-solving.

4.1.   Specific methodological features of HAs' applications in problem solving

It is well-known that fuzzy sets are computational objects that can well appropriately represent fuzzy data. Since L-words are fuzzy data, as discussed in Sect. 3, the ability to computationally handle L-variables' word semantics requires representing the L-variables' word-domains semantic structures by their appropriately constructed fuzzy set-based word-semantics structures. Thus, a natural question arises when L-variables' word-domains are axiomatically formalized, which and how are the different features between the fuzzy set-based methodology and the HA-methodology in problem solving. Therefore, before presenting the HA-based methods to solve application problems in some fields, some primary such specific features are summarized as follows.

1) *Dealing with L-variables words in the natural language directly*

∘ *L-variables' word-domains containing their semantic math-structures for manipulating the L-variables' words*: In the HA-approach, to deal with human words of the natural language directly means that we have a mathformalism, established in an axiomatic way, to manipulate the L-variables words properly. The 'properly' here means that the axioms used to establish the math-formalism are simple inherent semantic properties of the L-variables words discovered from the natural language. Hence, similar to other math theories developed in an axiomatic manner, the L-variables' HAs, developed in an axiomatic manner, can properly model their word domain's inherent qualitative semantic structures.

∘ *Quantifying the L-variables' word-domains – The numeric words' semantics or the interval-valued words' semantics*: Like the fuzzy set framework, fuzzy sets are essential for dealing with the fuzziness of the L-data and for manipulating the interactions between the L-data and their numerical semantics. To determine the fuzzy set-based semantic of the words of a given L-variable, the quantification of its word domain is necessary. It is observed that, in every human social community, there are genuine relationships between the word semantics of an L-variable and the numeric values of its numeric universe, U: For every its word, x, there are U's values that are more compatible with x's semantics than other ones; there are U's values are most compatible with x, and they represent the x's core semantics numerically, called the numeric x's semantics in the case of the HA, and called the interval-valued x's semantics, in the case of the EnHA. When a word-domain is modelled by an HA (or, respective EnHA), its words' numeric semantics (or the interval-valued semantics) is used to determine the (singleton) cores (or the interval-cores) of their fuzzy set-based semantics.

As presented in Sect. 2.2 and 2.3, it is interesting that these fuzzy sets' cores are determined by the L-variable semantically quantifying mapping (SQM), in the case of the HA, or by the L-variable interval-valued semantically quantifying mapping (I-SQM), in the

case of the EnHA. In the HA-approach, note that both SQM and I-SQM are determined in an axiomatic way and they are entirely determined by providing the L-variables' fuzziness parameters' values. They form a formal basis to ensure the objectiveness of the (infinite) L-variables' word-domains quantification when only the L-variables' fuzziness parameters' values are carefully determined. It offers an opportunity to produce from the words, the words' fuzzy set-based semantics themselves and ensures the objectiveness of the constructed fuzzy set-based semantics of the L-variables' words in the above sense.

○ *Declaration of the L-variables' semantics*: Unlike the fuzzy set-based approaches to problemsolving, the declaration of L-variables' multi-aspect semantics is necessary and essential for a given application.

The syntactical semantics: The word-domain of an L-variable consists of all its words, represented in the string form, say $x = h_n \ldots h_1 c$, $c \in G = \{c^-, c^+\}$, where the hedges h_js $\in H$, present in the natural language.

○ *For its atomic or primary words*: It is necessary to declare its atomic words' name, say *young* and *old* for the L-variable AGE.

○ *For its linguistic hedges*: There are many hedges with different semantic shades; however, we are interested in those hedges that can generate a semantically, linearly ordered of word domain structure. Therefore, for determining the syntactical semantics, one may select some negative hedges in H^-, which comprises *little* (L) and *rather* (R), and some positive hedges in H^+ which comprise *very* (V) and *extremely* (E).

- *The qualitative semantics*: As examined in Sect. 2, it is necessary to declare the semantic order of the atomic words, $c^- \leq c^+$, the sign of the atomic words and the hedges, and the order of the hedges of the same signs representing their modifying effects, e.g., $R \leq L$ and $V \leq E$, and the relative signs between all the hedges as given in Tab. 1.

Then, all the L-variable word-domains can be linearly arranged by a developed procedure.

- *The quantitative semantics*: As presented in Sect. 2 and 3, let be given only a semantic order-isomorphic mapping, f from the L-variable word-domain into the normalized universe $[0, 1]$, whose image under f is dense in $[0, 1]$, it does determine all quantitative semantic aspects of the L-variable words, including their fuzzy set-based semantics. Moreover, it was shown in that sections that, to define f of an L-variable, it is necessary to determine its appropriate $(|H| + |G| - 2)$ fuzziness parameter values if an HA is used to model the word-domain, or $(|H_{en}| + |G| - 2)$ fuzziness parameter values, if an EnHA is used instead. Therefore, the parameter number of tuning the L-variables' quantitative semantics is very small compared to the total number of the L-variable words.

○ *Ability to develop a procedure to construct the fuzzy set-based words' semantics from the declared L-variable word-set*: Sect. 3.2.1 proposed such procedure with $|H_{en}| = 3$, including the artificial hedge to generate the words' semantics cores. A generation to an arbitrary H_{en} of larger cardinality is quite straightforward

2) The interpretable and scalable fuzzy set representation of L-variables word-sets and linguistic rule bases of linguistic rule-based systems (LRBSs)

For the fuzzy rule based systems (FRBSs), a drawback is that the L-variables' words assigned to their constructed fuzzy sets are only considered as L-labels. Nevertheless, their advantage is that these constructed fuzzy sets form a computational basis for the proposed method to realize the interaction between the L-labels with the data in the given dataset to design FRBSs Therefore, no formal basis was established in the fuzzy set framework to

relate the L-labels' semantics and their fuzzy sets constructed by the designer. Thus the FRBs extracted from the given dataset strongly depend on how the designer constructs fuzzy sets, while they, as human L-sentences, convey their inherent contents that may be different from those computed from the dataset and assigned to these dataset L-sentences by the proposed method.

Thus by Sect. 3, for ensuring the proposed method realizing this interaction with just the inherent word-semantics soundly, the set of the constructed fuzzy sets associated with the given words of an L-variable, $\mathfrak{X}$, is required to form a math-structure isomorphic onto the multi-semantic aspect structure of the set of these given $\mathfrak{X}$'s words. In other words, the fuzzy set representations of the given L-variables' word-sets must be interpretable.

Then, the extracted FRBs, considered L-rule bases (LRBs,) can be considered human expert knowledge about the given dataset, which usually is scalable. The scalability of the designed FRBs has not been examined in the fuzzy set framework. The HA-approach offers a formalized basis to develop a method to genetically design LRBSs, for a given dataset so that their LRSs are scalable. That is, they can be allowed to grow by permitting their declared L-variables' word-sets to grow at the future moment of the dataset life-cycle to increase their performance recalling that the larger the human-domain experts, the more exactly the human experts make their decisions.

Therefore, the HA-approach does not restrict the L-variables' word-sets cardinality to about 7 ± 2.

3) *The computational LRBs' representations and reasoning methods in the HA-approach*

Let be given an LRB, denoted by $\mathcal{B}$ in the form of the following form

$$(r_j)\ \mathfrak{X}_1 = x_{1j}\ \&\ldots\&\ \mathfrak{X}_n = x_{nj} \Rightarrow \mathfrak{X} = y_j, \text{for } j = 1,\ldots,m \tag{1}$$

When the L-variables' word-domains are mathematically formalized, two methods can be applied to represent the given m L-rules computationally.

a) Representing a given LRB by a multi-dimensional numeric surface and approximately reasoning method in the fuzzy control: In this case, the words' numeric semantics are essential. Then, one can consider every L-rule r_j as an L-point in an $(n+1)$-dimensional L-Cartesian product space, P_{n+1}. Thus, the LRB given in (1) defines an L-hypersurface in P_{n+1}, denoted by S_B^L. For the fuzzy control problems, the L-variables' word-domains are modelled by ordinary HAs and their numerical SQMs are applied to transform the L-hypersurface into the respective numeric one defined in the ordinary $(n+1)$-dimensional L-Cartesian product, denoted by S_B^N.

One can now apply any method, say an extrapolation-interpolation method, considered as a reasoning method, to approximately compute the S_B^N-value (output) for a given n-dimensional input vector.

b) Fuzzy representation of a given LRB: In the HA-approach, one must also develop a method to construct the fuzzy set-based structures to represent the declared L-variables' word-sets, but there are two new features in the way to construct them compared to the one applied in the fuzzy set framework:

- Once all semantic aspects of a given L-variable are declared, a developed procedure with a few of its fuzziness parameters can automatically construct the fuzzy set-based semantics of the currently declared k-specificity word set's words, for any $k > 0$. It increases the

ability to ensure the objectiveness of the constructed fuzzy sets in representing their words' semantics.

- Instead of the words' fuzzy sets constructed in the fuzzy set framework and arranged to form a singularity structure, which cannot represent the words' generality-specificity relation, in the HA-approach, they are arranged to form a fuzzy multi-granularity structure. In Sect. 3, it is shown that this structure can properly model the multi-semantic aspect L-variable word-domain structure. Under this condition, the LRBs extracted from the given dataset can represent their dataset information contents properly in the fuzzy uncertainty environment, and, hence, it may be useful for human dataset users who are familiar with their linguistic sentences.

4.2. HA-methodology applied in fuzzy control and the performance of its application

The HA-methodology was applied first in the study [15], which proposes an HA-based method to design an optimized controller to solve an inverted pendulum control problem. It motivated many studies to develop HA-based controllers for controlling active vibration control of uncertain structures against earthquake [17, 18, 19, 20, 21, 22, 23], a self-excited induction generator [24], a grid synchronization [25], and the speed in a Stand-Alone WECS [26, 27], and in solving control problems in the industry [28, 29, 30]. In the framework of this section, we summarize only some of these studies

4.2.1. The HA-methodology in fuzzy control and its specific features

This section presents the general control HA-methodology and exposes its primary specific features different from the general fuzzy control.

1) *General fuzzy control methodology in the fuzzy set-framework*

The primary methodology idea of fuzzy control is to simulate how human beings solve their daily problems based on their experiential knowledge. Let us consider a simple fuzzy control problem with two input variables x and $\dot{x}$, and a fuzzy rule base consisting of two fuzzy rules

$$(r_1) \text{ IF } x = PSmall \text{ and } \dot{x} = NSmall, \text{ THEN } u = PMedium;$$
$$(r_2) \text{ IF } x = Z \text{ and } \dot{x} = NBig, \text{ THEN } u = PSmall. \tag{2}$$

○ *Representation of the fuzzy rule base (FRB)*: Assume that the fuzzy variables x and $\dot{x}$ have five L-labels, while the control variable has seven and their fuzzy sets are constructed by the designer as represented in Fig. 3. The fuzzy rule (r_1) given in (2), described by the triple (*PSmall, NSmall, PMedium*), is represented by their respective triangular fuzzy sets, figured by the bolded triangles, while three dashed lines triangles represent the triple (*Z, NBig, PSmall*) of (r_2) in (2).

○*A reasoning method in the fuzzy set-framework*: Usually, in the fuzzy control, the fuzzy reasoning on such an FRB representation can be roughly described as follows:

Assume that (a_1, b_1) is an input vector. For the rule (r_1) in (2), the input value a_1 fires the triangle 'PS' with a degree smaller than the degree that the input b_1 fires the triangle 'NS'. Then, the smaller firing degree is used to define the lower part, a trapezoid T_{1U} with
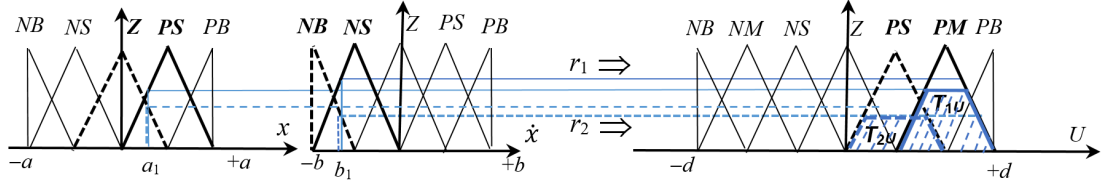
Figure 3. Computational representation of the rules r_1 and r_{15} of the LRB $\mathfrak{B}$ given in Tab. 1

continuous blue lines, of the output L-label 'PM' triangle. Similarly, for the rule (r_2), the input vector (a_1, b_1) determines a trapezoid T_{2U} in this figure.

Thus, the output fuzzy set of the input vector (a_1, b_1), computed by this fuzzy reasoning method, is defined by the fuzzy union of the two such obtained trapezoid fuzzy sets, and a defuzzification method is used to calculate the numeric control action, say by the centroid method.

We emphasize that, following our knowledge, most fuzzy control-based application studies found in the literature were applied this fuzzy reasoning method. It justifies the effectiveness and usefulness of the fuzzy reasoning method. However, in our opinion, it strongly depends on the designers' intuition. Moreover, the number of the variables' L-labels strictly limited, usually no larger than 7 ± 2.

2) *HA-methodology-based reasoning methods*

We will see that HA-methodology-based reasoning methods can be established based on a strict math-formalism. Firstly, note that such methods developed can directly handle L-words and, therefore, the L-rules in the following linguistic rule base (LRB) are viewed as human experts' knowledge pieces

$$(r_j) \; \mathfrak{X}_1 = x_{1j} \; \& \ldots \; \& \; \mathfrak{X}_n = x_{nj} \Rightarrow \mathfrak{X} = y_j, \; j = 1, \ldots, m. \tag{3}$$

Since the words $x_{ij} \in Dom(\mathfrak{X}_i) = X^{\mathfrak{X}_i}_{(k_i)}$ and $y_j \in Dom(\mathcal{Y})$ are elements of the HAs, $\mathcal{A}^{\mathfrak{X}_i}$'s and $\mathcal{A}^{\mathcal{Y}}$, every L-rule defines a point in the n-dimensional L-Cartesian space, $\mathbb{S}_n = \prod_{1 \le i \le n} X^{\mathfrak{X}_i}_{(k_i)}$. Thus, m L-rules in (3) define m points in $\mathbb{S}_n$, which, in turn, define an $n-$dimensional L-surface S^L_n in $\mathbb{S}_n$.

As discussed in Sect. 2 and 3, when declaring all the multi-semantic aspects of $\mathfrak{X}_I$'s, the quantifying semantic mappings (SQMs) $f^{\mathfrak{X}_i} : X^{\mathfrak{X}_i}_{(k_i)} \to [0, 1]$, $i = 1, \ldots, n$ and $f^{\mathcal{Y}} : Y^{\mathcal{Y}}_{(l)} \to [0, 1]$ are defined. These SQMs transform S^L_n into the Euclid surface S^N_n defined based on the m points $(f^{\mathfrak{X}_i}(x_{1j}), \ldots, f^{\mathfrak{X}_i}(x_{nj}), f^{\mathcal{Y}}(y_j))$, $j = 1, \ldots, m$, in $[0, 1]^{n+1}$. As example, Fig. 4 represents a surface S^N_2 based on nine L-rules, in which s denotes '*small*', W denotes '*medium*', and l denotes '*large*'.

Now, an approximate reasoning method in the HA-approach can be developed applying, for instance, an interpolation-exploration reasoning method based on the m points obtained above, for every given numeric input vector, $\mathcal{A} = (a_1, \ldots, a_n) \in [0, 1]^n$. In general, there is a wide selection of such methods, including linear interpolation-exploration methods, e.g., bi-linear interpolation-exploration-based methods [17, 18, 19, 20, 23, 24], explicit linear formulas [21, 22], or non-linear ones whose ideas are described as follows, when $\mathcal{A}$ falls in an n-cube, one can develop methods reasoning based on a weighted sum of the $f^{\mathcal{Y}}$-values at the n-cube's
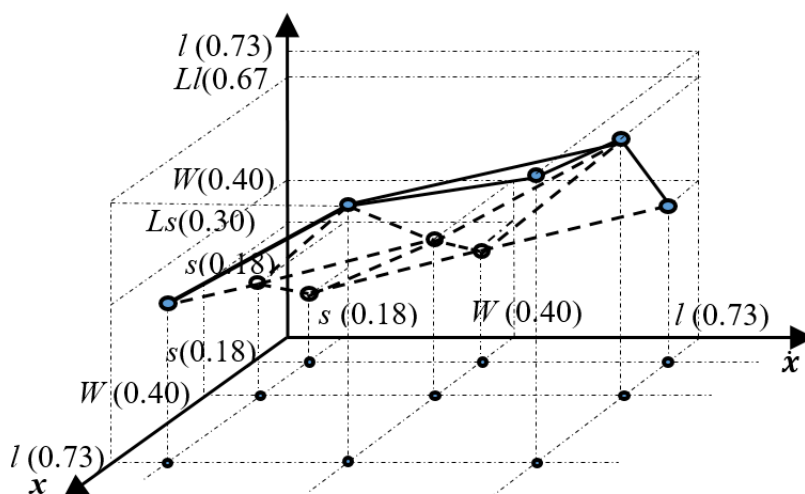
Figure 4. Numeric graphical representation of the given LRB based on nine L-points

vertices [25, 27]. Especially, these interpolation-exploration methods can preserve the monotonicity of their LRB. I.e., if the L-surface S_n^L is monotonic, then so is its respective numeric surface S_n^N, see [10]. It is an essential feature of the HA-based interpolation-exploration reasoning method.

4.2.2. HA-methodology's applicability in fuzzy control

The study [15] to solve an inverted pendulum problem is the first one to show the applicability of HAs in fuzzy control. Then, there are many works applying the HA-methodology to solve many different problems in this field, such as a swing-up control problem [16], vibration control problems of high-rise buildings subjected to seismic loads [17, 18, 19, 20, 21, 22, 23], and fuzzy control problems in the industry [28, 29, 30].

In the framework of this study, we introduce only an HA method to develop, for the first time in electrical engineering, a HA controller (HAC) to control the self-excited induction generator (SEIG) generated in the indirect rotor-flux oriented (IRFO) control system.

1) *The problem determination*

The HAC mentioned above aims to keep the DC voltage constant and equal to the reference value in the self-excited induction generator system (SEIG). The basic configuration of the proposed IRFO (indirect rotor-flux-oriented) control system is shown in Fig.5, in which the components IG is the squirrel-cage induction generator, the IGBT power converter, the proposed IRFO controller, and the DC link. The main objective of the control is to keep the DC voltage constant and equal to the reference value regardless of changes in the rotor speed and load. Such an approach allows a DC load to be connected directly at the excitation capacitor terminals.

The HAC controller was developed to implement for "Voltage controller." The steady, controlled voltage is Udc across load Rdc (Fig 5). The LRB with three L-variables, *Le*, *Lce*, and *Lu* (the output control signal *u*) was designed as given in Tab. D1. The *Le*'s, *Lce*'s,
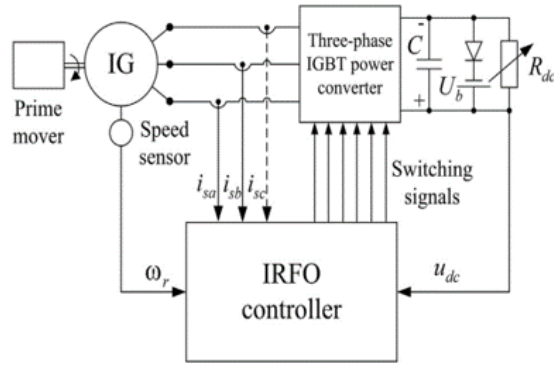
Figure 5. Basic configuration of the SEIG vector control system
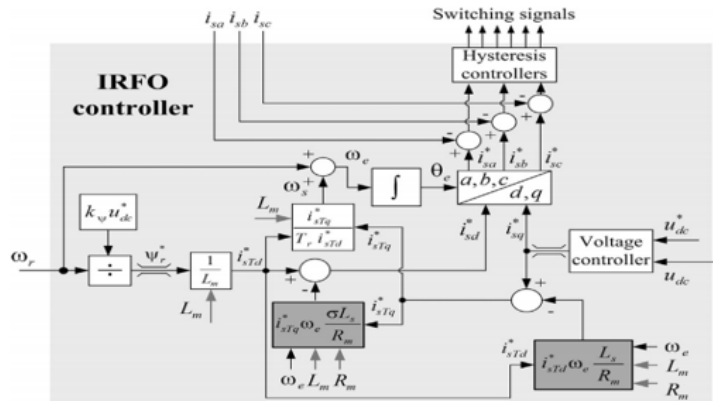
Figure 6. IRFO controller schematic diagram

and Lu's semantics were declared as follows:

- The syntactical semantics of all three Le, Lce, and Lu are declared the same as follows: the atomic words are *Small* (S) and *Big* (B), and the set of hedges is $H = \{L, V\}$;

- Then, the Le's, Lce's, and Lu's qualitative semantics are determined the same: the signs and the relative signs of the hedges are declared the same as previously (see Tab. 1).

- Their quantitative semantics are declared by specifying their fuzziness parameters' values given in Tab. 3.

When all L-variables' semantic aspects are declared as above, their semantically quantifying mappings f_{Le}, f_{Lce}, and f_{Lu} are defined, and, as presented in Sect. 4.1, they transform the LRB given in Tab. 2 into 25 points in $[0, 1]^3$, defining a surface in it.

That study applied a bi-linear extrapolation-interpolation method as the approximate reasoning method, *which preserves the monotonicity* of the given LRB considered an L-functions. The simulation model of the proposed control system was built in the MAT-LAB/Simulink environment. The IRFO control algorithm was also programmed in the MATLAB/Simulink environment and executed in real-time using the DS1104 R&D Controller Board, manufactured by DSpace, which was integrated within a PC. The three-phase

power converter was constructed of three dual IGBT modules – type SKM 100GB125DN, manufactured by Semikron – each used in one of the phases and driven by the hybrid dual IGBT driver SKHI 22B, also manufactured by Semikron.

2) *Simulation and Experimental result analysis*

The experimental setup of the considered SEIG control system was built for the experimental validation of the simulation results.

2.a) *First case: changes in load resistance*

The load resistance was varied in a step manner, as shown in Tab. 4. The following two cases were considered: fixed rotor speed and variable rotor speed at $n = 1200$ rpm and the reference DC voltage u*dc = 300 V.

Table 2. Step changes in load resisitance

t [s]	0-2	2-5	5-8	8-10
Rdc [Ω]	10^{12}	500	220	500

Observe on Fig. 7 (a), found that the response of DC voltage for FLC and HAC is quite similar and much better than PI. The voltage dips noted for these two controllers are about twice smaller than the PI, whereas the restoration times are about four times shorter. The FLC response has an overshoot (green line), but the HAC response has no (red line).

Observe on Fig. 7 (b), at $t = 2$s, the following peak values of the reference $q-$axis stator current were noted: 0.9 A for the PI controller, 0.93 for the HA controller, and 1.01 for the FL controller. However, although higher than the PI controller, the HA controller's peak values are still lower than the FL controller. The response of HAC is much faster compared to PI controller.

The experimental results, shown in Fig. 8, are equivalent to the simulation results shown in Fig. 7. Comparing the results presented in these two figures points to a disagreement in the experimental and simulation steady-state values of the reference $q-$axis stator current. Other than that, the experimental results in Fig. 8 are in good agreement with the simulation results in Fig. 7.

2.b) *Second case: changes in DC voltage u*dc*

In the second set of simulations, the reference DC voltage was varied in a step manner, as shown in Tab. 5. Again, the cases with fixed and variable rotor speeds were considered. Fig. 9 shows the actual DC voltage simulation responses and the reference $q-$axis stator current to step changes in reference DC voltage. The responses were obtained for fixed values of the rotor speed and the load resistance.

Table 3. Step changes in reference DC voltage

t [s]	0-2	2-4	4-6
u*dc [V]	250	300	250

In Fig. 9a, the HA controller's response to step changes about DC voltage can be declared the most satisfactory in the speed of response and overshoot/undershoot. It is important to note that the HA controller manages to reconcile two generally conflicting requirements: the

Figure 7. Simulation responses of the actual DC voltage (a) and reference q-axis stator current (b) to step changes in load resistance for fixed rotor speed and reference DC voltage: $n = 1200$ rpm and $u * dc = 300$ V.

Figure 8. Experimental responses of the actual DC voltage (a) and reference q-axis stator current (b) to step changes in load resistance for fixed rotor speed and reference DC voltage: $n = 1200$ rpm, $u * dc = 300$ V.

requirement for fast response on the one hand (i.e., short settling time and rise time) and the requirement for small overshoot/undershoot another hand. This conflict is more obvious than in the case of the other two controllers.

Namely, although the FL controller's response is notably faster than the PI controller and, in this sense, similar to that of the HA controller, the price is paid in higher over-shoots/undershoots in the DC voltage compared to both the PI controller and the HA

controller. On the other hand, the PI controller exhibits smaller overshoots/undershoots in the DC voltage than the FL controller, though still higher than the HA controller, but this time, the price is paid in notably slower response compared to both the FL controller and the HA controller. For example, the voltage overshoots obtained at t^1/42 s are: 10.5% for the FL controller, 5% for the PI controller, and 2% for the HA controller; the corresponding settling times are: 0.10 s for the FL controller, 0.28 s for the PI controller, and 0.09 s for the HA controller.
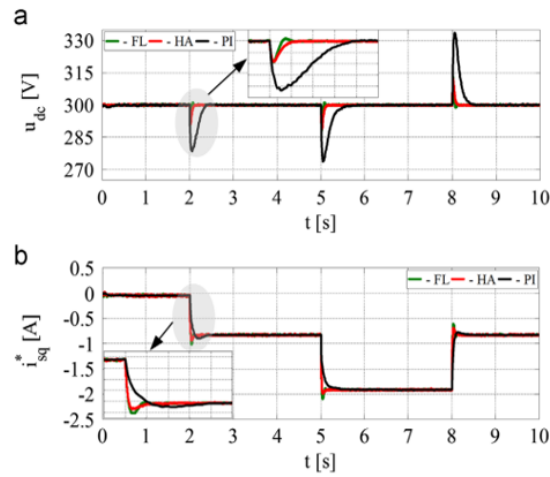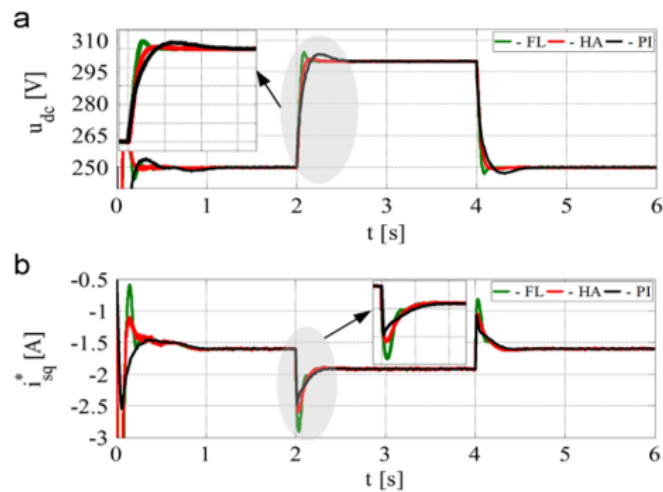

Figure 9. Simulation responses of the actual DC voltage (a) and reference q-axis stator current (b) to step changes in reference DC voltage for fixed rotor speed and load resistance: $n = 1200$ rpm and $Rdc = 220\Omega$.
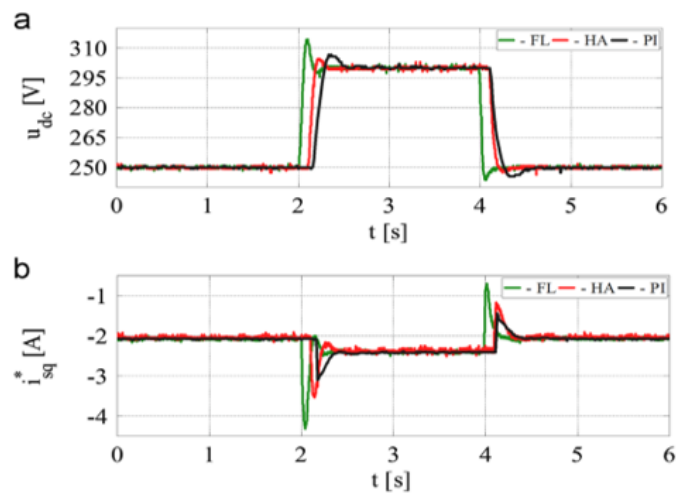
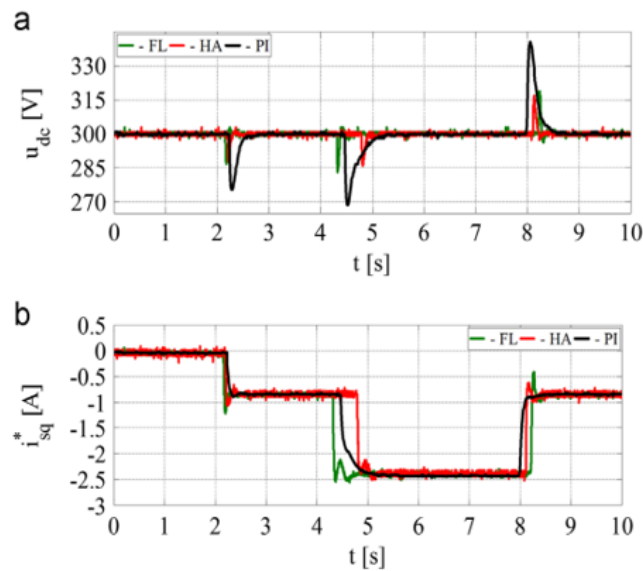*Figure 10.* Experimental responses of the actual DC voltage (a) and reference q−axis stator current (b) to step changes in reference DC voltage for fixed rotor speed and load resistance: $n = 1200$ rpm and $Rdc = 220\Omega$.

In Fig. 9b, all three controllers exhibit similar steady-state performance, whereas the differences become more obvious during transients. In this case, the peak values of the reference $q-$axis stator current obtained for the HA controller and especially for the FL controller are somewhat higher compared to the PI controller (HA: $> +4.5\%$, FL: $> +17.5\%$). However, changes in the reference $q-$axis stator currently obtained for the PI controller at step times are more abrupt and steep compared to the cases of the other two controllers. It is a direct consequence of the PI controller's operating principle – specifically, the P part of it. Such abrupt changes in the reference $q-$axis stator current may lead to high stator phase current derivation values and, consequently, to dangerously high voltages induced in the stator phase windings.

Fig. 10 shows the results equivalent to those shown in Fig. 9. The highest over-shoots/undershoots in the actual DC voltage were again obtained for the FL controller, although the speed of response to some extent compensates them. For instance, the following voltage overshoots were obtained at $t \approx 2$ s: 24.5% for the FL controller, 10% for the PI controller, and 7% for the HA controller; the corresponding settling times are: 0.14 s for the FL controller, 0.35 s for the PI controller, and 0.18 s for the HA controller. Similar values were obtained at $t \approx 4$ s. As for the reference $q-$axis stator current, the highest peak values were obtained for the FL controller, whereas those obtained for the HA controller are only slightly higher than the PI controller.

The results and analysis obtained above, and the results achieved in [15, 16, 17, 18, 19, 20, 21, 22, 23, 25, 26, 27, 28, 29, 30], most published in international journals ISI indexed, justify the meaning of discovering L-variables' word domain semantic structures and their quantification in problem-solving control.

4.3. HA-methodology in solving classification dataset problems

It is well-known that, in simulating the way human beings use their domain L-knowledge to solve their daily problems, the application of fuzzy rule base systems (FRBSs) to solve classification problems was intensively investigated, as it can be seen in an overview study, say [56, 57].

A question emerging is that, with the specific new features of the L-variables word sets and the fuzzy set-based structures of the HA-approach, as summarized in Sect. 4.1, based on which the LRBs are extracted from a given dataset by a proposed method, which advantages and benefits can the LRBSs designed by a method developed in the HA-approach offer? We show that when the inherent multi-semantic word domain structures are formalized and represented by their constructed interpretable and scalable fuzzy MGr-*structures* the proposed methods developed based on the HA-methodology can genetically design LRBSs more effective than their fuzzy counterparts, the FRBSs.

Let be given a classification problem $\mathcal{P}$ given by a set $P_{Cl} = \{p_l = (d_l,\ C_l) :\ d_l \in D,$ $C_l \in C,\ l = 1,\ ...,N_P\}$, where $d_l = (d_{l,1},\ d_{l,2},\ ...,\ d_{l,n}) \in D$ is of n dimensions, $C = \{C_l : l= 1,...,M\ \}$ is the set of class names. Assume that, the syntactical semantics of every L-variable $\mathfrak{X}_j$, $j = 1,\ ...,\ n$, is declared by giving $G_j = \{c_j^-,\ c_j^+\}$, $C_j = \{\boldsymbol{0}_j,\ W_j,\ \boldsymbol{1}_j\}$ and $H_{en,j} = \{L_j,\ h_j,\ V_j\}$, where L_j and V_j stand, respectively, for "*Little*" and "*Very*". With its qualitative semantics determined as presented in Sect. 3, it is associated with $k-$multi-semantic aspect *structure*, $\mathcal{S}_{en,k}^{\mathfrak{X}} = (X_{en,k}^{\mathfrak{X}},\ \leq,\ g)$. The problem is how one can develop a genetic method, an Evolutionary Multi-objective Optimization (EMO-) method, to design

an optimized LRBS to solve a classification problem, called a Class-LRBS.

1) Develop an EMO-PSO-HA-method to design optimized Class-LRBSs for a given $\mathcal{P}$

Because, in the HA-approach, determining the L-variables' multi-semantic aspect is essential, the PSO-based genetic HA method proposed in [32] has two phases. The first one is to design an optimized semantics of individual $\mathfrak{X}_j$'s and the second one is to design optimized Class-LRBSs, i.e., those whose LRBs are optimized in solving the given classification problem, based on the already optimized L-variables' semantics. They have the following features:

a) Phase I: Designing optimized $\mathfrak{X}_j$'s semantic parameters using a simple classification problem EMOPSO algorithm. In the HA approach, a distinguishing feature is that the fuzzy set-based semantics of the $\mathfrak{X}_j$'s words can be *algorithmically constructed by a procedure* while preserving their words semantics, when all their semantic aspects, including their fuzziness parameters' values and their declared word-set specificity degrees are given. Thus, when the syntactic semantics of every L-variable $\mathfrak{X}_j$ is declared and fixed for an application dataset, only its word set's semantic parameters, consisting of the $\mathfrak{X}_j$'s fuzziness parameters values and its word-set $k-$specificity degree, are necessary to be optimally determined.

Thus, for genetically designing the optimized $\mathfrak{X}_j$s' semantics, study [32] proposed an EMO-PSO-HA-algorithm utilizing a simple Clas-LRBS denoted by $\mathcal{C}l(S_0(\pi,\ P_{Cl}))$. It is called here the $\mathcal{C}l(S_0(\pi,\ P_{Cl}))$-HA-algorithm, where

- P_{Cl} is the given classification dataset, π is an individual consisting of the parameter k_j, for determining the $\mathfrak{X}_j$'s word-set, $X_{(k_j)}$, and the $\mathfrak{X}_j$'s fuzziness parameters, $fm(c_j^-)$, $fm(c_j^+)$, $\mu(L_j)$, $\mu(V_j)$, $\mu(h_{j,0})$, for $j = 1, \ldots, n$;

- $S_0(\pi,\ P_{Cl})$ is its LRB, consisting of N_0 L-rules (N_0 is prespecified) selected from the set of *initial* L-rules directly generated from the P'_{Cl}s data with highest values using the prescreening criterion $c.s$. Where c is the confidence measure of every initial L-rule, measured by the degree its P'_{Cl}s datum fires the initial L-rule, and s is its support measure.

- The algorithm's optimal problem: Seek for optimized $\mathfrak{X}_j$s' semantics π_{opt} for $\mathcal{P}$: For a given π, let the Clas-LRBS, $\mathcal{C}l(S_0(\pi,\ P_{Cl}))$, be determined. Then, π_{opt} can be found by solving the following optimization problem:

Maximize the performance $Perf(\mathcal{C}l(S_0(\pi,\ P_{Cl})))$

Subject to

- Constraints on the fuzziness parameters: $0 < \{fm(c_j^-),\ fm(c_j^+)\} \le a_j$, $0 < \{\mu(L_j),\ \mu(h_{j,0}),\ \mu(V_j)\} \le b_j$, $fm(\mathbf{0}) + fm(c_j^-) + fm(W) + fm(c_j^+) + fm(\mathbf{1}) = 1$ and $\mu(L_j) + \mu(h_{j,0}) + \mu(V_j) = 1$, $j = 1, \ldots, n$.

- Constraints on the positive integer k_j : $0 < k_j \le \kappa_j$, where the values of κ_j's are pre-specified, for $j = 1, \ldots, n$.

b) Phase II: Designing an optimized Class-LRBS with an optimal rule base. By Sect. 3, from the optimized $\mathfrak{X}_j$'s semantic parameters' vector, π_{opt}, which Phase I provides, one can determine, for every $\mathfrak{X}_j$ the word-set $X_{(k_j)}$ and generate the fuzzy k_j-MGr*structure* representing the multi-semantic aspect k_j-structure, $\mathcal{S}_{en,k_j}^{\mathfrak{X}_j}$. It provides a sufficient basis for the desired EMO-PSO-HA-algorithm, denoted by LRB-$\mathcal{A}$, which performs the following tasks:

○ Generating an initial L-rule-set S, using the π_{opt} provided, as described above;

○ Determining an optimized LRB, $KB_{opt}(\pi_{opt})$ to design an optimized Class-LRBS, $Cl(KB_{opt}(\pi_{opt}))$, whose reasoning method is the single-winner rule, with maximized accuracy (or, performance) of $Cl(KB_{opt}(\pi_{opt}))$, denoted by $Perf(Cl(KB_{opt}(\pi_{opt})))$, and minimized the complexity of $KB_{opt}(\pi_{opt})$ measured by the total numbers of the conditions appearing in every its rule and denoted by $Comp(KB_{opt}(\pi_{opt}))$.

Thus, the LRB-EMO-algorithm aims to solve the following multi-objective optimization problem:

Find an optimal solution, S_{opt}, such that

Maximize $Perf(Cl(S))$, Minimize $Comp(S)$,

Subject to: $S \subseteq S_0(\pi_{opt}, P)$

Table 4. Simulation results of the proposed method and the counterpart ones.
#R and #C indicate the number of the rules and the number the conditions of the designed rule bases, and #R#C represents the average of the FRBs or LRBs complexities. The *Diff*-columns represent the difference between the test-phase (*Ts*) and the training-phase (*Tr*) accuracy or complexity of their respective methods. The italic and bold values are aimed to emphasize that these values are negative.

Data sets	LRB-𝒜			Triangle GSA-based method			*Diff.* Comp.	*Diff.* Perf.	All Granularities			*Diff.* Comp.	*Diff.* Perf.	Product/1-ALL TUN			*Diff.* Comp.	*Diff.* Perf.
Name	#R#C	Tr	Ts	#R#C	Tr	Ts			#R#C	Tr	Ts			#R#C	Tr	Ts		
Australian	46.50	88.56	87.15	43.00	87.83	86.18	*-39.68*	0.97	4.00	85.51	85.51	42.5	1.64	62.43	89.18	85.65	*-15.93*	1.50
Bands	58.20	78.19	73.46	83.40	75.57	70.63	*-12.43*	2.83	57.18	71.36	68.73	1.02	4.73	104.09	71.18	65.80	*-45.89*	7.66
Bupa	181.19	79.78	72.38	196.37	77.40	67.71	113.48	4.67	112.59	69.50	63.99	68.6	8.39	210.91	78.59	67.19	*-29.72*	5.19
Cleveland	468.13	66.64	62.39	966.04	77.08	61.20	406.93	1.19	1132.14	73.11	55.11	*-664.0*	7.28	1020.7	77.21	58.80	*-552.53*	3.59
Dermato.	182.84	96.37	94.40	194.61	98.82	95.52	87.32	*-1.12*	220.36	99.07	94.12	*-37.52*	0.28	185.28	99.28	94.48	*-2.44*	*-0.08*
Glass	474.29	78.78	72.24	468.8	80.90	73.95	400.34	*-1.71*	408.83	78.65	60.48	65.46	11.76	534.88	83.68	71.28	*-60.59*	0.96
Haberman	12.00	77.12	77.10	11.30	76.78	75.11	*-63.11*	1.99	90.55	79.46	71.89	*-78.55*	5.21	21.13	76.82	71.88	*-9.13*	5.22
Heart	123.29	89.19	84.57	134.13	89.12	84.94	38.35	*-0.37*	109.45	90.19	83.46	13.84	1.11	164.61	91.87	82.84	*-41.32*	1.73
Ionosphere	88.03	94.69	91.56	91.73	94.60	90.21	*-2.18*	1.35	141.33	95.64	88.62	*-53.3*	2.94	86.75	96.25	90.79	1.28	0.77
Mammogr.	73.84	85.49	84.20	81.71	85.72	84.33	*-10.49*	*-0.13*	102.46	83.07	81.04	*-28.62*	3.16	106.74	83.90	80.49	*-32.9*	3.71
Pima	56.12	78.69	77.01	51.17	79.03	75.70	*-19.58*	1.31	95.01	77.80	74.92	*-38.89*	2.09	57.20	79.06	77.05	*-1.08*	*-0.04*
Saheart	59.28	75.51	70.05	107.57	74.91	68.99	*-9.71*	1.06	76.24	76.70	71.14	*-16.96*	*-1.09*	110.84	77.73	70.13	*-51.56*	*-0.08*
Sonar	49.31	87.59	78.61	84.00	88.59	76.73	*-27.42*	1.88	70.67	86.54	78.88	*-21.36*	*-0.27*	47.59	87.91	78.90	1.72	*-0.29*
Vehicle	195.07	70.74	68.20	324.98	70.59	67.46	127.61	0.74	492.55	69.34	62.81	*-297.5*	5.39	382.12	71.11	66.16	*-187.05*	2.04
Wdbc	25.04	97.08	96.78	45.86	96.51	94.90	*-69.86*	1.88	55.74	97.12	94.90	*-30.7*	1.88	44.27	97.33	94.90	*-19.23*	1.88
Wine	40.39	99.60	98.49	65.17	99.79	98.30	*-57.91*	0.19	32.10	100.0	96.08	8.29	2.41	58.99	99.92	93.03	*-18.6*	5.46
Wisconsin	69.81	97.78	96.95	67.42	98.38	96.72	*-26.91*	0.23	77.41	98.22	96.07	*-7.6*	0.88	69.11	98.33	96.35	0.7	0.60
Mean	**129.61**	**84.81**	**81.50**	**177.49**	**85.39**	**80.50**			**192.86**	**84.19**	**78.10**			**192.21**	**85.84**	**79.16**		

2. Computer simulation results [32]

For justifying the performance of the proposed LRB-𝒜, it is compared with the performance of three algorithms, denoted by All-Granularities and Product/1-ALL TUN, examined in [58], and Triangle GSA-based method, examined in [31]. The algorithm name, 'All-Granularities' means that all singularities of its 4-level multi-granularity, formed from the fourteen constructed fuzzy sets, whose levels consists, in turn, of 2, 3, 4, and 5 fuzzy sets, are used to represent the L-variables' word-sets. Meanwhile, the name 'Product/1-ALL TUN' means that only one of its singularities is used to represent the L-variables' word-sets.

Table 5. Comparison of FLRBCS's ACCURACY using the Wilcoxon's signed-ranks test at level α = 0.05

Comparison (α = 0.05)	R^+	R^-	Exact P-value	Asymp. P-value	Confidence interval	Exact confidence
LRB-$\mathcal{A}$ vs Triangle GSA	128.0	25.0	0.012864	0.013278	[0.19 , 1.615]	0.95524
LRB-$\mathcal{A}$ vs All Granularities	148.0	5.0	1.5258E-4	0.000654	[1.495 , 5.135]	0.95524
LRB-$\mathcal{A}$ vs Product/1-ALL TUN	143.0	10.0	6.562E-4	0.00142	[0.865 , 3.615]	0.95524

Table 6. Comparison of fuzzy rule base COMPLEXITY using the Wilcoxon's signed-ranks test at level α = 0.05

Comparison (α = 0.05)	R^+	R^-	Exact P-value	Asymp. P-value	Confid. interval	Exact confid.
LRB-$\mathcal{A}$ vs Triangle GSA	136.0	17.0	0.003158	0.004507	[-41.48905, -5.06995]	0.95524
LRB-$\mathcal{A}$ vs All Granularities	107.0	46.0	0.15938	0.142245	[-65.925, 8.29]	0.95524
LRB-$\mathcal{A}$ vs Product/1-ALL TUN	145.0	8.0	3.814E-4	0.00109	[-56.0751, -10.83605]	0.95524

Seventeen classification datasets, examined in [31], taken from , are also considered in this simulation study.

For running the proposed LRB-$\mathcal{A}$ the semantics constraints and the running EMO-PSO-algorithm's parameters are specified as follows:

○ For running its Phase I:

- *The semantics constraints*: $0.00001 \leq fm(\boldsymbol{0})$, $fm(\boldsymbol{1}) \leq 0.01$, $0.2 \leq fm(c^-) \leq 0.6$, $0.0001 \leq fm(W) \leq 0.2$, $0.2 \leq \mu(L) \leq 0.6$, $0.0001 \leq \mu(h) \leq 0.5$ and $1 \leq k_j \leq 3$.

- *The running PSO-parameters*: inertia coefficient = 0.4, self-cognitive factor = 2.0, social cognitive factor = 2.0, population size = 600, and generation number = 250.

○ For running its Phase II:

The running RBO-parameters: inertia coefficient = 0.4, self-cognitive factor = 1.0, social cognitive factor = 1.0, the population size = 600, and the generation number = 1000 (for some simple datasets we use population size = 300, and generation number = 500).

The simulation results of all four genetic algorithms are represented in Tab. 6 with 17 datasets under consideration. It is sufficient to perform the statistical comparison study using Wilcoxon's test, whose results are represented in Tables 7 and 8.

As all p-values are smaller than α, Wilcoxon's signed-ranks test at level α = 0.05 for comparing LRBs/FRBs performance given in Tab. 7 shows that the Clas-LBCSs, designed by the LRB-$\mathcal{A}$, statistically outperform the ones designed by the counterpart methods. Moreover, by the Wilcoxon's signed-ranks test at level α = 0.05 for comparing LRB/FRB complexity given in Tab. 4,3, it follows, in general, that the complexities of the LRBs designed by the LRB-$\mathcal{A}$ cannot be higher than those designed by the three counterpart algorithms. Since the richer the LRBs/FRBs, the more exact the designed Clas-LRBSs/FRBSs make their decisions, we can eliminate the complexity of the LRBs, designed by the LRB-$\mathcal{A}$, to be the factor to make the proposed LRB-$\mathcal{A}$ outperform its counterparts.

It shows the HA-methodology's performance and advantages.

4.4.   HA-methodology in linguistic data summarization

The linguistic data summarization aims to extract, from a given dataset $\boldsymbol{D}$ the so-called linguistic summaries (LSs) of the following form, called classic prototypes:

$$\text{(i) ``}\mathcal{Q} \ ys \text{ are/have } A_s\text{''; and (ii) ``}\mathcal{Q} \ B_q ys \text{ are/have } A_s\text{''} \tag{4}$$

In (4), $\mathcal{Q}$ is an L-quantifier like '*most*', '*all most*', '*few*', '*little*'; '*y*' denotes a real-world object represented by a record in a given dataset $\boldsymbol{D}$; B_q, called its qualifier, is a linguistic logical conditional expression that y is required to satisfy. E.g., AGE of y is '*rather young*'; A_s, called a summarizer, is a linguistic logical condition that y must satisfy, e.g., y's salary is '*high*'.

For example, the simple LSs are as follows:
    "*Very little* workers have a *high* salary"
or "*Very, very, very little* workers who are *young*, have *high* salary"

4.4.1.   The linguistic summaries' dataset content problem and Tarski et al.'s interpretability concept

This section shows that the HA method has a major advantage over the other fuzzy methods to solve the so-called LSs' dataset content problem because the latter have no formalized formalism to deal with L-words in the natural language directly.

1) The LS' dataset content problem

For ensuring the $\mathfrak{X}$'s words interacting with the $\mathfrak{X}$'s numeric universe values, similar to the L-labels in the fuzzy set framework, words must be represented by fuzzy sets defined on the $\mathfrak{X}$'s numeric universe. Recall that every LS S_L, extracted from $\boldsymbol{D}$, as an ordinary sentence in the natural language, conveys its inherent semantics describing an information content about $\boldsymbol{D}$, denoted by $Cont_{\mathcal{D}}(S_L)$. Thus, it is *objective* because every dataset user understands the same when he reads S_L. Then, a problem emerges: whether the information content conveyed by an LS, S_L, extracted from the dataset $\boldsymbol{D}$, based on its fuzzy set-based representation, $\mathcal{R}_{fuz}(S_L)$, and the information content of $\boldsymbol{D}$, computed by a human-made L-summarization method, $\mathcal{M}$, and denoted by $Cont_{D,\mathcal{M}}(S_L)$, are '*identical*' in the uncertain environment? They are different because $Cont_{D,\mathcal{M}}(S_L)$ depends on $\mathcal{R}_{fuz}(S_L)$ constructed by a human expert based on his intuition, i.e., *subjective*. Thus, under which condition we have the following 'uncertain' equality ($\approx$), called the LS-content problem:

$$Cont_{\mathcal{D},\mathcal{M}}(R_C(S_L)) \approx \ Cont_{\mathcal{D}}(S_L) \tag{5}$$

Thus, (5) is called the *LS-content uncertain equality* in this study. It depends on several factors such as how one can represent words by fuzzy sets, as discussed in Sect. 2 and 3, which operations on the constructed fuzzy set-based words' semantics can be used to model the linguistic logical connectives in the natural language, AND, OR, NEGATION, and IF-THEN. One can adopt to interpret AND and OR as the respective operations MIN and MAX on the fuzzy sets, but it is not easy to define operations on the fuzzy sets to model the NEGATION and IF-THEN behavior because they may relate to the close-/open-world assumption. Therefore, it may also be a hard and complex problem.

Thus, the question is under which condition the 'uncertain' equality in (5) is valid?

2) The condition to ensure the LS-content uncertain equality validity and Tarski et al.'s interpretability concept

Restrict to the classic prototypes, in which occur only the connectives AND and OR, the study [36] proposed, for the first time, a methodology for developing LS-algorithm to extract LSs from a given dataset $\boldsymbol{D}$ satisfying LS-content problem, based on the formalized formalism established in Sect. 2 and 3. Based on this formalism, the developed LS algorithm can manipulate relationships between the words' semantics of an L-variable and its numeric universe values utilizing the interpretable fuzzy $k-$MGr-*structures* representing the L-variables $k-$specificity word-domain multi-semantic aspect *structures*

One can imagine that the way a developed LS algorithm extracts its LSs from $\boldsymbol{D}$ is similar to how human dataset experts accumulate their knowledge pieces of the LS's form from $\boldsymbol{D}$ using their 'natural language-based formalism' when its cardinality is small. Thus, when $\boldsymbol{D}$ is a very large dataset, one may consider the developed LS-algorithm as a tool to solve the human experts' $\boldsymbol{D}$-L-data summarization problem, denoted by P_{Ls} based on the fuzzy set-based formalism established in Sect.3. Instead of solving P_{Ls} in the human experts' natural language-based formalism, one has to solve the respective problem $P_{Ls,Fs}$, translated from P, in the established fuzzy set-based formalism. Because these two formalisms are entirely different, a question now emerging is under which condition the uncertain equality is valid?

Tarski et al.'s interpretability concept in the framework of math-logics [56] states roughly that a theory $\mathcal{S}$ is said to be interpretable in another theory $\mathcal{T}$ if:

(i) $\mathcal{S}$ can be transformed into $\mathcal{T}$;

(ii) Every statement τ_S of $\mathcal{S}$ can be proved in $\mathcal{S}$ if and only if τ_T in $\mathcal{T}$, translated from τ_S, is proved in $\mathcal{T}$. Mathematically, $\mathcal{S}$ should be isomorphically embedded in $\mathcal{T}$.

Thus, instead of solving a problem P of $\mathcal{S}$, one can solve it in $\mathcal{T}$ by translating P of $\mathcal{S}$ into a problem P' in $\mathcal{T}$, only when $\mathcal{S}$ is isomorphic into $\mathcal{T}$.

Therefore, as mentioned above, instead of solving P_{Ls} in the human experts' natural language-based formalism, one can transform P_{Ls} into a problem $P_{Ls,Fs}$ and solve it in a fuzzy set-based formalism developed based on the constructed fuzzy $k-$MGr-*structures*, when and only when they are isomorphic images of their respective L-variables' word-sets $k-$multi-semantic aspect *structures*, $\mathcal{S}_{en,k}^{\mathfrak{X}} = (X_{en,k}^{\mathfrak{X}}, \leq, g)$. I.e., when the L-variables' *structures*, $\mathcal{S}_{en,k}^{\mathfrak{X}}$'s are *interpretable* in their constructed fuzzy $k-$MGr-*structures*. In other words, these constructed fuzzy $k-$MGr-*structures* are interpretable, see Sect. 3.2. Thus, Sect. 2 and 3 offer a fuzzy set-based formalism to solve the LS-content problem formulated above.

Since the LS algorithm proposed in [36] is developed based on the above formalism, its LSs extracted from a given dataset satisfy the LS-content uncertain equality (5).

4.4.2. The proposed LS algorithm's ability and its advantages and benefits

Since, restricted to the logical schema of the LS-algorithm, denoted by LS-$\mathcal{A}$, proposed for the first time in [36] is, in general, similar to those examined in the fuzzy sets' framework, to reduce the presentation space, it is not necessary to recall all here. Thus, only its primary specific features are emphasized as follows:

a. The words appearing in prototypes (4) are not L-labels as in the fuzzy set framework, but the L-variables' words in the natural language, whose semantic *structures* are modelled by their EnHAs, as discussed in Sect. 2. However, in application problem

solving, the only restricted L-variables' word-sets of finite cardinalities and their multi-semantic aspects must be declared for human users' use at the current application life cycle moment. Thus, in the HA approach the application L-variables' all aspect semantics declaration is essential.

b. The constructed fuzzy set-based semantics of each L-variable words forms an interpretable trapezoid fuzzy $k-$MGr-*structure*, which is generated from its current $k-$multi-semantic aspect word set's *structures*, $\mathcal{S}_{en,k}^{\mathfrak{X}}$. Moreover, it must be scalable when k (or its $k-$word set's size) grows.

c. By the interpretability of the currently declare L-variables word-sets the LSs mined by the LS-$\mathcal{A}$ satisfy the *LS-content uncertain equality*, one can consider them as forming a current dataset users' knowledge about the given dataset $\boldsymbol{D}$. Notice that the ordinary experts' domain knowledge in practice is scalable and, hence, it usually grows in time. Study [36] justified that the proposed LS-$\mathcal{A}$ can also simulate how human experts can accumulate their new domain knowledge pieces in their current life cycle while maintaining *their existing knowledge*.

With these specific features, we show that it can extract many useful dataset contents from a given dataset in terms of the LSs satisfying the LS-content uncertain equality.

1) Dataset and its L-variables syntax and semantics

To reach the section's purpose, we apply the LS-$\mathcal{A}$ to the *Bank Marketing* dataset, denoted by D_{BM}, taken from the UCI dataset repository at ; the data are direct marketing campaign data based on phone calls of a Portuguese banking institution. The dataset consists of 41,188 records and 21 attributes. Each record is a set of individual personal client information, such as *age, job, marital, education, housing, loan*. For illustration, in this study, it is restricted to consider only the attributes AGE, JOB, CCI (Consumer Confidence Index – monthly indicator), and Y (indicating an answer to the following question: has the client subscribed to a term deposit?) More details can be found in [36].

Let us consider only simple LSs' prototypes dealing with the attributes AGE, JOB, CCI, and Y.

a) The scalar domains of these attributes are as follows:

∘ Dom(AGE) = [17, 98] in years;

∘ Dom(JOB) = {admin., blue-collar, entrepreneur, housemaid, management, retired, self-employed, services, student, technician, unemployed, unknown};

∘ Dom(CCI) = [-50.8, -26.9];

∘ Dom(Y) = {yes, no}.

b) Determining the L-variables' syntactic, qualitative and quantitative semantics

In this experimental study, the fuzziness parameters' values of each L-variables is specified as follows:

∘ *For the attribute AGE*: Its the independent fuzziness parameters values consists of:

(i) The fuzziness measure of the atomic words and constants: $fm(\boldsymbol{0}) = 0.037$; $fm(y) = 0.259$; $fm(middle) = 0.062$, $fm(\mathfrak{o}) = 0.346$, hence, $fm(\boldsymbol{1}) = 0.296$.

(ii) The fuzziness measure of hedges: $\mu(L) = 0.4$, $\mu(V) = 0.35$, and $\mu(h) = 0.25$.

∘ *For the attribute CCI*: In a similarly way of the case of the attribute AGE, one can also determine the values of the fuzziness measures of the words of its 1-specificity word-set,

$F_{CCI,1} = \{\boldsymbol{0}, l, middle, h, \boldsymbol{1}\}$, where l denotes "*low,*" and h denotes '*high*' of CCI. And, its fuzziness parameters values are given as follows:

$fm(\boldsymbol{0}) = 0.05$; $fm(l) = 0.4$; $fm(medium) = 0.1$, $fm(h) = 0.4$; and, hence, $fm(\boldsymbol{1}) = 0.05$;

The hedges' fuzziness parameters' values are the same as for AGE, $\mu(L) = 0.4$, $\mu(V) = 0.35$, and $\mu(h_0) = 0.25$.

◦ *For the quantifier variable* $\mathcal{Q}$: Similarly, we assume that

For $F_1 = \{\boldsymbol{0}, f, haft, m, \boldsymbol{1}\}$, where f denotes "*few,*" and m denotes '*many*' of $\mathcal{Q}$, $fm(\boldsymbol{0}) = 0.02$; $fm(l) = 0.43$; $fm(haft) = 0.1$, $fm(h) = 0.43$; and, hence, $fm(\boldsymbol{1}) = 0.02$;

The fuzziness parameters' values of the hedge of $\mathcal{Q}$ are: $\mu(L) = 0.5$, $\mu(V) = 0.3$, and $\mu(h_0) = 0.2$.

2) Algorithmically generating the trapezoid-fuzzy k-MGr-structures representing their L-variables' k-multi-semantic aspect structures

As presented in Sect. 3.2.1, when all L-variables semantic aspects are determined, by the procedure proposed in [36], their trapezoid-fuzzy $k-$MGr-*structure*s representing their L-variables' $k-$multi-semantic aspect *structure*s, in the form given in Fig. 2, are algorithmically generated. This feature is crucial for developing a *scalable* LS-$\mathcal{A}$, i.e., it can extract the new LSs from the given dataset, containing higher specificity words, when some of the current L-variables' word-sets are required to grow. Meanwhile, the already extracted LSs are maintained (even though they may not appear in the output list by certain criteria under consideration, sometimes).

Thus, the proposed *scalable* LS-$\mathcal{A}$ can handle the growth of some existing L-variables' word-sets to extract additional LSs from the given dataset similar to human domain experts accumulating additional consistent knowledge pieces in their daily activities.

4.4.3. Experimental study results

To justify the proposed *scalable* LS-algorithm, LS-$\mathcal{A}$, we exhibit some of the simulation study results in this point [36].

Let us consider LSs of the prototypes given in (4), whose qualifier is $(o(\text{JOB}) = 'j')$ AND $(o(\text{AGE}) = x)$, and whose summarizer is $((o(\text{CCI}) = y)$ AND $(o(\text{Y}) = \{yes, no\}))$, where $j \in \text{Dom(JOB)}$ and the word $y \in \text{Dom(AGE)}$. Its qualifier defined a *fuzzy* client-group, whose job is 'j' and age is 'x'; its summariser describes the client-group behavior by answering '*yes*' or '*no*' when CCI $= y$.

Considering the L-attributes AGE and CCI as usual variables, this experiment can show that the proposed LS-$\mathcal{A}$ can discover L-distributions of a fuzzy client-group to another $D_{BM}-$attribute word, say CCI. They represent the dependents of the fuzzy client-groups, defined by the AGE-words, on the CCI-words. For example, these L-distributions extracted by LS-$\mathcal{A}$ from the *Bank Marketing* dataset are represented by Tab. 9 to 12, in which the index numbers of the L-quantifier words indicate their position in the linear chain of the Q-word-set of 4-specificity, i.e., its maximal words' lengths are 4.

These tables provide much useful linguistic information for $D_{BM}-$users. Indeed,

- Since D_{BM} under consideration is a large dataset, it can be viewed as a data cloud, from which it is not easy to draw useful L-information for dataset users. However, each table can represent a 2-dimensional L-distribution, whose values are L-quantifiers. Its L-quantifiers variation describes the clients' behavior with different ages to the variation of the CCI's L-values. E.g., since the darker the color, the larger the L-quantifies of the clients whose

Table 9. The distribution of the L-fuzzy client group defined by "(JOB IS 'SERVICE') AND (AGE IS 'u')" to CCI-L-VALUES with Y = 'YES' ($F_{Q,3}$-words: $0_0 \leq VVf_2 \leq Vf_4 \leq LVf_6 \leq few_8 \leq LLf_{10} \leq Lf_{12} \leq VLf_{14} \leq half_{16} \leq VLm_{18} \leq Lm_{20} \leq LLm_{22} \leq m_{24} \leq LVm_{26} \leq Vm_{28} \leq VVm_{30} \leq 1_{32}$) (in the context of 33 L-quantifiers)

AGE \ CCI	0	VV_l	V_l	LV_l	low	LL_l	L_l	VL_l	Medium	VL_h	L_h	LL_h	High	LV_h	V_h	VV_h	1
V_y	0_0	0_0	0_0	0_0	V_f_4	VV_f_2	VV_f_2	0_0	V_f_4	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0
LV_y	0_0	0_0	0_0	0_0	V_f_4	VV_f_2	VV_f_2	0_0	V_f_4	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0
young	0_0	0_0	0_0	0_0	VV_f	0_0	VV_f_2	0_0	VV_f_2	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0
LL_y	0_0	0_0	0_0	0_0	VV_f_2	0_0	VV_f_2	0_0	VV_f_2	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0
L_y	0_0	0_0	0_0	0_0	VV_f_2	0_0	VV_f_2	0_0	VV_f_2	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0
VL_y	0_0	0_0	0_0	0_0	VV_f_2	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0
Middle	0_0	0_0	0_0	0_0	VV_f_2	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0
VL_o	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0	VV_f_2	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0
L_o	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0
old	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0	0_0

Table 10. The distribution of the L-fuzzy client group defined by "(JOB IS 'SERVICE') AND (AGE IS 'u')" to CCI-L-VALUES with Y = 'NO' ($F_{Q,3}$-words: $0_0 \leq VVf_2 \leq Vf_4 \leq LVf_6 \leq few_8 \leq LLf_{10} \leq Lf_{12} \leq VLf_{14} \leq half_{16} \leq VLm_{18} \leq Lm_{20} \leq LLm_{22} \leq m_{24} \leq LVm_{26} \leq Vm_{28} \leq VVm_{30} \leq 1_{32}$) (in the context of 33 L-quantifiers)

AGE \ CCI	0	VV_l	V_l	LV_l	low	LL_l	L_l	VL_l	Medium	VL_h	L_h	LL_h	High	LV_h	V_h	VV_h	1
V_y	0_0	0_0	few_8	few_8	VL_m_{18}	VL_f_{14}	$half_{16}$	few_8	$half_{16}$	few_8	few_8	V_f_4	few_8	0_0	0_0	0_0	0_0
LV_y	0_0	0_0	few_8	few_8	VL_m_{18}	VL_f_{14}	$half_{16}$	few_8	$half_{16}$	few_8	few_8	V_f_4	LV_f_6	0_0	0_0	0_0	0_0
young	0_0	0_0	few_8	few_8	$half_{16}$	L_f_{12}	$half_{16}$	few_8	VL_f_{14}	few_8	few_8	V_f_4	LV_f_6	0_0	0_0	0_0	0_0
LL_y	0_0	0_0	LL_f_{10}	LL_f	$half_{16}$	L_f_{12}	$half_{16}$	few_8	$half_{16}$	few_8	LL_f_{10}	V_f_4	few_8	0_0	0_0	0_0	0_0
L_y	0_0	0_0	LL_f_{10}	few_8	$half_{16}$	L_f_{12}	$half_{16}$	few_8	$half_{16}$	few_8	LL_f_{10}	V_f_4	LV_f_6	0_0	0_0	0_0	0_0
VL_y	VV_f_2	0_0	LL_f_{10}	few_8	$half_{16}$	L_f_{12}	$half_{16}$	few_8	$half_{16}$	few_8	LL_f_{10}	V_f_4	few_8	0_0	0_0	0_0	0_0
Middle	VV_f_2	0_0	few_8	few_8	$half_{16}$	L_f_{12}	$half_{16}$	few_8	$half_{16}$	LL_f_{10}	L_f_{12}	LV_f_6	few_8	0_0	0_0	0_0	0_0
VL_o	0_0	0_0	few_8	few_8	$half_{16}$	L_f_{12}	VL_f_{14}	few_8	$half_{16}$	L_f_{12}	VL_f_{14}	few_8	few_8	0_0	0_0	0_0	0_0
L_o	0_0	0_0	few_8	few_8	VL_f_{14}	LL_f_{10}	VL_f_{14}	few_8	$half_{16}$	L_f_{12}	L_f_{12}	few_8	few_8	0_0	0_0	0_0	0_0
old	VV_f_2	0_0	few_8	few_8	$half_{16}$	L_f_{12}	VL_f_{14}	few_8	VL_m_{18}	L_f_{12}	L_f_{12}	few_8	few_8	0_0	0_0	0_0	0_0

Table 11. The distribution of the L-fuzzy client group defined by "(JOB IS 'STUDENT') AND (AGE IS 'u')" to CCI-L-VALUES with Y = 'NO' ($F_{Q,3}$-words: $0_0 \leq VVf_2 \leq Vf_4 \leq LVf_6 \leq few_8 \leq LLf_{10} \leq Lf_{12} \leq VLf_{14} \leq half_{16} \leq VLm_{18} \leq Lm_{20} \leq LLm_{22} \leq m_{24} \leq LVm_{26} \leq Vm_{28} \leq VVm_{30} \leq 1_{32}$) (in the context of 33 L-quantifiers)

AGE \ CCI	0	VV_l	V_l	LV_l	low	LL_l	L_l	VL_l	Medium	VL_h	L_h	LL_h	High	LV_h	V_h	VV_h	1
Ab_y	V_f_4	0_0	few_8	few_8	LL_f_{10}	V_f_4	few_8	V_f_4	few_8	0_0	V_f_4	V_f_4	few_8	LV_f_6	LV_f_6	0_0	VV_f_2
VV_y	V_f_4	0_0	few_8	few_8	LL_f_{10}	LV_f_6	few_8	V_f_4	few_8	VV_f_2	V_f_4	V_f_4	LV_f_6	V_f_4	LV_f_6	0_0	VV_f_2
V_y	V_f_4	0_0	few_8	few_8	L_f_{12}	LV_f_6	few_8	V_f_4	few_8	V_f_4	LV_f_6	V_f_4	few_8	V_f_4	V_f_4	0_0	0_0
LV_y	V_f_4	0_0	few_8	few_8	L_f_{12}	few_8	few_8	V_f_4	few_8	V_f_4	few_8	V_f_4	few_8	V_f_4	V_f_4	0_0	0_0
young	V_f_4	0_0	few_8	few_8	L_f_{12}	few_8	few_8	V_f_4	LL_f_{10}	V_f_4	few_8	V_f_4	few_8	V_f_4	V_f_4	0_0	0_0
LL_y	V_f_4	0_0	few_8	few_8	L_f_{12}	few_8	L_f_{12}	few_8	VL_f_{14}	few_8	few_8	LV_f_6	few_8	VV_f_2	V_f_4	0_0	0_0
L_y	V_f_4	0_0	few_8	few_8	L_f_{12}	few_8	L_f_{12}	few_8	L_f_{12}	few_8	few_8	LV_f_6	few_8	V_f_4	V_f_4	0_0	0_0

answer is 'NO,' the behavior of the clients' group 'STUDENT", represented by Tab. 10, is notably different from the one of the clients' group 'SERVICE", represented by Tab. 11.

- The rows of every table represent the L-distributions of a fuzzy linguistic age clients' group to CCI's linguistic values. They can be considered *non-linear L-regression* functions, drawn from the given data cloud D_{BM}, which may provide D_{BM}−users valuable information in L-words. Each of them represents a fuzzy L-age client groups' behavior when the linguistic

Table 12. The distribution of the L-fuzzy client group defined by "(JOB IS 'STUDENT') AND (AGE IS 'u')" to CCI-L-VALUES with Y = 'NO' ($F_{Q,4}$-words: $0_0 \leq VVVf_1 \leq VVf_2 \leq LVVf_3 \leq Vf_4 \leq LLVf_5 \leq LVf_6 \leq VLVf_7 \leq few_8 \leq VLLf_9 \leq LLf_{10} \leq LLLf_{11} \leq Lf_{12} \leq LVLf_{13} \leq VLf_{14} \leq VVLf_{15} \leq half_{16} \leq VVLm_{17} \leq VLm_{18} \leq LVLm_{19} \leq Lm_{20} \leq LLLm_{21} \leq LLm_{22} \leq VLLm_{23} \leq m_{24} \leq VLVm_{25} \leq LVm_{26} \leq LLLm_{27} \leq Vm_{28} \leq LVVm_{29} \leq VVm_{30} \leq VVVm_{31} \leq 1_{32}$)

CCI / AGE	0	VV_l	V_l	LV_l	low	LL_l	L_l	VL_l	Medium	VL_h	L_h	LL_h	High	LV_h	V_h	VV_h	1
Ab_y	V_f4	0_0	few_8	few_8	LL_f10	LLV_f5	few_8	V_f4	few_8	0_0	V_f4	V_f4	few_8	LV_f6	LV_f6	0_0	VV_f2
VV_y	LVV_f3	0_0	few_8	few_8	LL_f10	LV_f6	few_8	V_f4	few_8	VV_f2	V_f4	V_f4	VLV_f7	LLV_f5	LV_f6	0_0	VV_f2
V_y	V_f4	0_0	few_8	few_8	LLL_f11	LV_f6	few_8	V_f4	few_8	V_f4	LV_f6	V_f4	few_8	V_f4	V_f4	0_0	0_0
LV_y	V_f4	0_0	few_8	few_8	L_f12	VLV_f7	few_8	V_f4	VLL_f9	V_f4	VLV_f7	V_f4	few_8	V_f4	V_f4	0_0	0_0
young	V_f4	0_0	few_8	few_8	LLL_f11	few_8	VLL_f9	LLV_f5	LL_f10	LLV_f5	LV_f5	V_f4	VLV_f7	V_f4	V_f4	0_0	0_0
LL_y	V_f4	0_0	few_8	few_8	L_f12	few_8	LLL_f11	few_8	LVL_f13	few_8	VLL_f9	LV_f6	few_8	LVV_f3	V_f4	0_0	0_0
L_y	V_f4	0_0	few_8	few_8	L_f12	few_8	LLL_f11	few_8	L_f12	few_8	few_8	LV_f6	few_8	V_f4	V_f4	0_0	0_0

CCI-values increase. Moreover, when transforming their word-values to numeric, quantitative values, we obtain piecewise-linear regression functions also very useful for $D_{BM}-$users' decision-making activities.

- It can be observed that the 2-dimensional L-regression functions represented in Tab.'s 9 - 11 above vary roughly and, hence, not much information can be drawn from them. A specific, very useful feature of the proposed LS-$\mathcal{A}$ is its scalability, i.e., it can handle the growth of the currently declared L-variables' word-sets to refine them. Therefore, the proposed LS-$\mathcal{A}$ can improve, for example, the L-regression functions presented in Tab. 11 by allowing the current L-variable $\mathcal{Q}$'s 3-specificity word-set $F_{Q,3}$ to grow to the 4-specificity word-set, $F_{Q,4}$, in Tab. 12. They provide useful information when paying attention to the $\mathcal{Q}$-words with red.

Note that the proposed LS-$\mathcal{A}$ maintains their respective words in Tab. 11. However, they are replaced by their respective higher specific $\mathcal{Q}$-words in Tab.'s 12, which provide more information. Considering the LSs in each table extracted by the proposed LS-$\mathcal{A}$ from the banking dataset, D_{BK}, as D_{BK}knowledge, we can conclude that the proposed LS-$\mathcal{A}$ can simulate how human experts accumulate the domain knowledge pieces in their daily lives activities.

5. CONCLUSIONS AND SOME DISCUSSIONS

The natural language has been formed and developed in parallel with the struggles for human society's existence and development. It plays an essential human beings' tool for communicating between communities' people, interacting with reality, accumulating their knowledge, and making decisions. As a result, the natural language has become the human tool for representing and conveying necessary information in daily human activities. Thus, based on the essential structure and information relation standpoint, there must be many distinct structures in the natural language, the logical structure in the natural language, for instance. However, until 1990 and 1992, the L-variables' word-domain semantic structures were discovered and mathematically modeled by the so-called hedge algebras. Recall that there are still many people who doubt that there are mathematical structures in the natural language. This fact emphasizes the essential role of the structure and information relationships more strongly, which Prof Phan Dinh Dieu usually mentioned in his talks in the 70s

and 80s - decades.

In order to justify this essential relationship again, the survey describes how one can discover and mathematically formalize the potentially infinite L-variables' word domains in an axiomatic manner. Since the L-variables' word semantics is a complex concept, it should emphasize that the axiomatic manner is ideal for revealing the L-variables' word domain semantic structures soundly.

The survey presented necessary facts to establish an HA formalism with its essential specific features that make the HA formalisms, established to develop HA methods for problem-solving, entirely different from the respective formalisms established in the fuzzy set framework. It shows that to handle L-words directly, following Tarski et al."s interpretability concept in the math-logic field, the L-variables' word domains, with no numerically computational features, must be mathematically formalized and represented by their *interpretable* and *scalable* fuzzy set *structures.*

With the above novel features, the survey introduced also selected research results applying HA-formalism to develop some methods to solve application problems in fuzzy control, classification, and, in particular, in linguistic data summarization to show the performance and advantages of the HA formalism. A distinguished application result is solving the problem of extracting linguistic knowledge from a given large dataset is an important and meaningful problem. A question arising from this problem is that the dataset contents represented by human L-sentences (LS) are objective because when they read them, the human experts capture the same meaning. While the fuzzy sets used to represent the dataset's L-variables' words depend on how they are constructed, it usually depends on the designers' intuition and, hence, it is subjective. Thus, the content extracted from a given dataset D by a human-made extracting method, $\mathcal{M}$, and assigned to the LS $\mathcal{L}_s$, denoted by $Cont_{\mathcal{M},\mathcal{D}}(\mathcal{L}_s)$, and the dataset content represented by $\mathcal{L}_s$, denoted by $Cont(L_s)$, when it is considered a linguistic sentence, are different in general. The question is that when they are approximately equal, in the uncertainty environment, called the LS-content problem in this survey.

Following our knowledge, it could not be solved in the fuzzy set framework by Tarski et al."s interpretability concept, which requires at least that the L-variables' word domains must be mathematically formalized. Study [36] proved that it could be solved in the HA approach. Only this achieved result is sufficient to show the performance and the methodological advantage of the HA approach.

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