ROBUST FINITE-TIME SUPOPTIMAL CONTROL OF LARGE-SCALE SYSTEMS WITH INTERACTED STATE AND CONTROL DELAYS

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Abstract. This paper concerns with a problem of supoptimal finite-time control for a class of linear large-scale delay systems. The system under consideration is subjected to the state and control delays interacted between subsystems. Based on improved LMI approach combining with new estimation techniques, we derive sufficient conditions for solving $H_{\infty}$ finite-time control and guaranteed cost control of the system. A numerical example is given to illustrate the validity and effectiveness of the theoretical results.

Keywords. Finite-time control; Supoptimal control; Large-scale system; State and control delay; Lyapunov function; Linear matrix inequality.

1. INTRODUCTION

Over the past decades stability and control theory of large-scale systems has been extensively studied due to its useful applications in various areas of applied science such as signal processing, communications, power systems and telecommunication networks (see, e.g. [1, 2] and the references therein). The concept of large-scale systems has been introduced when it became clear that there are real world control problems, which cannot be solved by using one-shot approaches. In general, a large-scale system can be characterized by a large number of variables representing the system, a strong interaction between subsystem variables. The control and stability analysis of large-scale systems have become complicated owing to the high dimensionality of the system equation, uncertainties, and time-delays. In the framework of large-scale interconnected systems, the problem of $H_{\infty}$ control has been received considerable attention [3, 4, 5, 6]. On the other hand, the $H_{\infty}$ finite-time control problem consists of the design of a state feedback control, which stabilizes the closed-loop system and guarantees an adequate level of system performance over a finite-time interval. It is notable

Dedicated to Professor Phan Dinh Dieu on the occasion of his 85th birth anniversary.

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that the aforementioned results are only devoted to asymptotic stability, i.e. the stability criteria are defined on an infinite-time interval. However, in practice we are interested not only in the system behavior in infinite-time interval, but also in finite-time interval. Then, instead of asymptotic stability, it is preferable to use the stability defined over a finite-time interval, i.e. finite-time stability (FTS) [7]. The authors of [8, 9, 10, 11] proposed some sufficient conditions for $H_\infty$ finite-time control of linear large-scale systems, however, either the state delays or control delays were not considered. It is worth pointing out that almost the existing results on finite-time control of large-scale systems were studied for systems without control delays in interconnection. It is difficult to design feedback controllers for large-scale systems when the state and control delays are both interacted between all subsystems. The reason is that the interacted state and control delays are of high dimensions and thus require extensive computations to implement the centralized procedure. In [12, 13, 14] the authors studied problem of finite-time stability and stabilization for large-scale systems with delays, however the control delays are not considered. To the best of our knowledge, the problem of guaranteed cost finite-time control for large-scale systems with delays on both the state and control has not yet been studied in the literature. Therefore, this problem for large-scale systems with control and state delays in interconnection still remains open, which motivates our present research.

In this paper, we consider the problem of robust $H_\infty$ finite-time control for linear large-scale systems with state and control delays in interconnection. Our purpose is to design state feedback controllers which guarantee not only the robust finite-time stability of the closed-loop system, but also provide an optimal level of the cost performance. The contribution of our paper lies in three aspects. (a) The time delays are interacted in both the state and control variables; (b) The disturbance is norm-bounded; (c) Based on Lyapunov function method combining with linear matrix inequality technique, we provide new sufficient conditions for solving the control problem. The conditions are formulated in terms of linear matrix inequalities (LMIs), which can be easily implemented by numerical algorithms [15]. Finally, a numerical example is given to show the validity and effectiveness of the theoretical results.

**Notations.** $R^{n\times r}$ denotes the space of all $(n \times r)$- matrices; $\lambda(A)$ denotes the set of all eigenvalues of $A$; $\lambda_{\text{max}}(A) = \max\{\text{Re} \lambda : \lambda \in \lambda(A)\}$; $\lambda_{\text{min}}(A) = \min\{\text{Re} \lambda : \lambda \in \lambda(A)\}$; $C([a, b], R^n)$ denotes the set of all $R^n$-valued continuous functions on $[a, b]$; $L_2([0, T], R^r)$ stands for the set of all square-integrable $R^r$-valued functions on $[0, T]$; A matrix $P$ is symmetric positive definite, $P > 0$, if $P = P^T$ and $x^T P x > 0$ for all $x \in R^n, x \neq 0$, $P > Q$ means that $P - Q > 0$. The symmetric terms in a matrix are denoted by $*$; The segment of the trajectory $x(t)$ is denoted by $x_t = \{x(t+s) : s \in [-\tau, 0]\}$ with its norm $\|x_t\| = \sup_{s \in [-\tau, 0]} \|x(t+s)\|$. 

2. PRELIMINARIES

Consider the following linear large-scale control system with delays
\[
\begin{aligned}
\dot{x}_i(t) &= A_i x_i(t) + \sum_{j=1, j\neq i}^{N} A_{ij} x_j(t - d_{ij}) + B_i w_i(t) \\
&\quad + \sum_{j=1, j\neq i}^{N} B_{ij} u_j(t - m_{ij}) + D_i w_i(t), \\
\dot{z}_i(t) &= E_i x_i(t) + F_i x_i(t - d_{ii}), \\
x_i(t) &= \varphi_i(t), \quad u_i(t) = \phi_i(t), \quad t \in [-\tau, 0],
\end{aligned}
\]

where \( x_i(t) \in \mathbb{R}^{n_i} \) is the state vector, \( u_i(t) \in \mathbb{R}^{r_i} \) is the control vector; \( w_i(t) \in \mathbb{R}^{r_i} \) is the disturbance vector, \( z_i(t) \in \mathbb{R}^{n_i} \) is the observation vector; \( \tau = \max\{d, m\} \); \( A_i, E_i, F_i \in \mathbb{R}^{n_i \times n_i} \), \( B_i \in \mathbb{R}^{n_i \times m_i} \), \( A_{ij} \in \mathbb{R}^{n_i \times n_j} \), \( B_{ij} \in \mathbb{R}^{n_i \times m_j} \), \( D_i \in \mathbb{R}^{n_i \times r_i} \) are constant matrices of appropriate dimensions; the delays satisfy

\[
0 \leq d_{ij} \leq d, \quad 0 \leq m_{ij} \leq m, \quad i, j = 1, N;
\]

the disturbance \( w_i(t) \) satisfies

\[
\exists \eta > 0 : \max_{i=1, N} \left\{ \sup_{t>0} \{w_i^\top(t)w_i(t)\} \right\} \leq \eta. \tag{2}
\]

A solution of large-scale system (1) is a vector function \( x(t) = [x_1(t), x_2(t), \ldots, x_N(t)] \in \mathbb{R}^{Nn} \), which satisfies system of differential equations (1). It is well-known from [1], for \( u_i(t) \in L_2([0, T], \mathbb{R}^{m_i}) \), \( w_i(t) \in L_2([0, T], \mathbb{R}^{r_i}) \), \( i = 1, 2, \ldots, N, T > 0 \), the large-scale system (1) under the initial condition \( \varphi_i(.) \in C([-\tau, 0], \mathbb{R}^{n_i}) \) has a unique solution \( x(t) \).

Associate with system (1), we consider the following cost function

\[
J(t, u) = \int_{0}^{t} \sum_{i=1}^{N} \left[ x_i^\top(t)U_i x_i(t) + \sum_{j=1, j\neq i}^{N} x_j^\top(t - d_{ji}) V_i x_i(t - d_{ji}) + u_i^\top(t) W_i u_i(t) \right] dt, \tag{3}
\]

where \( U_i > 0, V_i > 0, \) and \( W_i > 0, \) \( i = 1, N \) are given symmetric matrices. Let us set

\[
Q = \text{diag}\{Q_1, \ldots, Q_N\}, \quad x^\top(t) = [x_1(t)^\top, \ldots, x_N(t)^\top],
\]

\[
u(t) = [u_1(t), \ldots, u_N(t)], \quad \varphi^\top(t) = [\varphi_1(t)^\top, \ldots, \varphi_N(t)^\top].
\]

**Definition 1.** (Robust finite-time stability) Given positive numbers \( T, c_1, c_2 \) and a symmetric matrix \( Q > 0 \), the unforced system (1) \( (u(t) = 0) \) is said to be robustly finite-time stable w.r.t. \( (c_1, c_2, T, Q) \) if for all disturbances \( w(t) \) satisfying (2) the solution \( x(t) \) of the system satisfies the following relation

\[
\sup_{s \in [-\tau, 0]} \{\varphi^\top(s)Q\varphi(s)\} \leq c_1 \rightarrow x^\top(t)Qx(t) < c_2, \forall t \in [0, T].
\]

**Definition 2.** (Finite-time stabilization) For given positive numbers \( T, c_1, c_2 \), and a symmetric matrix \( Q > 0 \), system (1) is robustly finite-time stabilizable with respect to \( (c_1, c_2, T, Q) \)
if there exist state feedback controllers \( u_i(t) = K_i x_i(t) \), \( i = 1, N \), such that the closed-loop system is robustly finite-time stable w.r.t. \((c_1, c_2, T, Q)\).

**Definition 3.** (Robust \textit{H}_\infty finite-time control). For given positive numbers \( T, c_1, c_2, \gamma > 0 \), and a symmetric matrix \( Q > 0 \), the robust \textit{H}_\infty finite-time control problem for system (1) is solvable if:

(i) System (1) is robustly finite-time stabilizable w.r.t. \((c_1, c_2, T, Q)\).

(ii) There is a number \( c_0 > 0 \) such that

\[
\sup_{\varphi_i(t) \in C([−\tau, 0], R^n), w_i(t) satisfying (2)} \int_0^T \sum_{i=1}^N \|z_i(t)\|^2 dt \leq c_0 \|\varphi\|^2 + \sum_{i=1}^N \int_0^T \|w_i(t)\|^2 dt \leq \gamma,
\]

where the supremum is taken over all \( \varphi_i(t) \in C([−\tau, 0], R^n) \) and non-zero disturbances \( w_i(t) \) satisfying (2).

**Definition 4.** (Guaranteed cost control) For given positive numbers \( T, c_1, c_2 \) and symmetric matrices \( Q > 0 \), the guaranteed cost finite-time control for system (1) is solvable if there exist state feedback controllers \( u_i = K_i x_i(t) \), \( i = 1, N \) and a number \( J^* > 0 \) such that the closed-loop system of (1) is robustly finite-time stable w.r.t \((c_1, c_2, T, Q)\) and \( J(T, u) \leq J^* \).

Number \( J^* \) is the guaranteed cost value, control \( u(t) \) is the guaranteed cost controller. The following technical lemmas are introduced for the proof of the main result.

**Lemma 1.** (Cauchy matrix inequality [16]) For given \( a, b \in R^n, 0 < P \in R^{n \times n} \), we have

\[
2a^\top b \leq a^\top P^{-1} a + b^\top P b.
\]

**Lemma 2.** (Schur complement lemma [16]) For given matrices \( U, V, Q \) with appropriate dimensions satisfying \( V = V^\top > 0, U = U^\top \), we have

\[
U + Q^\top V^{-1} Q < 0 \iff \begin{bmatrix} U & Q^\top \\ Q & -V \end{bmatrix} < 0.
\]

**Lemma 3.** For given matrices \( M_1, M_2, Z, Y, Q \) with appropriate dimensions satisfying \( M_1 = M_1^\top, Q = Q^\top > 0 \) and \( Y = Y^\top > 0 \), we have

\[
\begin{bmatrix} M_1 + Z^\top Y^{-1} Z & M_2 \\ M_2^\top & -Q \end{bmatrix} < 0 \iff \begin{bmatrix} M_1 & M_2^\top \\ M_2 & -Q \end{bmatrix} < 0.
\]

**Proof.** We have

\[
\begin{bmatrix} M_1 + Z^\top Y^{-1} Z & M_2 \\ M_2^\top & -Q \end{bmatrix} = \begin{bmatrix} M_1 & M_2^\top \\ M_2 & -Q \end{bmatrix} + \begin{bmatrix} Z^\top Y^{-1} Z & 0 \\ 0 & 0 \end{bmatrix}.
\]
which is equivalent to
\[ Z^1 Y^1 Z 0 \] and obtain that
\[ \begin{bmatrix} M_1 & M_2^T \\ M_2 & -Q \end{bmatrix} + \begin{bmatrix} Z^T \\ 0 \end{bmatrix} Y^{-1} \begin{bmatrix} Z & 0 \end{bmatrix} < 0, \]
which is equivalent to
\[ \begin{bmatrix} M_1 & M_2^T & Z^T \\ M_2 & -Q & 0 \\ Z & 0 & -Y \end{bmatrix} < 0. \]

3. MAIN RESULTS

In this section, we provide sufficient conditions for solving finite-time supoptimal finite-time control of (1). Before presenting the theorem, we introduce the following notations of several matrix variables for simplicity.

\[ P_i = P_i^{-1}, R_i = P_i^{-1} R_i P_i^{-1}, \]

\[ H_i^1, 1 = A_i P_i + P_i A_i^T + \frac{1}{\gamma} D_i D_i^T + B_i Y_i + Y_i^T B_i^T + (N - 1) R_i + \sum_{j=1,j \neq i}^{N} A_{ij} A_{ij}^T, \]

\[ H_i^{1+i, i+1} = -I, H_i^{1+i, i+1} = \sqrt{N - 1} P_i, H_i^{1+j, 1+j} = -R_j, H_i^{1+j, 1+j} = B_{ij} Y_j, i, j = 1, N, j \neq i, \]

\[ H_i^{1, N+2} = P_i E_i^T, H_i^{1, N+3} = P_i F_i^T, H_i^{1, N+3, N+3} = -\beta I, \]

\[ H_{ij} = 0, \text{ for all other } i, j, j \neq i, \]

\[ \alpha_1 = \min_{i=1,N} \left( \frac{\lambda_{\min}(P_{11})}{\lambda_{\max}(Q_i)} \right), \]

\[ \alpha_2 = \max_{i=1,N} \left( \frac{\lambda_{\min}(Q_i)}{\lambda_{\min}(Q_i)} \right) + 2 \beta \tau \max_{i=1,N} \left( \frac{\lambda_{\max}(F_i^T F_i)}{\lambda_{\max}(Q_i)} \right) + (N - 1) \tau \max_{i=1,N} \left( \frac{1}{\lambda_{\min}(Q_i)} \right), \]

\[ \alpha_3 = \max_{i=1,N} \left( \frac{\lambda_{\min}(Q_i)}{\lambda_{\min}(Q_i)} \right) + 2 \beta \tau \max_{i=1,N} \left( \frac{\lambda_{max}(R_{11})}{\lambda_{\max}(Q_i)} \right) + (N - 1) \tau \left[ 1 + \max_{i=1,N} \left( \frac{\lambda_{\max}(R_{11})}{\lambda_{\max}(Q_i)} \right) \right]. \]

Next theorem gives sufficient conditions for \( H_\infty \) finite-time control of system (1).

**Theorem 1.** For given positive numbers \( T, c_1, c_2 \), and a symmetric matrix \( Q > 0 \), the robust \( H_\infty \) finite-time control problem for the systems (1) is solvable if there exist symmetric matrices \( P_i > 0, R_i > 0, i = 1, N \), matrices \( Y_i, i = 1, N \), and a number \( \beta > 0 \) satisfying the following conditions

\[ \begin{bmatrix} H_{11} & H_{12} & \cdots & H_{1(N+3)} \\ \ast & H_{22} & \cdots & H_{2(N+3)} \\ \ast & \ast & \cdots & H_{i(N+3)(N+3)} \end{bmatrix} < 0, \quad i = 1, N, \]

\[ \frac{\alpha_2 c_1 + N \gamma \beta T}{\alpha_1} \leq e^{-\beta T} c_2. \]

Moreover, the state feedback controllers are defined by \( u_i(t) = Y_i P_i^{-1} x_i(t), i = 1, N \).
Proof. Consider the following Lyapunov-Krasovskii functional

\[ V(t, x_t) = \sum_{i=1}^{N} [V_{i1}(t, x_t) + V_{i2}(t, x_t) + V_{i3}(t, x_t)], \]

where

\[ V_{i1}(t, x_t) = e^{\beta t} x_i(t)^T P_{i1} x_i(t), \]
\[ V_{i2}(t, x_t) = e^{\beta t} \int_{t-d_{ii}}^{t} 2\beta x_i(s)^T F_i^T F_i x_i(s) ds, \]
\[ V_{i3}(t, x_t) = e^{\beta t} \sum_{j=1, j \neq i}^{N} \left( \int_{t-d_{ji}}^{t} x_i(s)^T x_j(s) ds + \int_{t-m_{ji}}^{t} x_i(s)^T R_{1i} x_j(s) ds \right). \]

Taking the derivative of \( V(t, x_t) \) in \( t \), we have

\[
\dot{V}_{i1}(t, x_t) = \beta V_{i1}(t, x_t) + e^{\beta t} 2x_i(t)^T P_{i1} \left[ A_i x_i(t) \\
+ \sum_{j=1, j \neq i}^{N} \left( A_{ij} x_j(t - d_{ij}) + B_{ij} Y_j P_{j1} x_j(t - m_{ij}) \right) + B_i Y_i P_{1i} x_i(t) + D_i w(t) \right],
\]
\[
\dot{V}_{i2}(t, x_t) = \beta V_{i2}(t, x_t) + e^{\beta t} 2\beta \left[ x_i(t)^T F_i^T F_i x_i(t) - x_i(t - d_{ii})^T F_i^T F_i x_i(t - d_{ii}) \right],
\]
\[
\dot{V}_{i3}(t, x_t) = \beta V_{i3}(t, x_t) + e^{\beta t} (N - 1) \left[ x_i(t)^T x_i(t) + x_i(t)^T R_{1i} x_i(t) \right]
- e^{\beta t} \sum_{j=1, j \neq i}^{N} \left[ x_i(t - d_{ji})^T x_j(t - d_{ji}) + x_i(t - m_{ji})^T R_{1i} x_j(t - m_{ji}) \right].
\]

Applying Cauchy matrix inequality, Lemma 1, for the following inequalities

\[
2x_i(t)^T P_{i1} \left[ \sum_{j=1, j \neq i}^{N} A_{ij} x_j(t - d_{ij}) \right] \leq \sum_{j=1, j \neq i}^{N} \left[ x_i^T(t) P_{i1} A_{ij} A_{ij}^T P_{i1}^T x_i(t) + x_j(t - d_{ij})^T x_j(t - d_{ij}) \right],
\]
\[
2x_i^T(t) P_{i1} \left[ \sum_{j=1, j \neq i}^{N} B_{ij} Y_j P_{j1} x_j(t - m_{ij}) \right] \leq \sum_{j=1, j \neq i}^{N} \left[ x_i^T(t) P_{i1} B_{ij} Y_j R_j^{-1} Y_j^T B_{ij}^T P_{i1}^T x_i(t) + x_j(t - m_{ij})^T R_{j1} x_j(t - m_{ij}) \right],
\]
\[
2x_i^T(t) D_i w_i(t) \leq \frac{1}{\gamma \beta} x_i^T(t) P_{i1} D_i D_i^T P_{i1} x_i(t) + \gamma \beta w_i^T(t) w_i(t),
\]
we obtain that

\[
\dot{V}_{i1}(t, x_t) \leq \beta V_{i1}(t, x_t) + e^{\beta t} x_i(t)^T [P_{i1} A_i + A_i^T P_{i1} + \frac{1}{\gamma \beta} P_{i1} D_i D_i^T P_{i1} + P_{i1}(B_i Y_i + Y_i^T B_i^T) P_{i1}] x_i(t) + e^{\beta t} \sum_{j=1, j \neq i}^N x_j(t)^T P_{i1} A_{ij} A_{ij}^T P_{i1} x_j(t)
\]

Moreover, using the following identities

\[
\sum_{i=1}^N \sum_{j=1, j \neq i}^N x_j(t - d_{ij})^T x_j(t - d_{ij}) = \sum_{i=1}^N \sum_{j=1, j \neq i}^N x_i(t - d_{ji})^T x_i(t - d_{ji}),
\]

\[
\sum_{i=1}^N \sum_{j=1, j \neq i}^N x_j(t - m_{ij})^T R_{ij} x_j(t - m_{ij}) = \sum_{i=1}^N \sum_{j=1, j \neq i}^N x_i(t - m_{ji})^T R_{ij} x_i(t - m_{ji}),
\]

\[
\|z_i(t)\|^2 \leq 2x_i^T(t) E_i^T E_i x_i(t) + 2x_i^T(t - d_{ii}) F_i^T F_i x_i(t - d_{ii}).
\]

we obtain from (7)-(9) that

\[
\dot{V}(t, x_t) - \beta V(t, x_t) \leq e^{\beta t} \sum_{i=1}^N \left[ x_i(t)^T [P_{i1} A_i + A_i^T P_{i1} + \frac{1}{\gamma \beta} P_{i1} D_i D_i^T P_{i1} + P_{i1}(B_i Y_i + Y_i^T B_i^T) P_{i1}] x_i(t) + e^{\beta t} \sum_{j=1, j \neq i}^N x_j(t)^T P_{i1} A_{ij} A_{ij}^T P_{i1} x_j(t) \right.
\]

\[
+ \sum_{j=1, j \neq i}^N x_i(t)^T P_{i1} B_{ij} Y_{j} - Y_{j}^T B_{ij}^T P_{i1} x_i(t) + (N - 1)[x_i(t)^T x_i(t) + x_i(t)^T R_{i1} x_i(t)]
\]

\[
+ \gamma \beta e^{\beta t} \sum_{i=1}^N w_i^T(t) w_i(t) - \beta e^{\beta t} \sum_{i=1}^N z_i^T(t) z_i(t)
\]

\[
\leq e^{\beta t} \sum_{i=1}^N y_i(t)^T M y_i(t) + \gamma \beta e^{\beta t} \sum_{i=1}^N w_i^T(t) w_i(t) - \beta e^{\beta t} \sum_{i=1}^N z_i^T(t) z_i(t),
\]

\[(10)\]
where \( y_i(t) = P_{i1}x_i(t) \), \( i = 1, \ldots, N \), and

\[
\mathcal{M}^i = A_iP_i + P_i^TA_i^\top + B_iY_i + \frac{1}{\gamma\beta}D_iD_i^\top + Y_i^TB_i^\top + 2\beta P_i^T E_i^\top E_iP_i + 2\beta P_i^T F_i^\top F_iP_i \\
+ (N - 1)[R_i + P_i^2] + \sum_{j=1, j\neq i}^N [A_{ij}A_{ij}^\top + B_{ij}Y_j^{-1}B_{ij}^\top].
\]

Applying Lemma 3, the condition (5) is equivalent to \( \mathcal{M}^i < 0 \), \( i = 1, N \), and hence from (10) it follows that

\[
\dot{V}(t, x_t) - \beta V(t, x_t) \leq \gamma\beta e^{\beta t} \sum_{i=1}^N w_i^\top(t)w_i(t).
\]

Moreover, since

\[
\frac{d}{dt} e^{-\beta t} V(t, x_t) = e^{-\beta t} \dot{V}(t, x_t) - \beta e^{-\beta t} V(t, x_t),
\]

we get

\[
V(t, x_t) \leq V(0, x_0) + N\gamma\beta\eta T e^{\beta T}, \quad \forall t \in [0, T]. \tag{11}
\]

Next, in the view of \( V(t, x_t) \) we have

\[
V(t, x_t) \geq \sum_{i=1}^N x_i(t)^\top P_{i1}x_i(t) \geq \sum_{i=1}^N \lambda_{\min}(P_{i1})x_i(t)^\top x_i(t) \geq \alpha_1 \sum_{i=1}^N x_i(t)^\top Q_i x_i(t) = \alpha_1 x(t)^\top Q x(t). \tag{12}
\]

On the other hand, the following estimation for \( V(0, x_0) \) can be derived

\[
V(0, x_0) = \sum_{i=1}^N x_i(0)^\top P_{i1}x_i(0) + \sum_{i=1}^N \int_{-d_i}^0 2\beta x_i(s)^\top F_i^\top F_i x_i(s) ds \\
+ \sum_{i=1}^N \sum_{j=1, j\neq i}^N \left( \int_{-d_{ij}}^0 x_i(s)^\top x_i(s) ds + \int_{-m_{ij}}^0 x_i(s)^\top R_{ij} x_i(s) ds \right) \leq \alpha_2 \sup_{s \in [-\tau, 0]} \{ \phi(s)^\top Q \phi(s) \} \leq \alpha_2 c_1. \tag{13}
\]

Therefore, combining the derived inequalities (10)-(13) with the condition (6) gives

\[
x(t)^\top Q x(t) \leq \frac{1}{\alpha_1} (V(0, x_0) + N\beta\gamma\eta T e^{\beta t} \\
\leq \frac{1}{\alpha_1} (\alpha_2 c_1 + N\gamma\beta\eta T e^{\beta t} \leq c_2, \quad \forall t \in [0, T]).
\]
To complete the proof of the theorem, it remains to show the $\gamma$–optimal level condition (4). For this, we consider the following relation

$$
\sum_{i=1}^{N} \int_{0}^{T} \beta \|z_i(t)\|^2 - \gamma \beta \|w_i(t)\|^2 dt = \int_{0}^{T} \left\{ \sum_{i=1}^{N} \left[ \beta \|z_i(t)\|^2 - \gamma \beta \|w_i(t)\|^2 \right] + \frac{d}{dt} \left[ e^{-\gamma t} V(t, x_t) \right] \right\} dt
$$

$$
- \int_{0}^{T} \frac{d}{dt} \left[ e^{-\beta t} V(t, x_t) \right] dt.
$$

Since $V(t, x_t) \geq 0$, we have

$$
- \int_{0}^{T} \frac{d}{dt} \left[ e^{-\beta t} V(t, x_t) \right] dt = -e^{-\beta T} V(T, x_T) + V(0, x_0) \leq \alpha_3 \|\varphi\|^2.
$$

It follows from (10) that

$$
\sum_{i=1}^{N} \left[ \beta \|z_i(t)\|^2 - \gamma \beta \|w_i(t)\|^2 \right] + \frac{d}{dt} \left[ e^{-\beta t} V(t, x_t) \right] < 0,
$$

hence

$$
\sum_{i=1}^{N} \left[ \int_{0}^{T} (\beta \|z_i(t)\|^2 - \gamma \beta \|w_i(t)\|^2) dt \right] \leq \alpha_3 \|\varphi\|^2.
$$

Setting $c_0 = \frac{\alpha_3}{\gamma \beta} > 0$, the above inequality yields

$$
\sup_{\varphi} \frac{\int_{0}^{T} \sum_{i=1}^{N} \|z_i(t)\|^2 dt}{c_0 \|\varphi\|^2 + \sum_{i=1}^{N} \int_{0}^{T} \|w_i(t)\|^2 dt} \leq \gamma.
$$

This condition holds for all non-zero $w_i \in L_2([0, T], R^n)$, $\varphi \in C([-\tau, 0]; R^n)$, and then the condition (4) is derived. This completes the proof of the theorem.

Next theorem gives sufficient conditions for solving the guaranteed cost finite-time control of system (1). Let us denote

$$
H_{1,1}^i = A_i P_i + P_i A_i^T + D_i D_i^T + B_i Y_i + Y_i^T B_i^T + (N-1) R_i + \sum_{j=1, j \neq i}^{N} A_{ij} A_{ij}^T,
$$

$$
H_{1,i+1}^i = -I, \quad H_{1,i+1}^i = \sqrt{N-1} P_i, \quad H_{1+1,i+1}^i = -R_j, \quad H_{1,i+1+j}^i = B_{ij} Y_j, \quad i, j = 1, N, \quad j \neq i,
$$

$$
\alpha_2 = \max_{i=1,N} \left( \frac{\lambda_{\max}(P_i)}{\lambda_{\min}(Q_i)} \right) + (N-1) \tau \left[ \max_{i=1,N} \left( \frac{\lambda_{\max}(U_i+V_i)}{\lambda_{\min}(Q_i)} \right) + \max_{i=1,N} \left( \frac{\lambda_{\max}(R_i)}{\lambda_{\min}(Q_i)} \right) \right].
$$

**Theorem 2.** For given positive numbers $T$, $c_1$, $c_2$, and symmetric matrices $Q_i > 0$, $W_i > 0$, $U_i \geq 0$, $V_i \geq 0$, $i \in 1,N$, the guaranteed cost finite-time control for system (1) is solvable if there exist symmetric matrices $P_i > 0$, $R_i > 0$, $i \in 1,N$, matrices $Y_i$, $i \in 1,N$, and a number $\beta > 0$ satisfying the following conditions
Moreover, the guaranteed cost controllers are defined by $u_i(t) = Y_i P_i^{-1} x_i(t)$, $i = 1, \ldots, N$ and the guaranteed cost value is defined by $J^* = \alpha_2 c_1 + N \eta T$.

**Proof.** We take the following Lyapunov-Krasovskii functional

$$V(t, x_t) = \sum_{i=1}^{N} \left[ V_{i1}(t, x_t) + V_{i2}(t, x_t) \right],$$

where $V_{i1}(t, x_t) = e^{\beta t} x_i(t)^T P_{i1} x_i(t)$,

$$V_{i2}(t, x_t) = e^{\beta t} \sum_{j=1, j \neq i}^{N} \left( \int_{t-d_{ji}}^{t} x_j(s)^T (I + V_t) x_i(s) ds + \int_{t-m_{ji}}^{t} x_i(s)^T R_{i1} x_i(s) ds \right).$$

By the same arguments used in the proof of Theorem 1 we have derived the following estimation:

$$\dot{V}(t, x_t) - \beta V(t, x_t) \leq e^{\beta t} \sum_{i=1}^{N} \left[ x_i(t)^T [P_{i1} A_i + A_i^T P_{i1} + P_{i1} D_i D_i^T P_{i1}] x_i(t) + P_{i1} (B_i Y_i + Y_i^T B_i^T) P_{i1} x_i(t) \right] + \sum_{j=1, j \neq i}^{N} x_i(t)^T P_{i1} A_{ij} A_{ij}^T P_{i1} x_i(t)$$

$$+ \sum_{j=1, j \neq i}^{N} x_i(t)^T P_{i1} B_{ij} Y_j R_{j}^{-1} Y_j^T B_{ij}^T P_{i1} x_i(t)$$

$$+ (N-1) x_i(t)^T [I + V_i + R_{i1}] x_i(t) + e^{\beta t} \sum_{i=1}^{N} w_i^T(t) w_i(t)$$

$$- e^{\beta t} \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} x_i(t - d_{ji})^T V_t x_i(t - d_{ji})^T.$$

Adding and substituting the value $e^{\beta t} f^0(t)$, where $f^0(t)$ is denoted by

$$f^0(t) := \sum_{i=1}^{N} \left[ x_i(t)^T U_i x_i(t) + \sum_{j=1, j \neq i}^{N} x_i(t - d_{ji})^T V_i x_i(t - d_{ji}) + u_i^T(t) W_i u_i(t) \right],$$

on the right hand-side of (16), we have

$$\dot{V}(t, x_t) - \beta V(t, x_t) \leq e^{\beta t} \sum_{i=1}^{N} y_i(t)^T M y_i(t) + e^{\beta t} \sum_{i=1}^{N} w_i^T(t) w_i(t) - e^{\beta t} f^0(t),$$

(17)
where \( y_i(t) = P_i x_i(t) \) and

\[
M^i = A_i P_i + P_i A_i^T + B_i Y_i + D_i D_i^T + P_i U_i + Y_i^T B_i^T + (N - 1) [P_i V_i + R_i + P_i^2] + Y_i^T W_i Y_i + \sum_{j=1,j \neq i}^N [A_{ij} A_{ij}^T + B_{ij} Y_j R_j^{-1} Y_j^T B_{ij}^T].
\]

Applying the Schur complement lemma, Lemma 2, and Lemma 3, the condition (14) is equivalent to \( M^i < 0, \forall i = 1, N \), and hence from the inequality (17) it follows that

\[
\dot{V}(t, x) - \beta V(t, x) \leq -e^{\beta t} f^0(t) + e^{\beta t} \sum_{i=1}^N w_i^T(t) w_i(t), \forall t \in [0, T]. \tag{18}
\]

Therefore, using the same arguments used in the proof in Theorem 1, we can prove the robust finite-time stabilization of the closed system by the feedback controllers \( u_i(t) = Y_i P_i^{-1} x_i(t) \).

To obtain the guaranteed cost value, we derive from (18) that

\[
e^{-\beta t} \dot{V}(t, x) - \beta e^{-\beta t} V(t, x) \leq -f^0(t) + \sum_{i=1}^N w_i^T(t) w_i(t). \tag{19}
\]

Integrating both sides of (19) from 0 to \( T \) leads to

\[
\int_0^T f^0(t) dt \leq V(0, x_0) - e^{-\beta t} V(t, x) + N \eta T \leq V(0, x_0) + N \eta T,
\]

due to \( V(t, x) \geq 0 \). On the other hand, we see that

\[
V(0, x_0) = \sum_{i=1}^N x_i(0)^T P_i x_i(0) + \sum_{i=1}^N \sum_{j=1,j \neq i}^N \left( \int_{-d_i}^0 x_i(s)^T [I + V_i] x_i(s) ds \right)
\]

\[
+ \int_{-m_i}^0 x_i(s)^T R_i x_i(s) ds \leq \alpha_2 c_1,
\]

hence, we obtain that

\[
J(u) \leq \alpha_2 c_1 + N \eta T = J^*.
\]

This completes the proof of the theorem.

**Remark 1.** Note that the condition (6) (or (15)) is not a LMI w.r.t. \( \beta > 0 \). Since \( \beta \) is not included in LMI (5) (or in LMI (14)), we first find solutions \( P_i, R_i, Y_i \) from the LMI (5) (or from the LMI (14)), and then define \( \beta > 0 \) from condition (6) (or from (15)).

### 4. A NUMERICAL EXAMPLE

In this section, an illustrative example is given to show the validity and effectiveness of the theoretical result.
Consider the dynamic of the mechanical machine (see, e.g. [2]) described by the large-scale complex system (1), where \( N = 3 \) and the absolute rotor angle, acceleration and angular velocity in each subsystem are respectively denoted by \( x_i = (x_{i1}, x_{i2})^\top \), \( i = 1, 2, 3 \); the observation vector \( z_i \), \( i = 1, 2, 3 \); the coefficient matrices \( A_i, A_{ij} \), the perturbation coefficient matrices \( D_i \), the time-delays \( d_{ij} \) are given by

\[
\begin{align*}
\dot{x}_1(t) &= A_1x_1(t) + A_{12}x_2(t - d_{12}) + A_{13}x_3(t - d_{13}) + B_1u_1(t) + B_{12}u_2(t - m_{12}) + B_{13}u_3(t - m_{13}) + D_1w_1(t), \\
\dot{x}_2(t) &= A_2x_2(t) + A_{21}x_1(t - d_{21}) + A_{23}x_3(t - d_{23}) + B_{21}u_1(t - m_{21}) + B_{23}u_3(t - m_{23}) + D_2w_2(t), \\
\dot{x}_3(t) &= A_3x_3(t) + A_{31}x_1(t - d_{31}) + A_{32}x_2(t - d_{32}) + B_{31}u_1(t - m_{31}) + B_{32}u_2(t - m_{32}) + D_3w_3(t), \\
\end{align*}
\]

\[
\begin{align*}
z_1(t) &= E_1x_1(t) + F_1x_1(t - d_{11}), \\
z_2(t) &= E_2x_2(t) + F_2x_2(t - d_{22}), \\
z_3(t) &= E_3x_3(t) + F_3x_3(t - d_{33}),
\end{align*}
\]

\[
x_1(t) = \phi_1(t), \quad u_1(t) = \phi_1(t), \quad t \in [-\tau, 0],
\]

\[
x_2(t) = \phi_2(t), \quad u_2(t) = \phi_2(t), \quad t \in [-\tau, 0],
\]

\[
x_3(t) = \phi_3(t), \quad u_3(t) = \phi_3(t), \quad t \in [-\tau, 0],
\]

and \( d_{12} = 0.4; d_{13} = 0.09; d_{21} = 0.06; d_{23} = 0.2; d_{31} = 0.3; d_{32} = 0.04; m_{12} = 0.42; m_{13} = 0.5; m_{21} = 0.28; m_{23} = 0.37; m_{31} = 0.45; m_{32} = 0.1; \tau = 0.5. \)

\[
A_1 = \begin{bmatrix}
-2 & 0.1 \\
-0.1 & -4
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
-3 & 0 \\
-1 & 1
\end{bmatrix}, \quad A_3 = \begin{bmatrix}
-5 & 0.1 \\
0 & 1
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
1 & 0 \\
2 & 0.1
\end{bmatrix},
\]

\[
B_2 = \begin{bmatrix}
-1 & 1 \\
1.9 & -5
\end{bmatrix}, \quad B_3 = \begin{bmatrix}
2 & 0.1 \\
0.4 & -2
\end{bmatrix}, \quad A_{12} = \begin{bmatrix}
0.2 & 1 \\
0.1 & 0
\end{bmatrix}, \quad A_{13} = \begin{bmatrix}
0.5 & 1 \\
0 & 3
\end{bmatrix},
\]

\[
A_{21} = \begin{bmatrix}
0.5 & 0 \\
-0.1 & 0.5
\end{bmatrix}, \quad A_{23} = \begin{bmatrix}
0.4 & -0.1 \\
0 & 1
\end{bmatrix}, \quad A_{31} = \begin{bmatrix}
-0.02 & -1 \\
0.1 & -0.2
\end{bmatrix},
\]

\[
A_{32} = \begin{bmatrix}
1 & 0.4 \\
0 & -2
\end{bmatrix}, \quad B_{12} = \begin{bmatrix}
-1 & 0 \\
0.1 & 3
\end{bmatrix}, \quad B_{13} = \begin{bmatrix}
-2 & 0 \\
1 & 0.2
\end{bmatrix}, \quad B_{21} = \begin{bmatrix}
-1 & 0.1 \\
0 & -1
\end{bmatrix},
\]

\[
B_{23} = \begin{bmatrix}
-1 & 0 \\
-0.5 & 0
\end{bmatrix}, \quad B_{31} = \begin{bmatrix}
-2 & 0 \\
0.2 & -3
\end{bmatrix}, \quad B_{32} = \begin{bmatrix}
1 & 0 \\
-1 & 3
\end{bmatrix}, \quad D_1 = \begin{bmatrix}
0.1 & 0 \\
0 & 0.2
\end{bmatrix},
\]

\[
D_2 = \begin{bmatrix}
0.05 & 0.1 \\
0.1 & 0.01
\end{bmatrix}, \quad D_3 = \begin{bmatrix}
-0.1 & 0.1 \\
0.2 & -0.2
\end{bmatrix}, \quad E_1 = \begin{bmatrix}
0.1 & 0.2 \\
0.1 & 0.1
\end{bmatrix}, \quad E_2 = \begin{bmatrix}
1 & 0.2 \\
0 & -0.1
\end{bmatrix},
\]

\[
E_3 = \begin{bmatrix}
-1 & 0 \\
-1 & -2
\end{bmatrix}, \quad F_1 = \begin{bmatrix}
0.2 & 0 \\
0 & 0.1
\end{bmatrix}, \quad F_2 = \begin{bmatrix}
1 & 0 \\
0 & -0.1
\end{bmatrix}, \quad F_3 = \begin{bmatrix}
-0.1 & 0 \\
-0.5 & 0.5
\end{bmatrix},
\]

and the matrices

\[
Q = diag(Q_1, Q_2, Q_3), \quad Q_1 = \begin{bmatrix}
0.5 & 0 \\
0 & 0.5
\end{bmatrix},
\]
\[ Q_2 = \begin{bmatrix} 0.64 & 0 \\ 0 & 0.64 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 0.42 & 0 \\ 0 & 0.42 \end{bmatrix}. \]

For \( \eta = 0.1; \ \gamma = 4, \ \beta = 0.01, \ c_1 = 1, \ c_2 = 115, \ T = 10, \) using the LMI algorithm in Matlab [15] to find solutions of the LMI (5), we have

\[
\begin{align*}
P_1 &= \begin{bmatrix} 0.9327 & 0.0107 \\ 0.0107 & 1.8988 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1.5631 & 0.0416 \\ 0.0416 & 0.2447 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 2.3185 & -0.0769 \\ -0.0769 & 1.7110 \end{bmatrix}, \\
R_1 &= \begin{bmatrix} 0.6955 & 1.0743 \\ 1.0743 & 2.2041 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 1.1333 & -1.0559 \\ -1.0559 & 3.1772 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 0.4498 & 0.4507 \\ 0.4507 & 60.3971 \end{bmatrix}, \\
Y_1 &= \begin{bmatrix} -0.9953 & -2.0142 \\ -2.0142 & -0.5320 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0.4243 & -1.1587 \\ -1.1587 & 0.4800 \end{bmatrix}, \quad Y_3 = \begin{bmatrix} -0.0363 & -2.2345 \\ -2.2345 & 37.5152 \end{bmatrix}.
\]

By Theorem 1, the system is robust finite-time \( H_\infty \) stabilizable, and the state feedback control \( u_i(t) = Y_i P_i^{-1} x_i(t) \) are given by

\[
\begin{align*}
u_1(t) &= \begin{bmatrix} -1.0550 \\ -0.2449 \\ -0.0590 \\ 0.3992 \\ -0.0590 \\ 0.8708 \end{bmatrix} x_1(t), \\
u_2(t) &= \begin{bmatrix} -1.0548 \\ -0.2788 \\ -4.8024 \\ -1.5861 \\ -1.3086 \\ 21.9651 \end{bmatrix} x_2(t), \\
u_3(t) &= \begin{bmatrix} -0.0769 \\ 0.4507 \\ -0.2314 \\ -0.1585 \\ -2.2345 \\ 0.4507 \end{bmatrix} x_3(t).
\]

For the guaranteed cost control, we take \( c_1 = 1, \ c_2 = 83, \ T = 10 \) and the cost matrices

\[
\begin{align*}
V_1 &= \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.04 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 0.04 & 0 \\ 0 & 0.04 \end{bmatrix}, \\
U_1 &= \begin{bmatrix} 0.01 & 0 \\ 0 & 0.0225 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0.04 & 0 \\ 0 & 0.01 \end{bmatrix}, \quad U_3 = \begin{bmatrix} 0.04 & 0 \\ 0 & 0.0625 \end{bmatrix},
\end{align*}
\]
\[
W_1 = \begin{bmatrix} 0.0225 & 0 \\ 0 & 0.0225 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}, \quad W_3 = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.04 \end{bmatrix},
\]

the solutions of the inequalities (14), (15) are defined as

\[
P_1 = \begin{bmatrix} 0.9743 & 0.0060 \\ 0.0060 & 1.9313 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1.5824 & -0.0107 \\ -0.0107 & 0.3980 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 2.3362 & -0.0215 \\ -0.0215 & 1.2289 \end{bmatrix},
\]

\[
R_1 = \begin{bmatrix} 0.7025 & 0.9511 \\ 0.9511 & 1.9673 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 1.5972 & -1.6245 \\ -1.6245 & 5.5039 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 1.2160 & 0.0388 \\ 0.0388 & 14.3699 \end{bmatrix},
\]

\[
Y_1 = \begin{bmatrix} -0.8592 & -1.7635 \\ -1.222 & -0.3062 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0.6132 & -1.7502 \\ -1.7502 & 0.8320 \end{bmatrix}, \quad Y_3 = \begin{bmatrix} -0.0447 & -0.6375 \\ -0.6375 & 14.6352 \end{bmatrix}.
\]

By Theorem 2, the guaranteed cost control problem is solvable and the guaranteed cost value is \( J^* = 66.7 \), the guaranteed cost controllers are

\[
\begin{align*}
  u_1(t) &= \begin{bmatrix} -0.8763 & -0.9104 \\ -0.1245 & -0.1582 \end{bmatrix} x_1(t), \\
  u_2(t) &= \begin{bmatrix} 0.3578 & -4.3874 \\ -0.1566 & 2.0862 \end{bmatrix} x_2(t), \\
  u_3(t) &= \begin{bmatrix} -0.0239 & -0.5192 \\ -0.1077 & 11.9073 \end{bmatrix} x_3(t).
\end{align*}
\]

Figure 1 shows the time history \( x^\top(t)Rx(t) \) of the system with the initial conditions \( \varphi_1(t) = (0.5 \sin t, 0.5); \ \varphi_2(t) = (e^t, 0.02t); \ \varphi_3(t) = (0.01e^t, 0.1 \cos t), \ t \in (-0.5; 0) \) and \( c_1 = 1, \ c_2 = 83. \)

5. CONCLUSIONS

In this paper, we have studied the problem of supoptimal finite-time control for linear large-scale systems with the state and control delays in interconnection. Exploiting the Lyapunov function method and linear matrix inequality technique, we have proposed new LMI conditions for the robust \( H_\infty \) finite-time control and guaranteed cost finite-time control of such systems. The conditions are presented in terms of tractable LMIs, which can be solved by standard computational LMI toolbox algorithms. The validity and effectiveness of the proposed results have been illustrated by a numerical example.

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REFERENCES


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