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NUMMERICAL MODEL OF NON HYDROSTATIC VERTICAL BI-DIMENSIONAL FLOW

TRAN GIA LICH, PHAN NGOC VINH

Abstract. A splitting technique in the x and z directions to solve Navier–Stokes equations for simulation of non-hydrostatic vertical bi-dimensional flows is presented. The finite-difference method in a Cartesian coordinate system and an UPWIND scheme for the convection terms are applied in the model. Several test cases are done to validate the model on the aspects of the qualitative property. Calculated results have been compared to the analytical solution of problem in a special case.

Tóm tắt. Bài báo trình bày một kỹ thuật phân rã theo các phương x và z để giải hệ phương trình Navier–Stockes mô phỏng bài toán dòng chảy 2 chiều đứng phi thủy tĩnh. Phương pháp sai phân hữu hạn trong hệ tọa độ Đê cac và sơ đồ ngược dòng đối với thành phần tải được dùng trong mô hình. Các tác giả đã thực hiện một vài phương án tính kiểm tra mô hình về mặt định tính, so sánh kết quả tính toán từ mô hình với nghiệm giải tích của bài toán trong một trường hợp đặc biệt.

1. MATHEMATICAL MODEL

1.1. Governing equations

It is well known that the Navier–Stokes equations describing bi-dimensional vertical flow for the viscous incompressible fluid consist of 2 momentum equations as follows (see [1, 2, 3, 7, 10]):

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + w\dfrac{\partial u}{\partial z} + \dfrac{1}{\rho}\dfrac{\partial p}{\partial x} = v\Delta u \\ \dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + w\dfrac{\partial w}{\partial z} + \dfrac{1}{\rho}\dfrac{\partial p}{\partial z} = v\Delta w \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = 0 \end{cases} \tag{1.1}$$

The system of equations (1.1) with the initial condition: $U_1(x,z,0) = U_1^0(x,z)$ and the boundary conditions on the boundary ∂G of the considered region G:

$$U_1(x,z,t)\big|_{\partial G} = 0,$$

where $U_1 = (u, w)$ has a unique solution in the space of generalized functions (see [1], [7]). The equation system with the gravitational force g can be written as:

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + w\dfrac{\partial u}{\partial z} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \dfrac{\partial}{\partial x}\Big(v\dfrac{\partial u}{\partial x}\Big) + \dfrac{\partial}{\partial z}\Big(\mu\dfrac{\partial u}{\partial z}\Big) \\ \dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + w\dfrac{\partial w}{\partial z} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} - g + \dfrac{\partial}{\partial x}\Big(v\dfrac{\partial w}{\partial x}\Big) + \dfrac{\partial}{\partial z}\Big(\mu\dfrac{\partial w}{\partial z}\Big) \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = 0 \end{cases} \tag{1.2}$$

Place the coordinate origin to be at the mean water level. The x axis is horizontal and the z axis is taken positive upwards.

Replace the variable p by P, with $P = p - \rho g(\xi - z)$ and suppose that density of the fluid does not vary in time and space, i.e. $\rho = \text{const}$, say, $\rho = 1$. The system of equations (1.2) now becomes:

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + w\dfrac{\partial u}{\partial z} = -\dfrac{1}{\rho}\dfrac{\partial P}{\partial x} - g\dfrac{\partial \xi}{\partial x} + \dfrac{\partial}{\partial x}\Big(v\dfrac{\partial u}{\partial x}\Big) + \dfrac{\partial}{\partial z}\Big(\mu\dfrac{\partial u}{\partial z}\Big) \\ \dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + w\dfrac{\partial w}{\partial z} = -\dfrac{1}{\rho}\dfrac{\partial P}{\partial z} + \dfrac{\partial}{\partial x}\Big(v\dfrac{\partial w}{\partial x}\Big) + \dfrac{\partial}{\partial z}\Big(\mu\dfrac{\partial w}{\partial z}\Big) \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = 0 \end{cases} \tag{1.3}$$

In this case the system of equations has the same structure as (1.1).

1.2. Initial conditions

$$u(x,z,0) = u^0(x,z), \quad w(x,z,0) = w^0(x,z), \quad p(x,z,0) = p^0(x,z), \tag{1.4}$$

where, $u^0(x,z)$, $w^0(x,z)$, $p^0(x,z)$ are known functions.

1.3. Boundary conditions

In addition to the initial conditions, the solution of the equation system (1.3) requires the following boundary conditions (see [6]):

At the water surface:

$$\begin{aligned} &\frac{\partial \xi}{\partial t} + u_s\frac{\partial \xi}{\partial x} = w_s, \\ &\rho\mu\Big(\frac{\partial u}{\partial z}\Big)_{z=\xi} = \tau_s, \quad \tau_s = \rho_a C_d |W| W. \end{aligned} \tag{1.5}$$

At the bottom:

$$\begin{aligned} &w_b + u_b\frac{\partial h}{\partial x} = 0, \\ &\rho v\Big(\frac{\partial u}{\partial z}\Big)_{z=-h} = \tau_b, \quad \tau_b = \rho g\frac{|u|}{Ch^2}u. \end{aligned} \tag{1.6}$$

At the open boundary:

The normal velocity component at the land boundary is null, that means:

$$\frac{\partial u}{\partial n} = 0. \tag{1.7}$$

At the solid boundary:

Commonly, the water level is given at all times, if water level is unknown at the boundary, the radiation condition is applied. For the outflow, the flux can be calculated from the model and for the inflow, the flux is either given or calculated from the radiation condition.

2. ALGORITHM AND NUMERICAL BACKGROUND

It is difficult to directly find pressure P in system of equations (1.3). In the case of quasi-stationary flows, to overcome this, according to the authors Yanenko N. N. and Belotserkovsky O. M., the artificial compression component is added to the continuity equation and the modified Navier–Stokes equations are obtained (see [2, 7, 10])

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + w\dfrac{\partial u}{\partial z} = -\dfrac{1}{\rho}\dfrac{\partial P}{\partial x} - g\dfrac{\partial \xi}{\partial x} + \dfrac{\partial}{\partial x}\Big(v\dfrac{\partial u}{\partial x}\Big) + \dfrac{\partial}{\partial z}\Big(\mu\dfrac{\partial u}{\partial z}\Big) \\ \dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + w\dfrac{\partial w}{\partial z} = -\dfrac{1}{\rho}\dfrac{\partial P}{\partial z} + \dfrac{\partial}{\partial x}\Big(v\dfrac{\partial w}{\partial x}\Big) + \dfrac{\partial}{\partial z}\Big(\mu\dfrac{\partial w}{\partial z}\Big) \\ \dfrac{\partial\Big(P + \rho\frac{u^2+w^2}{4}\Big)}{\partial t} + \dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = 0 \end{cases} \tag{2.1}$$

The system of equations (2.1) is solved by the splitting method. In other words, each one-dimensional equation system in the x and z directions are successively solved as follows (see [8, 10])

$$\begin{cases} \dfrac{1}{2}\dfrac{\partial u}{\partial t} + \dfrac{\partial q_1}{\partial x} = -g\dfrac{\partial \xi}{\partial x} + \dfrac{\partial}{\partial x}\Big(v\dfrac{\partial u}{\partial x}\Big) \\ \dfrac{1}{2}\dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} = \dfrac{\partial}{\partial x}\Big(v\dfrac{\partial w}{\partial x}\Big) \\ \dfrac{1}{2}\dfrac{\partial q_1}{\partial t} + \dfrac{\partial u}{\partial x} = 0 \end{cases} \tag{2.2}$$

$$\begin{cases} \dfrac{1}{2}\dfrac{\partial u}{\partial t} + w\dfrac{\partial u}{\partial z} = \dfrac{\partial}{\partial z}\Big(\mu\dfrac{\partial u}{\partial z}\Big) \\ \dfrac{1}{2}\dfrac{\partial w}{\partial t} + \dfrac{\partial q_2}{\partial z} = \dfrac{\partial}{\partial z}\Big(\mu\dfrac{\partial w}{\partial z}\Big) \\ \dfrac{1}{2}\dfrac{\partial q_2}{\partial t} + \dfrac{\partial w}{\partial z} = 0 \end{cases} \tag{2.3}$$

where, $q_1 = P + \rho\dfrac{u^2}{2}, \quad q_2 = P + \rho\dfrac{w^2}{2}$.

Assuming that, the solutions of (2.1) at the time step t_k are known. In order to find its solutions at the next time step $t_{k+1} = t_k + dt$, two main calculation steps are carried out as follows:

a. Step 1: To specify the surface water elevation $\xi(x,t)$.

Integrating the continuity equation of (1.3) in the z direction from the bottom to the water surface, we obtain:

$$\int_{-h}^{\xi} \frac{\partial u}{\partial x} dz + \int_{-h}^{\xi} \frac{\partial w}{\partial z} dz = 0 \tag{2.4}$$

According to Leibniz's formula, it follows from (2.4):

$$w_\xi - w_h = -\int_{-h}^{\xi} \frac{\partial u}{\partial x} dz = -\frac{\partial}{\partial x}\int_{-h}^{\xi} u dz + u_\xi\frac{\partial \xi}{\partial x} + u_h\frac{\partial h}{\partial x}.$$

Using the conditions at the bottom and the water surface, we obtain:

$$\frac{\partial \xi}{\partial t} + \frac{\partial H\overline{u}}{\partial x} = 0, \tag{2.5}$$

where $\overline{u} = \dfrac{1}{H}\displaystyle\int_{-h}^{\xi} u dz$ is the depth-averaged velocity.

For ensuring the stability of numerical results, the viscosity artificial term is added into equation (2.5) (see [4]) and one gets:

$$\frac{\partial \xi}{\partial t} + \overline{u}\frac{\partial H}{\partial x} + H\frac{\partial \overline{u}}{\partial x} = \varepsilon dt\frac{\overline{u}^2}{2}\frac{\partial^2 H}{\partial x^2}, \quad 0 < \varepsilon \le 1. \tag{2.6}$$

And, there are two ways to specify the surface water elevation $\xi(x,t)$:

$*$ For equation (2.5), an explicit finite-difference scheme is used:

$$\frac{\xi_n^{k+1} - \xi_n^k}{dt} + \frac{(H\overline{u})_{n+1}^k - (H\overline{u})_{n-1}^k}{2dx} = 0. \tag{2.7}$$

$*$ For equation (2.6), an implicit finite-difference scheme is used:

$$\frac{\xi_n^{k+1}-\xi_n^k}{dt}+\frac{(u+|u|)_{n,m}^k}{2}\frac{(H)_n^{k+1}-(H)_{n-1}^{k+1}}{dx}+\frac{(u-|u|)_{n,m}^k}{2}\frac{(H)_{n+1}^{k+1}-(H)_n^{k+1}}{dx}$$
$$+H_n^k\frac{\overline{u}_{n+1}^k-\overline{u}_{n-1}^k}{2dx}-\varepsilon dt\frac{(\overline{u}_n^k)^2}{2}\frac{H_{n+1,m}^{k+1}-2H_{n,m}^{k+1}+H_{n-1,m}^{k+1}}{dx^2}=0 \tag{2.8}$$

or in the reduced form:

$$a_nH_{n-1}^{k+1}+b_nH_n^{k+1}+c_nH_{n+1}^{k+1}=d_n\,. \tag{2.9}$$

b. Step 2: Calculation of the velocity field (u,w) and the pressure P.

Discretizing the equation system (2.2) by an implicit finite-difference scheme in the x direction, and an UPWIND scheme is used for advection terms:

$$\frac{1}{2}\frac{u_{n,m}^{k+1/2}-u_{n,m}^k}{dt/2}+\frac{(q_1)_{n+1,m}^{k+1/2}-(q_1)_{n,m}^{k+1/2}}{dx}=v\frac{u_{n+1,m}^{k+1/2}-2u_{n,m}^{k+1/2}+u_{n-1,m}^{k+1/2}}{dx^2}$$
$$-g\frac{\xi_{n+1,m}^{k+1}-\xi_{n-1,m}^{k+1}}{2dx}, \tag{2.10}$$

$$\frac{1}{2}\frac{w_{n,m}^{k+1/2}-w_{n,m}^k}{dt/2}+\frac{(u+|u|)_{n,m}^k}{2}\frac{w_{n,m}^{k+1/2}-w_{n-1,m}^{k+1/2}}{dx}+\frac{(u-|u|)_{n,m}^k}{2}\frac{w_{n+1,m}^{k+1/2}-w_{n,m}^{k+1/2}}{dx}$$
$$=v\frac{w_{n+1,m}^{k+1/2}-2w_{n,m}^{k+1/2}+w_{n-1,m}^{k+1/2}}{dx^2}, \tag{2.11}$$

$$\frac{(q_1)_{n,m}^{k+1/2}-(q_1)_{n,m}^k}{dt/2}+\frac{u_{n,m}^{k+1/2}-u_{n-1,m}^{k+1/2}}{dx}=0. \tag{2.12}$$

Replacing $(q_1)_{n,m}^{k+1/2}$ resulted from (2.12) into (2.10), we have:

$$\alpha_nu_{n-1,m}^{k+1/2}+\beta_nu_{n,m}^{k+1/2}+\gamma_nu_{n+1,m}^{k+1/2}=\delta_n\,, \tag{2.13}$$

where:

$$\begin{cases}\alpha_n=-\dfrac{dt}{dx^2}-\dfrac{v}{dx^2}\\ \beta_n=\dfrac{1}{dt}+\dfrac{2dt}{dx^2}+\dfrac{2v}{dx^2}\\ \gamma_n=-\dfrac{dt}{dx^2}-\dfrac{v}{dx^2}\\ \delta_n=\dfrac{u_{n,m}^k}{dt}-\dfrac{(q_1)_{n+1,m}^k-(q_1)_{n,m}^k}{dx}-g\dfrac{\xi_{n+1,m}^{k+1}-\xi_{n-1,m}^{k+1}}{2dx}\end{cases}$$

Also, equation (2.11) has the following reduced form:

$$\alpha_n^*w_{n-1,m}^{k+1/2}+\beta_n^*w_{n,m}^{k+1/2}+\gamma_n^*w_{n+1,m}^{k+1/2}=\delta_n^*\,, \tag{2.14}$$

where,

$$\begin{cases}\alpha_n^*=-\dfrac{(u+|u|)_{n,m}^k}{2dx}-\dfrac{v}{dx^2}\\ \beta_n^*=\dfrac{1}{dt}+\dfrac{|u|_{n,m}^k}{dx}+\dfrac{2v}{dx^2}\\ \gamma_n^*=\dfrac{(u-|u|)_{n,m}^k}{2dx}-\dfrac{v}{dx^2}\\ \delta_n^*=\dfrac{w_{n,m}^k}{dt}\end{cases}$$

The value of $(q_1)_{n,m}^{k+1/2}$ can be drawn either from equation (2.12):

$$(q_1)_{n,m}^{k+1/2} = (q_1)_{n,m}^{k} - \frac{dt}{dx}(u_{n,m}^{k+1/2} - u_{n-1,m}^{k+1/2})$$

or from equation (2.10):

$$(q_1)_{n+1,m}^{k+1/2} = (q_1)_{n,m}^{k+1/2} - \frac{dt}{dx}(u_{n,m}^{k+1/2} - u_{n,m}^{k}) + v\frac{u_{n+1,m}^{k+1/2} - 2u_{n,m}^{k+1/2} + u_{n-1,m}^{k+1/2}}{dx^2}$$

In the same way, from equation (2.3), one has:

$$\frac{1}{2}\frac{u_{n,m}^{k+1} - u_{n,m}^{k+1/2}}{dt/2} + \frac{(w+|w|)_{n,m}^{k+1/2}}{2}\frac{wu_{n,m}^{k+1} - u_{n,m-1}^{k+1}}{dz} + \frac{(w-|w|)_{n,m}^{k+1/2}}{2}\frac{u_{n,m+1}^{k+1} - u_{n,m}^{k+1}}{dz} = \mu\frac{u_{n,m+1}^{k+1} - 2u_{n,m}^{k+1} + u_{n,m-1}^{k+1}}{dz^2}\,, \tag{2.15}$$

$$\frac{1}{2}\frac{w_{n,m}^{k+1} - w_{n,m}^{k+1/2}}{dt/2} + \frac{(q_2)_{n,m+1}^{k+1} - (q_2)_{n,m}^{k+1}}{dz} = \mu\frac{w_{n,m+1}^{k+1} - 2w_{n,m}^{k+1} + w_{n,m-1}^{k+1}}{dz^2}\,, \tag{2.16}$$

$$\frac{(q_2)_{n,m}^{k+1} - (q_2)_{n,m}^{k+1/2}}{dt/2} + \frac{w_{n,m}^{k+1} - w_{n,m-1}^{k+1}}{dz} = 0. \tag{2.17}$$

From equation (2.15) and (2.17), one also has:

$$\alpha_m w_{n,m-1}^{k+1} + \beta_m w_{n,m}^{k+1} + \gamma_m w_{n,m+1}^{k+1} = \delta_m\,, \tag{2.18}$$

where,

$$\begin{cases} \alpha_m = -\dfrac{dt}{dz^2} - \dfrac{\mu}{dz^2} \\ \beta_m = \dfrac{1}{dt} + \dfrac{2dt}{dz^2} + \dfrac{2\mu}{dz^2} \\ \gamma_m = -\dfrac{dt}{dz^2} - \dfrac{\mu}{dz^2} \\ \delta_m = \dfrac{w_{n,m}^{k+1/2}}{dt} - \dfrac{(q_2)_{n,m+1}^{k+1/2} - (q_2)_{n,m}^{k+1/2}}{dz} \end{cases}$$

and

$$(q_2)_{n,m}^{k+1/2} = (P)_{n,m}^{k+1/2} + \frac{\left(w_{n,m}^{k+1/2}\right)^2}{2} = (q_1)_{n,m}^{k+1/2} - \frac{\left(u_{n,m}^{k+1/2}\right)^2}{2} + \frac{\left(w_{n,m}^{k+1/2}\right)^2}{2}\,.$$

Also, from equation (2.16), the followings are obtained:

$$\alpha_m^* u_{n,m-1}^{k+1} + \beta_m^* u_{n,m}^{k+1} + \gamma_m^* u_{n,m+1}^{k+1} = \delta_m^*\,, \tag{2.19}$$

where,

$$\begin{cases} \alpha_m^* = -\dfrac{(w+|w|)_{n,m}^{k+1/2}}{2dz} - \dfrac{\mu}{dz^2} \\ \beta_n^* = \dfrac{1}{dt} + \dfrac{|w|_{n,m}^{k+1/2}}{dz} + \dfrac{2\mu}{dz^2} \\ \gamma_n^* = \dfrac{(w-|w|)_{n,m}^{k+1/2}}{2dz} - \dfrac{\mu}{dz^2} \\ \delta_m^* = \dfrac{u_{n,m}^{k+1/2}}{dt} \end{cases}$$

The value of $(q_2)^{k+1}_{n,m}$ can be calculated as follows:

$$(q_2)^{k+1}_{n,m} = (q_2)^{k+1/2}_{n,m} - \frac{dt}{dx}(w^{k+1}_{n,m} - u^{k+1}_{n,m-1})$$

or:

$$(q_2)^{k+1}_{n,m+1} = (q_2)^{k+1}_{n,m} - \frac{dz}{dt}(w^{k+1}_{n,m} - w^{k+1/2}_{n,m}) + \mu\frac{w^{k+1}_{n,m+1} - 2w^{k+1}_{n,m} + w^{k+1}_{n,m-1}}{dz^2}.$$

Equations (2.9), (2.13), (2.14), (2.18) and (2.19) can be solved by the double sweep method (see 4]). The coefficients of these equations satisfy the condition on prominent 3-diagonal matrix: i.e. $|\beta_n| \succ |\alpha_n| + |\gamma_n|$, so the linear algebraic equation systems have the unique solution and do not accumulate a numerical error.

Determining solution values at the boundaries:

For calculating the solution values at the boundary we use the following complementary equations (see [8, 9]):

At the left boundary:

$$\left(\frac{du}{dt}\right)_{x_2} - \left(\frac{dq_1}{dt}\right)_{x_2} = 2v\frac{\partial^2 u}{\partial x^2}. \tag{2.20}$$

If $u \leq 0$, one more equation is added:

$$\left(\frac{dw}{dt}\right)_{x_3} = 2v\frac{\partial^2 w}{\partial x^2}. \tag{2.21}$$

At the right boundary:

$$\left(\frac{du}{dt}\right)_{x_1} + \left(\frac{dq_1}{dt}\right)_{x_1} = 2v\frac{\partial^2 u}{\partial x^2}. \tag{2.22}$$

If $u \geq 0$, one more equation (2.21) is added.

Where, $\left(\frac{df}{dt}\right)_{x_i} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\lambda_i$ $(i = 1,\ 2,\ 3)$, $\lambda_1 = 2$, $\lambda_2 = -2$ and $\lambda_3 = 2u$.

At the bottom:

$$\left(\frac{dw}{dt}\right)_{z_2} - \left(\frac{dq_2}{dt}\right)_{z_2} = 2\mu\frac{\partial^2 w}{\partial z^2}. \tag{2.21}$$

At the water surface:

$$\left(\frac{dw}{dt}\right)_{z_1} + \left(\frac{dq_2}{dt}\right)_{z_1} = 2\mu\frac{\partial^2 w}{\partial z^2}. \tag{2.22}$$

Where, $\left(\frac{df}{dt}\right)_{z_i} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial z}\sigma_i$ $(i = 1,\ 2,\ 3)$, $\sigma_1 = 2$, $\sigma_2 = -2$ and $\sigma_3 = 2w$, and f represents one of the following functions: u, w, q_1 *or* q_2.

3. NUMERICAL EXPERIMENTS AND DISCUSSION OF THE RESULTS

In [8], the steady pressure field on the valves of a sluice with the free surface being given has been calculated. In this work, the following test cases are carried out for validation of the model on the aspects of qualitative property and calculated results have been compared to the analytical solution of problem in a special case.

3.1. Test cases on qualitative property

1. Computed domain is a 2 open-end rectangular canal with a constant depth $h = 14$ m. Constant water level is given at 2 open boundaries. Fig. 1 shows that, in the vertical direction Z, velocity profile at the bottom layer varies under the Logarithmic law.

2. Computed domain is a close rectangular canal with a constant depth $h = 14\,\mathrm{cm}$. Wind strength $W = 10\,\mathrm{m/s}$ is taken. Fig. 2 shows the conservation of water mass in the canal. In other words, at the surface layer, flow direction is the wind's. On the contrary, at bed layer, flow has to have the opposite direction.

3. Computed domain is a 2 open-end rectangular canal with a varied depth. The first part of the domain has a constant depth of 5 m, next, depth changes sharply and increase to 14 m and then remains a constant depth of 5 m as long as the end of the rectangular. Constant discharge is given at the open boundary on the left, and at the other one constant water level is given. An eddy flow is detected just after the terrace where the depth changes sharply (see Fig. 3).

Fig. 1. Velocity profile for computational case 1

Fig. 2. Velocity profile for computational case 2

Fig. 3. Velocity profile for computational case 3

4. Computed domain is a 2 open-end rectangular canal with a varied depth. The first part of the domain has a constant depth of 5 m, depth then changes sharply and increase up to 14 m and remains that depth as long as the end of the rectangular. A constant discharge is given at the open

boundary on the left, and at the other one a constant water level is also given. Velocity field at the bottom layer and just after the terrace where the depth change sharply has a direction opposite to mean velocity's (see Fig. 4).

5. Computed domain is a 2 open-end rectangular canal with a gradually-varied depth. Constant discharge is given at open boundary on the left, and at the one constant water level is given. Velocity profile is indicated in Fig. 5, an eddy flow is also found at the deepest part of the rectangular.

Fig. 4. Velocity profile for computational case 4

Fig. 5. Velocity profile for computational case 5

Fig. 6. Velocity profile for computational case 6

For all above-mentioned computational cases, length of the canal is $L = 1500\,\mathrm{m}$. The domain is covered by a grid of 31×15. Time step $dt = 2\,\mathrm{s}$, coefficient of Chezy $= 40$ are taken. Numerical results reach their stable values after around 1.5 h of calculation.

As an example, the calculated water level along the canal is shown separately in Fig. 6 for the computational case 5, whereas, for the rest cases, the calculated water level are indicated in the very same figures (1 - 4) of velocity profile. In Fig. 6, for purpose of tracing the water level only, their values are multiplied to 100 so that its change along the canal can be observed more obviously. Additionally, the dashed line just below the water level one is the mean water level.

3.2. Test case for analytical solution

Computed domain is a close rectangular canal with a constant depth $h = 20\,\mathrm{m}$, a length $L = 95\,\mathrm{km}$. With constant wind stress τ_0, at the central point of the canal, velocity profile in the vertical direction has the following form (see [5]):

$$\begin{aligned} u &= \frac{3H\tau_0}{4v}\Big(1 - \frac{z}{H}\Big)\Big(\frac{1}{3} - \frac{z}{H}\Big), \\ v &= 0, \end{aligned}$$

where z is distance from the water surface to considered site.

Comparison of calculated results and the analytical solution of problem shows in Fig. 7.

Fig. 7. Comparison of calculatred result with analitical solution

4. CONCLUSIONS

A simple algorithm has been presented and applied for test cases on the qualitative properties as well as on the analytical solution with the different calculated domains.

Calculated results are in agreement with the analytical solution of problem in a special case.

The model should be developed to 3D for application to practical domains with complicated depth.

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NOMENCLATURE

x, z: Cartesian coordinates in the longitudinal and the vertical directions, respectively, [m]
t: time variable, [s]
u, w: velocity components in the x and z directions, respectively, [m/s]
p: fluid pressure, [N/m^2]
ρ: density of fluid, [KGB/m^3]
g: acceleration of gravity, [m/s^2]
v, μ: horizontal and vertical turbulent viscosity coefficients, [m^2/s]

Laplace's operator: $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial z^2}$
ξ: water surface elevation, [m]
h: bottom depth, [m]
$H = h + \xi$: flow depth, [m]
τ_s, τ_b: wind stress at the water surface and bed shear stress [N/m^2]
ρ_a: air density ($\approx 1.3\,\mathrm{kg/m^3}$)
C_d: empirical coefficient ($= 2.5 \times 10^{-3}$)
W: wind strength in the x direction at the height of 10 m above the ground, [m/s]
Ch: Chezy coefficient, [m$^{1/2}$/s]
u_s, w_s: velocity components the water surface [m/s]
u_b, w_b: velocity components at the bottom, [m/s]
$\vec{n}$: the normal vector to the considered boundary
k: the superscript referring to the time step
n, m: the subscripts referring to the space step

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Tran Gia Lich - Institute of Mathematics, Hanoi, Vietnam.
Phan Ngoc Vinh - Institute of Mechanics, Hanoi, Vietnam.