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FUZZY FUNCTIONAL DEPENDENCIES WITH LINGUISTIC QUANTIFIERS

HO THUAN, TRAN THIEN THANH

Abstract. In this paper, we give a definition of Fuzzy Functional Dependency with Linguistic Quantifiers (QFFD) on the fuzzy relational database models. QFFD is an extension of Fuzzy Functional Dependency (FFD) by substituste quantifier *for all* by linguistic quantifiers such as *most, a few, at least half...* In order to extend, we give a compatibility degree of a tuple with a fuzzy functional dependency based on Fuzzy Implication Operators (FIOs). In this work, a system of axioms for QFFDs similar to Armstrong's axioms is presented. The soundness of these axioms is proved. In order to testing membership in closures of set of QFFDs, we give some algorithms in polynomial time for computing minimum family of deppendency sets. The correctness of these algorithms is proved.

Tóm tắt. Trong bài báo này chúng tôi đưa ra khái niệm phụ thuộc hàm mờ với lượng từ (QFFD) trên mô hình cơ sở dữ liệu quan hệ mờ. Đây là một dạng phụ thuộc dữ liệu mở rộng từ khái niệm phụ thuộc hàm mờ bằng cách thay thế lượng từ chặt "với mọi" bằng các lượng từ ngôn ngữ. Để mở rộng, chúng tôi đưa ra một đánh giá độ thỏa phụ thuộc hàm của một bộ trong quan hệ nhờ các toán tử kéo theo mờ. Các lượng từ ngôn ngữ được xem như những tập mờ và được đánh giá qua các hàm thuộc. Việc mở rộng khái niệm phụ thuộc hàm cho phép mô tả nhiều phụ thuộc dữ liệu mềm dẻo hơn và gần với thực tế hơn, phục vụ cho việc khai thác tri thức từ các cơ sở dữ liệu mờ. Trong bài báo này chúng tôi cũng đưa ra một hệ tiên đề cho các QFFD với lượng từ tăng tương tự hệ tiên đề Armstrong cho các phụ thuộc hàm. Tính xác đáng của hệ tiên đề cũng được chứng minh. Để giải quyết bài toán thành viên cho tập các QFFD, chúng tôi đưa ra khái niệm tập phụ thuộc và các thuật toán có độ phức tạp đa thức để tính các tập phụ thuộc.

1. INTRODUCTION

Data dependencies are the most important topics in theory of relational database. Several authors have proposed extended dependencies in fuzzy relational database models. In [1, 4–6, 8, 9, 11, 12, 16, 17] various definitions of FFD and fuzzy multivalued dependencies (FMVD) were given. These dependencies are extension of dependencies in classical relational model.

An FFD $X \rightsquigarrow_{\varphi} Y$ holds in a relation r if and only if **for all** t_1, t_2 in r if equality measure of $t_1[X]$ and $t_2[X]$ more or less determines equality measure of $t_1[Y]$ and $t_2[Y]$. In this paper, we give an extension of FFD by using fuzzy linguistic quantifiers to replace quantifier *for all* and this dependency is called *Fuzzy Functional Dependency with Linguistic Quantifiers* (QFFD). The QFFD is a really extension of FFD in that allows to reflect more significant situations in real world.

This paper is organized as follows. Section 2 presents some of the basic definitions of the possibility-based relational database model. In Section 3, we introduce the concept of FFD based on FIOs. Definition of QFFD is presented in Section 4. In Section 5 we give some algorithms for testing membership in closure of QFFDs. Section 6 concludes this paper and give some perspectives of the present work.

2. POSSIBILITY-BASED FUZZY RELATIONAL DATABASE MODEL

2.1. Possibility distribution

Definition 2.1. Let X be a variable which takes values in a universe of discourse D. A *possibility distribution* of X, denoted by π_X is a map from D to $[0, 1]$.

Possibility-based relational database model represents data by means of possibility distributions. Let A be an attribute whose domain is D. Our entire knowledge about the value of A for an object x will be represented by a possibility distribution $\pi_{A(x)}$ on $D \cup \{e\}$, where e is an extraneous element denoting the case where A does not apply to x.

Definition 2.2. A *proximity relation* is a mapping $p : D \times D \to [0,1]$ such that for $x, y \in D$, $p(x,x) = 1$ (*reflexivity*), and $p(x,y) = p(y,x)$ (*symmetry*).

Definition 2.3. A *fuzzy relation scheme* is a triple (R, C, α), where $R = \{A_1, A_2, \ldots, A_n\}$ is the set of attributes, $C = (c_1, c_2, \ldots, c_n)$ is the set of associated proximity relations, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is the set of associated thresholds ($\alpha_i \in [0,1],\ 1 \le i \le n$).

Definition 2.4. A *fuzzy relation instance* r on scheme (R, C, α) is a subset of the cross product $\Pi(D_1) \times \Pi(D_2) \times \cdots \times \Pi(D_n)$, where $D_i = \mathrm{dom}(A_i) \cup \{e\}$, and $\Pi(D_i)$ is set of possibility distributions on D_i.

2.2. Equality measure

Let t, t' be fuzzy tuples in relation r on scheme (R, C, α), $t = (\pi_{A_1}, \pi_{A_2}, \ldots, \pi_{A_n})$, $t' = (\pi'_{A_1}, \pi'_{A_2}, \ldots, \pi'_{A_n})$, $X = \{A_{i_1}, A_{i_2}, \ldots, A_{i_k}\}$ is a set of attributes of R.
Equality measure of two values π_{A_i}, π'_{A_i}, denoted by $=_c (\pi_{A_i}, \pi'_{A_i})$, is defined by

$$=_c (\pi_{A_i}, \pi'_{A_i}) = \sup_{\substack{x,y \in D_i \\ c_i(x,y) \ge \alpha_i}} \min(\pi_{A_i}(x), \pi_{A_i}(y)).$$

Equality measure of two tuples t and t' on X, denoted by $\approx (t[X], t'[X])$, is defined by

$$\approx (t[X], t'[X]) = \min\left(=_c (\pi_{A_{i_1}}, \pi'_{A_{i_1}}), =_c (\pi_{A_{i_2}}, \pi'_{A_{i_2}}), \ldots, =_c (\pi_{A_{i_k}}, \pi'_{A_{i_k}})\right).$$

3. FUZZY FUNCTIONAL DEPENDENCY

Definition 3.1. Let r be a relation of scheme (R, C, α), X and Y be subsets of R, and I be a fuzzy implication operator. Relation r is said to satisfy the *fuzzy functional dependency* X determines Y to the degree $\varphi, \varphi \in [0,1]$, denoted by $X \leadsto_\varphi Y$, if and only if

$$\min_{t,t' \in r} I\left(\approx (t[X], t'[X]), \approx (t[Y], t'[Y])\right) \ge \varphi.$$

Based on FIOs of Gödel and Dienes, we suggets a FIO using for FFD as below:

$$I(a,b) = \begin{cases} 1 & , \text{ if } a \le b \\ \max(1-a, b) & , \text{ otherwise} \end{cases}$$

Let r be a fuzzy relation of scheme (R, C, α). X, Y be subsets of R, $\varphi \in [0,1]$.
Compatibility degree with a fuzzy functional dependency $X \leadsto Y$ of tuple t in r, denoted by $\sigma(t(r)|X \leadsto Y)$, is defined by

$$\sigma(t(r)|X \leadsto Y) = \min_{\substack{t' \in r \\ t' \ne t}} \left\{ I(\approx (t[X], t'[X]), \approx (t[Y], t'[Y])) \right\}.$$

Compatibility degree with a fuzzy functional dependency $X \leadsto_\varphi Y$ of relation r, denoted by $\sigma(r|X \leadsto_\varphi Y)$, is defined by

$$\sigma(r|X \leadsto_\varphi Y) = \frac{|r_{X \leadsto_\varphi Y}|}{|r|},$$

where $r_{X \leadsto_\varphi Y}$ is set of tuples of r such that $\sigma(t(r)|X \leadsto Y) \ge \varphi$.

4. FUZZY FUNCTIONAL DEPENDENCY WITH LINGUISTIC QUANTIFIERS

4.1. Linguistic quantifiers

Human discourse is very rich and diverse in its quantifiers, e.g, *about 5, almost all, a few, many, most, as many as possible, nearly half, at least half.* Zadeh, using Fuzzy logic, introduced the concept of *linguistic quantifier* to represent a large number of possible quantifiers. Zadeh suggested that the semantic of linguistic quantifier can be captured by using fuzzy subsets for its representation. He distinguished between two types of linguistic quantifiers, *absolute* and *proportional.* Absolute quantifiers are used to represent amounts that are absolute in nature such as *about 2, more than 5,...* These absolute linguistic quantifiers are closely related to the concept of the counting or number of elements. Proportional quantifiers are used to represent amounts that are relative in nature such as *most, at least half.* A proportional quantifier can be represented by a fuzzy subset Q in the unit interval, [0,1], such that for any $x \in [0,1]$, $\mu_Q(x)$ indicates the degree to which the proportion x is compatible with the meaning of the quantifier it represents.

A proportional quantifier Q have membership function $\mu_Q : [0,1] \to [0,1]$, satisfies: $\mu_Q(0) = 0$, and $\exists x \in [0,1]$ such that $\mu_Q(x) = 1$.

A non-decreasing quantifier satisfies: $\forall a, b$ if $a > b$ then $\mu_Q(a) \geq \mu_Q(b)$.

The membership function of a non-decreasing proportional quantifier can be represented as

$$\mu_Q(x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } x > b \end{cases}$$

with $a, b, x \in [0,1]$.

Some examples of proportional quantifiers, where the parameters, (a, b) are most $= (0.3, 0.8)$, at least half $= (0, 0.5)$ and as many as possible $= (0.5, 1)$.

In the rest of this paper, we suppose that Q is a non-decreasing quantifier with membership function μ_Q. Denote:

$$|Q| = \inf_{\mu_Q(x)=1} \{x\}$$

4.2. Fuzzy functional dependency with linguistic quantifiers

Definition 4.1. Let r be a relation of scheme (R, C, α), X and Y be subsets of R, and Q be a linguistic quantifier, $\varphi \in [0,1]$. Relation r is said to satisfy the *fuzzy functional dependency* X determines Y to the degree φ with quantifier Q, denoted by $X \rightsquigarrow_\varphi Y$, if and only if $\mu_Q(\sigma(r|X \rightsquigarrow_\varphi Y)) = 1$.

Remark 4.1. The FFD is a special case of QFFD with quantifier *for all* (**FA** for short), i.e, relation r holds QFFDs of the form $\mathbf{FA}(X \rightsquigarrow_\varphi Y)$, if and only if r holds FFD $X \rightsquigarrow_\varphi Y$.

4.3. Axioms of QFFDs

QFFD1 (*Reflexivity*): If $Y \subseteq X$ then $Q(X \rightsquigarrow_\varphi Y)$, for all Q and for all $\varphi \in [0,1]$.
QFFD2 (*Augmentation*): If $Q(X \rightsquigarrow_\varphi Y)$ then $Q(XW \rightsquigarrow_\varphi YW)$, for all $W \subseteq R$.
QFFD3 (*Transitivity 1*): If $Q(X \rightsquigarrow_\varphi Y)$ and $Y \rightsquigarrow_\varphi Z$ then $Q(X \rightsquigarrow_\varphi Z)$.
QFFD3' (*Transitivity 2*): If $X \rightsquigarrow_\varphi Y$ and $Q(Y \rightsquigarrow_\varphi Z)$ then $Q(X \rightsquigarrow_\varphi Z)$.
QFFD4 (*inclusion for threshold*): If $Q(X \rightsquigarrow_\varphi Y)$ then $Q(X \rightsquigarrow_{\varphi'} Y)$, for all $0 < \varphi' \leq \varphi$.
QFFD5 (*inclusion for quantifier*): If $Q(X \rightsquigarrow_\varphi Y)$ then $Q'(X \rightsquigarrow_\varphi Y)$, for all Q' such that $|Q'| \leq |Q|$.

Lemma 4.1 *Axioms* QFFD1 - QFFD5 *are sound.*

Proof.

Reflexivity: Let r be a relation of scheme (R, C, α), and $Y \subseteq X \subseteq R$. Since $Y \subseteq X$ then $\forall t_1, t_2 \in r$ we have $\approx (t_1[X], t_2[X]) \leq \approx (t_1[Y], t_2[Y])$. Hence $I(\approx (t_1[X], t_2[X]), \approx (t_1[Y], t_2[Y])) = 1$. Consequently $\sigma(r|X \rightsquigarrow_\varphi Y) = 1$, $\forall \varphi \in [0,1]$.

For any non-decreasing quantifier Q, we have $\mu_Q(\sigma(r|X \rightsquigarrow_\varphi Y)) = 1$. Hence $Q(X \rightsquigarrow_\varphi Y)$ holds in r.

Augmentation: Assume that $Q(X \rightsquigarrow_\varphi Y)$ holds in relation r. Since $Q(X \rightsquigarrow_\varphi Y)$ holds in r then

$$\mu_Q(\sigma(r|X \rightsquigarrow_\varphi Y)) = 1 \qquad (*)$$

$\forall t_1 \in r_{X \rightsquigarrow_\varphi Y}$, $\forall t_2 \in r$ we have $I(\approx (t_1[X], t_2[X]), \approx (t_1[Y], t_2[Y])) \geq \varphi$. It is easy to show that $I(\approx (t_1[XW], t_2[XW]), \approx (t_1[YW], t_2[YW])) \geq I(\approx (t_1[X], t_2[X]), \approx (t_1[Y], t_2[Y])) \geq \varphi$.
Since $(*)$ and Q is a non-decreasing quantifier then $\mu_Q(\sigma(r|XW \rightsquigarrow_\varphi YW)) = 1$.
Hence r holds $Q(XW \rightsquigarrow_\varphi YW)$.

Transitivity 1: Assume that $Q(X \rightsquigarrow_\varphi Y)$ and $Y \rightsquigarrow_\varphi Z$ holds in relation r.
From $Q(X \rightsquigarrow_\varphi Y)$ we have $\mu_Q(\sigma(r|X \rightsquigarrow_\varphi Y)) = 1$.
Hence $\forall t_1 \in r_{X \rightsquigarrow_\varphi Y}$, $\forall t_2 \in r$ we have $I(\approx (t_1[X], t_2[X]), \approx (t_1[Y], t_2[Y])) \geq \varphi$.
Since $Y \rightsquigarrow_\varphi Z$ holds in r then $I(\approx (t_1[Y], t_2[Y]), \approx (t_1[Z], t_2[Z])) \geq \varphi$.
Hence $I(\approx (t_1[X], t_2[X]), \approx (t_1[Z], t_2[Z])) \geq \varphi$. Consequently $Q(X \rightsquigarrow_\varphi Z)$ holds in r.

Transitivity 2: The proof is similar to that for Transitivity 1.

Inclusion for threshold: Assume that $Q(X \rightsquigarrow_\varphi Y)$ holds in relation r, and $\varphi' \leq \varphi$.
We have $\mu_Q(\sigma(r|X \rightsquigarrow_\varphi Y)) = 1$. Since $\varphi' \leq \varphi$ then $\mu_Q(\sigma(r|X \rightsquigarrow_{\varphi'} Y)) = 1$. Hence $Q(X \rightsquigarrow_{\varphi'} Y)$ holds in r.

Inclusion for quantifier: Assume that $Q(X \rightsquigarrow_\varphi Y)$ holds in relation r, and $|Q'| \leq |Q|$. We have $\mu_Q(\sigma(r|X \rightsquigarrow_\varphi Y)) = 1$. Since $|Q'| \leq |Q|$ then $\mu'_Q(\sigma(r|X \rightsquigarrow_\varphi Y)) = 1$. Hence $Q'(X \rightsquigarrow_\varphi Y)$ holds in r. ■

The following axioms are easily obtained from QFFD1 - QFFD5.

QFFD6 (*Union 1*): If $Q(X \rightsquigarrow_{\varphi_1} Y)$ and $X \rightsquigarrow_{\varphi_2} Z$ then $Q(X \rightsquigarrow_\varphi YZ)$, where $\varphi = \min(\varphi_1, \varphi_2)$.

QFFD6' (*Union 2*): If $X \rightsquigarrow_{\varphi_1} Y$ and $Q(X \rightsquigarrow_{\varphi_2} Z)$ then $Q(X \rightsquigarrow_\varphi YZ)$, where $\varphi = \min(\varphi_1, \varphi_2)$.

QFFD7 (*Pseudo-transitivity*): If $Q(X \rightsquigarrow_{\varphi_1} Y)$ and $WY \rightsquigarrow_{\varphi_1} Z$ then $Q(XW \rightsquigarrow_\varphi Z)$, where $\varphi = \min(\varphi_1, \varphi_2)$.

QFFD8 (*Decomposition*): If $Q(X \rightsquigarrow_\varphi Y)$ then $Q(X \rightsquigarrow_\varphi Z)$, for all $Z \subseteq Y$.

Proof.

Union 1:
By QFFD4, we have $Q(X \rightsquigarrow_\varphi Y)$ and $X \rightsquigarrow_\varphi Z$.
By QFFD2, we have $Q(X \rightsquigarrow_\varphi XY)$ and $XY \rightsquigarrow_\varphi YZ$.
By QFFD3, we have $Q(X \rightsquigarrow_\varphi Z)$.

Union 2: similar to the above proof.

Pseudo-transitivity:
By QFFD4, we have $Q(X \rightsquigarrow_\varphi Y)$ and $WY \rightsquigarrow_\varphi Z$.
From $Q(X \rightsquigarrow_\varphi Y)$ and by QFFD2, we have $Q(XW \rightsquigarrow_\varphi YW)$.
By QFFD3, we have $Q(XW \rightsquigarrow_\varphi Z)$.

Decomposition:
Since $Z \subseteq Y$ and by QFFD1 we have $Y \rightsquigarrow_\varphi Z$.
By transitivity axiom we have $Q(X \rightsquigarrow_\varphi Y)$.

4.4. Closure of QFFDs

Definition 4.2. Let F be set of QFFDs of scheme (R, C, α). *Closure* of F, denoted by F^+, is the set of all QFFDs that can be derived from F by application of the axioms QFFD1–QFFD8.

Remark 4.2. Since axioms QFFD4, QFFD5, QFFD8, we have if $Q(X \rightsquigarrow_\varphi Y) \in F^+$ then $Q'(X \rightsquigarrow_{\varphi'} Y') \in F^+$, $\forall Y' \subseteq Y$, $\varphi' \leq \varphi$, $|Q'| \leq |Q|$. Hence F^+ is an infinite set.

Definition 4.3. Let F be set of QFFDs of scheme (R, C, α). *Minimum Closure* of F, denoted by $\overline{F}^+$, is defined by

$$\overline{F}^+ = \left\{Q(X \rightsquigarrow_\varphi Y) \,\middle|\, Y \in \underset{Q(X \rightsquigarrow_\varphi Y_i) \in F^+}{\text{max-set}} \{Y_i\},\ Q = \sup_{Q_i(X \rightsquigarrow_\varphi Y) \in F^+} \{Q_i\},\ \varphi = \sup_{Q(X \rightsquigarrow_{\varphi_i} Y) \in F^+} \{\varphi_i\}\right\},$$

where $\underset{Q(X \rightsquigarrow_\varphi Y_i) \in F^+}{\text{max-set}} \{Y_i\}$ is family of maximum-set of Y_is, $\sup_{Q_i(X \rightsquigarrow_\varphi Y) \in F^+} \{Q_i\}$ is a quantifier Q such that $|Q| = \sup_{Q_i(X \rightsquigarrow_\varphi Y) \in F^+} \{|Q_i|\}$.

5. MEMBERSHIP PROBLEM

Membership problem: Let F be a set of QFFDs of scheme (R, C, α), $Q(X \rightsquigarrow_\varphi Y)$ be a QFFD. We have to determine whether that $Q(X \rightsquigarrow_\varphi Y) \in F^+$?

Now we introduce the following concepts.

5.1. Family of dependency sets

Definition 5.1. Let F be a set of QFFDs of scheme (R, C, α), X be a subset of attributes, Q be a linguistic quantifier, $\varphi \in [0, 1]$. *Family of dependency sets* for X (with respect to F, Q, φ), denoted by $Dep_F(X, Q, \varphi)$, is defined by

$$Dep_F(X, Q, \varphi) = \left\{Y \subseteq R | Q(X \rightsquigarrow_\varphi Y) \in F^+\right\}.$$

Remark 5.1. Since axioms QFFD4, QFFD5, QFFD8, we have if $Y \in Dep_F(X, Q, \varphi)$ then $Y' \in Dep_F(X, Q, \varphi)$, $\forall Y' \subseteq Y$.

Definition 5.2. Let F be a set of QFFDs of scheme (R, C, α), X be a subset of attributes, Q be a linguistic quantifier, $\varphi \in [0, 1]$. *Minimum family of dependency sets* for X (with respect to F, Q, φ), denoted by $\overline{Dep}_F(X, Q, \varphi)$, is defined by

$$\overline{Dep}_F(X, Q, \varphi) = \left\{Y \subseteq R \,|\, Y \in \text{max-set}_{Q(X \rightsquigarrow_\varphi Y_i) \in F^+} \{Y_i\}\right\}.$$

Proposition 5.1. *$Q(X \rightsquigarrow_\varphi Y) \in F^+$ if and only if $\exists Z \in \overline{Dep}_F(X, Q, \varphi)$ such that $Y \subseteq Z$.*

Proof. ($\Rightarrow$) Since $Q(X \rightsquigarrow_\varphi Y) \in F^+$ then $Y \in \overline{Dep}_F(X, Q, \varphi)$. By Definition 5.2. we have $\exists Z \in \overline{Dep}_F(X, Q, \varphi)$ such that $Y \subseteq Z$.
($\Leftarrow$) Assume that $\exists Z \in \overline{Dep}_F(X, Q, \varphi)$ such that $Y \subseteq Z$. Since $Z \in \overline{Dep}_F(X, Q, \varphi)$ then $Q(X \rightsquigarrow_\varphi Z) \in F^+$.
By Decomposition axiom, we have $Q(X \rightsquigarrow_\varphi Y) \in F^+$. ■

5.2. Algorithm for computing $\overline{Dep}_F(X, Q_0, \varphi_0)$

Denote $X_F^*(\varphi_0) = \{A \in R | X \rightsquigarrow_{\varphi_0} A \in F^+\}$.
It is easily to prove that $X_F^*(\varphi) \subseteq Y, \forall Y \in \overline{Dep}_F(X, Q, \varphi)$.

Algorithm 5.1. $X_F^*(\varphi_0)$

Input: X, φ_0, F

Output: $X_F^*(\varphi_0)$

Method:

Var OLDX, NEWX, XPLUS : Set of attributes;
NEWX := X;

Repeat

OLDX:=NEWX;

For each $V \rightsquigarrow_{\varphi} W$ **in** F **do**

If $(\varphi \geq \varphi_0)$ and ($V \subseteq$ NEWX) **Then**

NEWX := NEWX $\cup$ W ;

End If

End For

Until NEWX = OLDX;

XPLUS := NEWX;

Return (XPLUS)

Theorem 5.1. *Algorithm 5.1 is correct and time complexity is* $O(n.p.\min(n,p))$*, where* n *is number of attributes in* R *and* p *is number of QFFDs in* F.

Proof. The proof is similar to that for the algorithm computing the attributeset closure in classical relational model. See [13] for detail. ∎

Algorithm 5.2. $\overline{Dep}_F(X, Q_0, \varphi_0)$

Input: X, Q_0, φ_0, F.

Output: $\overline{Dep}_F(X, Q_0, \varphi_0)$.

Method:

Var XPLUS, Z : Set of attributes;

TEMP, OLDD, NEWD, DPLUS: Family of set of attributes;

XPLUS:=$X_F^*(\varphi_0)$;

NEWD:={XPLUS};

Repeat (1)

OLDD:=NEWD;

For each $Q(V \rightsquigarrow_{\varphi} W$ **in** F **do** (2)

TEMP:={∅};

If (Q = **FA**) and $(\varphi \geq \varphi_0)$ **Then**

For each Z **in** NEWD **do** (3)

If $V \subseteq Z$ **Then**

TEMP:=TEMP $\cup_{\subseteq}$ {$ZW \cup$ XPLUS} ;

End If

End For

Else If $(|Q| \geq |Q_0|)$ and $(\varphi \geq \varphi_0)$ and ($V \subseteq$ XPLUS) **Then**

TEMP:=TEMP $\cup_{\subseteq}$ {$W \cup$ XPLUS} ;

End If

End If

NEWD:=NEWD $\cup_{\subseteq}$ TEMP;

End For

Until NEWD = OLDD;

DPLUS := NEWD;

Return (DPLUS);

where $\cup_{\subseteq}$ is union operator for maximum-sets, and **FA** is quantifier *for all*.

Denote: NEWD$^{(i)}$, TEMP$^{(i)}$ are values of variables NEWD, TEMP of i^{th} step in Repeat-loop of Algorithm 5.2, respectively.

Example 5.1. $R = \{A, B, C, D, E, G\}$, $F = \{\, C \rightsquigarrow_{0.8} B, CD \rightsquigarrow_{0.85} B, \mathbf{M}(B \rightsquigarrow_{0.9} A), \mathbf{AMP}(C \rightsquigarrow_{0.85} D), BCD \rightsquigarrow_{0.9} E, \mathbf{M}(E \rightsquigarrow_{0.9} G), \mathbf{ALH}(G \rightsquigarrow_{0.95} C), C \rightsquigarrow_{0.9} G, \mathbf{M}(CG \rightsquigarrow_{0.95} A)\}$. Let's compute $\overline{Dep}_F(C, \mathbf{M}, 0.85)$, where $\mathbf{ALH}$ = *"at least half"* $\leq \mathbf{M}$ = *"most"* $\leq \mathbf{AMP}$ = *"as many as possible"* $\leq \mathbf{FA}$= *"for all"*.

XPLUS $= \{CG\}$

TEMP$^{(1)} = \{CDG, CGA\}$; NEWD$^{(1)} = \{CDG, CGA\}$

TEMP$^{(2)} = \{BCDG\}$; NEWD$^{(2)} = \{BCDG, CGA\}$

TEMP$^{(3)} = \{BCDGE, CGA\}$; NEWD$^{(3)} = \{BCDGE, CGA\}$

TEMP$^{(4)} = \{\varnothing\}$; NEWD$^{(4)} =$ NEWD$^{(3)}$.

$\overline{Dep}_F(C, \mathbf{M}, 0.85) = \{BCDGE, CGA\}$.

Theorem 5.2. *The time complexity of Algorithm 5.2 is $O(n.p^4)$, where n is number of attributes in R and p is number of QFFDs in F.*

Proof. Checking the inclusion and performing the union $\cup_{\subseteq}$ in the if-statement takes $O(n)+O(n.p) \approx O(n.p)$ time. Since For-loop (3) can be excuted at most p times then the excution of one cycle of For-loop (2) takes $O(n.p^2)$ time.

Since For-loop (2) can be excuted at most p times, for each step in this For-loop takes $O(n.p^2)$. Hence the excution of one cycle of Repeat-loop (1) takes $O(n.p^3)$ time.

The repeat-loop (1) can be excuted at most p times. Hence time complexity of this loop is $O(n.p^4)$.

The time complexity of Algorithm 5.1. for computing the $X_F^*(\varphi_0)$ is $O(n.p.\min(n, p))$. Hence the time complexity of Algorithm 5.2. is $O(n.p.\min(n, p)) + O(n.p^4) \approx O(n.p^4)$. ■

Lemma 5.1. *Algorithm 5.2. terminates.*

Proof. Since F and R are finite sets then Algorithm 5.2 terminates, i.e. there exists step k^{th} such that NEWD$^{(k)} =$ NEWD$^{(k-1)}$. ■

Now we denotes NEWD$^{(k)}$ as return of Algorithm 5.2.

Lemma 5.2. *For all $j \geq k$ we have NEWD$^{(j)} \cup_{\subseteq}$ TEMP$^{(j)}$ = NEWD$^{(j)}$.*

Proof. We prove this lemma by induction on j.
For $j = k$, since NEWD$^{(k-1)} \cup_{\subseteq}$ TEMP$^{(k-1)}$ = NEWD$^{(k)}$ = NEWD$^{(k-1)}$ then
TEMP$^{(k)}$ = TEMP$^{(k-1)}$. Hence NEWD$^{(k)} \cup_{\subseteq}$ TEMP$^{(k)}$ = NEWD$^{(k)}$. Consequently, the basis of induction is true.
Assume that lemma is true for j, i.e. NEWD$^{(j)} \cup_{\subseteq}$ TEMP$^{(j)}$ = NEWD$^{(j)}$. We have to show that lemma is true for $j+1$.
Since NEWD$^{(j)} \cup_{\subseteq}$ TEMP$^{(j)}$ = NEWD$^{(j)}$ then TEMP$^{(j+1)}$ = TEMP$^{(j)}$.
Hence NEWD$^{(j+1)} \cup_{\subseteq}$ TEMP$^{(j+1)}$ = NEWD$^{(j+1)}$.
The lemma is proved. ■

Lemma 5.3. *If $U \rightsquigarrow_{\varphi} Y \in F^+$, $\varphi \geq \varphi_0$ and $\exists Z \in$ NEWD$^{(k)}$ such that $U \subseteq Z$ then $Y \subseteq Z$.*

Proof. We prove this lemma by induction that for any G that $F \subseteq G \subseteq F^+$ obtained during this construction, we have:

If $U \rightsquigarrow_{\varphi} Y \in G$, $\varphi \geq \varphi_0$ and $\exists Z \in$ NEWD$^{(k)}$ such that $U \subseteq Z$ then $Y \subseteq Z$ $\quad (*)$

Basis of induction $G = F$: We consider two cases:

Case 1: $U \subseteq$ XPLUS. Since $U \rightsquigarrow_{\varphi} Y \in F$ and $\varphi \geq \varphi_0$ then $Y \subseteq$ XPLUS. Hence $Y \subseteq Z, \forall Z \in$ NEWD$^{(k)}$.

Case 2: $U \not\subseteq$ XPLUS. Since $U \subseteq Z$ and $U \rightsquigarrow_{\varphi} Y \in F$, by Algorithm 5.2 we have $YZ \in$ TEMP$^{(k)}$. Since Lemma 5.2 then $\exists Z' \in$ NEWD$^{(k)}$ such that $YZ \subseteq Z'$. Since Z is a maximum-set in NEWD$^{(k)}$

and $Z \not\subseteq$ XPLUS then $Z' = Z$. Hence $Y \subseteq Z$.

Hypothesis: Assume that G satisfies $(*)$.

Let G' be a set of QFFDs obtained from G by one application in {QFFD1 - QFFD5}. We have to show that:

If $U \rightsquigarrow_\varphi Y \in G'$, $\varphi \geq \varphi_0$ and $\exists Z \in \mathrm{NEWD}^{(k)}$ such that $U \subseteq Z$ then $Y \subseteq Z$ $\quad (*')$

If $U \rightsquigarrow_\varphi Y \in G$, by hypothesis then $(*')$ is true.

If $U \rightsquigarrow_\varphi Y \notin G$. There are four cases:

Case 1: $U \rightsquigarrow_\varphi Y$ is obtained from G by application reflexive axiom (QFFD1). We have $Y \subseteq U$. Hence $Y \subseteq Z$. Consequently $(*')$ is true.

Case 2: $U \rightsquigarrow_\varphi Y$ is obtained from G by application of augmentation axiom. There exists $U \rightsquigarrow_\varphi Y' \in G$ such that $Y = U'Y'$, $U' \subseteq U$.
By hypothesis, we have $Y' \subseteq Z$. Since $U' \subseteq U \subseteq Z$ then $Y \subseteq Z$. Hence $(*')$ is true.

Case 3: $U \rightsquigarrow_\varphi Y$ is obtained from G by application of transitivity axiom. There exists $U \rightsquigarrow_\varphi V \in G$ and $V \rightsquigarrow_\varphi Y \in G$. Since $U \rightsquigarrow_\varphi V \in G$ then by hypothesis $V \subseteq Z$. Since $V \rightsquigarrow_\varphi Y \in G$ and $V \subseteq Z$ then by hypothesis $Y \subseteq Z$. Hence $(*')$ is true.

Case 4: $U \rightsquigarrow_\varphi Y$ is obtained from G by application inclusion for threshold axiom (QFFD4). There exists $U \rightsquigarrow_{\varphi'} V \in G$, where $\varphi' \geq \varphi$. Since $\varphi' \geq \varphi_0$, then by hypothesis $Y \subseteq Z$. Hence $(*')$ is true.

The lemma is proved. ∎

Lemma 5.4. *If $U \subseteq XPLUS$, $Q(U \rightsquigarrow_\varphi Y) \in F^+, \varphi \geq \varphi_0$ and $|Q| \geq |Q_0|$ then $\exists Z \in NEWD^{(k)}$ such that $Y \subseteq Z$.*

Proof. Similar to the proof of Lemma 5.3. ∎

Lemma 5.5. *$NEWD^{(k)} = \overline{Dep}_F(X, Q_0, \varphi_0)$ if and only if following conditions holds*
 (i) *If $Y \in NEWD^{(k)}$ then $Q_0(X \rightsquigarrow_{\varphi_0} Y) \in F^+$.*
 (ii) *If $Q(X \rightsquigarrow_\varphi Y) \in F^+$, $|Q| \geq |Q_0|$, and $\varphi \geq \varphi_0$ then $\exists Z \in NEWD^{(k)}$ such that $Y \subseteq Z$.*

Proof. $(\Rightarrow)$ (i) It is trival by Priposition 5.1.
(ii) Since $Q(X \rightsquigarrow_\varphi Y) \in F^+$ and $|Q| \geq |Q_0|$, $\varphi \geq \varphi_0$ then $Q_0(X \rightsquigarrow_{\varphi_0} Y) \in F^+$. By Proposition 5.1 there exists $Z \in \mathrm{NEWD}^{(k)}$ $(= \overline{Dep}_F(X, Q_0, \varphi_0))$ such that $Y \subseteq Z$.

$(\Leftarrow)$ Assume that $\mathrm{NEWD}^{(k)}$ satisfies conditions (i) and (ii). In order to prove $\mathrm{NEWD}^{(k)} = \overline{Dep}_F(X, Q_0, \varphi_0)$ we have to show that:
a) $\forall Y \in \mathrm{NEWD}^{(k)}$ then $\exists Z \in \overline{Dep}_F(X, Q_0, \varphi_0)$ such that $Y \subseteq Z$ and
b) $\forall Z \in \overline{Dep}_F(X, Q_0, \varphi_0)$ then $\exists Y \in \mathrm{NEWD}^{(k)}$ such that $Z \subseteq Y$.
For a) $\forall Y \in \mathrm{NEWD}^{(k)}$, by (i) we have $Q_0(X \rightsquigarrow_{\varphi_0} Y) \in F^+$. By (ii) then $\exists Z \in \overline{Dep}_F(X, Q_0, \varphi_0)$ such that $Y \subseteq Z$.
For b) $\forall Z \in \overline{Dep}_F(X, Q_0, \varphi_0)$, by Proposition 5.1 we have $Q_0(X \rightsquigarrow_{\varphi_0} Y) \in F^+$. By (ii) then $\exists Y \in \mathrm{NEWD}^{(k)}$ such that $Z \subseteq Y$. Hence $\mathrm{NEWD}^{(k)} = \overline{Dep}_F(X, Q_0, \varphi_0)$.
The proof is complete. ∎

Theorem 5.3. *Algorithm 5.2. is correct.*

Proof. By Lemma 5.5 we have to show that:
(i) If $Y \in \mathrm{NEWD}^{(k)}$ then $Q_0(X \rightsquigarrow_{\varphi_0} Y) \in F^+$.
(ii) If $Q(X \rightsquigarrow_\varphi Y) \in F^+$, $|Q| \geq |Q_0|$, and $\varphi \geq \varphi_0$ then $\exists Z \in \mathrm{NEWD}^{(k)}$ such that $Y \subseteq Z$.

First we prove (i). We show that $\forall Y \in \mathrm{NEWD}^{(j)}$ then $Q_0(X \rightsquigarrow_{\varphi_0} Y) \in F^+$ (3) by induction on j.

Basis j=0: we have $\mathrm{NEWD}^{(0)} = \{\mathrm{XPLUS}\}$. Clearly $X \rightsquigarrow_{\varphi_0} \mathrm{XPLUS} \in F^+$.

Hypothesis: Assume that (3) is true for $j-1$. We have to show that (3) also is true for j.
$\forall Y \in \mathrm{NEWD}^{(j)} = \mathrm{NEWD}^{(j-1)} \cup_\subseteq \mathrm{TEMP}^{(j-1)}$.
If $Y \in \mathrm{NEWD}^{(j-1)}$, by hypothesis (3) is true.

If $Y \notin \mathrm{NEWD}^{(j-1)}$. We have $Y \in \mathrm{TEMP}^{(j-1)}$. By Algorithm 5.2, we consider two cases:

Case 1: $\exists V \rightsquigarrow_\varphi W \in F$ and $Z \in \mathrm{NEWD}^{(j-1)}$ such that $\varphi \geq \varphi_0$, $V \subseteq Z$ and $Y = ZW \cup \mathrm{XPLUS}$.
Since $Z \in \mathrm{NEWD}^{(j-1)}$, by hypothesis we have $Q_0(X \rightsquigarrow_{\varphi_0} Z) \in F^+$. From $V \subseteq Z$, by QFFD2 we have $Z \rightsquigarrow_{\varphi_0} ZW \in F^+$. By transitivity we have $Q_0(X \rightsquigarrow_{\varphi_0} ZW) \in F^+$.

Case 2: $\exists Q(V \rightsquigarrow_\varphi W) \in F$ such that $|Q| \geq |Q_0|$, $\varphi \geq \varphi_0$, $V \subseteq \mathrm{XPLUS}$, $Y = W \cup \mathrm{XPLUS}$.
Since $Q(V \rightsquigarrow_\varphi W) \in F$ and $|Q| \geq |Q_0|$, $\varphi \geq \varphi_0$ then by QFFD4, QFFD5 we have $Q_0(V \rightsquigarrow_{\varphi_0} W) \in F^+$. By $V \subseteq \mathrm{XPLUS}$ then $X \rightsquigarrow_{\varphi_0} V \in F^+$. By QFFD3, we have $Q_0(X \rightsquigarrow_{\varphi_0} W) \in F^+$. By QFFD2 then $Q_0(X \rightsquigarrow_{\varphi_0} W \cup \mathrm{XPLUS}) \in F^+$. Hence $Q_0(X \rightsquigarrow_{\varphi_0} Y) \in F^+$. Consequently (i) is proved. Now we prove (ii) by induction that for any G, that $F \subseteq G \subseteq F^+$ obtained during this construction, we have

$$\forall Q(X \rightsquigarrow_\varphi Y) \in G,\ |Q| \geq |Q_0| \text{ and } \varphi \geq \varphi_0 \text{ then } \exists Z \in \mathrm{NEWD}^{(k)} \text{ such that } Y \subseteq Z \tag{4}$$

Basis $G = F$: From $Q(X \rightsquigarrow_\varphi Y) \in F$ and $|Q| \geq |Q_0|$, $\varphi \geq \varphi_0$, by Algorithm 5.2 we have $Y \cup \mathrm{XPLUS} \in \mathrm{TEMP}^{(1)}$. Hence $\exists Z \in \mathrm{NEWD}^{(k)}$ such that $Y \subseteq Z$. Consequently (4) is true.

Hypothesis: Assume (4) is true.
Let G' be a set of QFFDs obtained from G by one application of axiom in {QFFD1-QFFD5}. We have to show that

$$\forall Q(X \rightsquigarrow_\varphi Y) \in G',\ |Q| \geq |Q_0| \text{ and } \varphi \geq \varphi_0 \text{ then } \exists Z \in \mathrm{NEWD}^{(k)} \text{ such that } Y \subseteq Z \tag{4'}$$

If $Q(X \rightsquigarrow_\varphi Y) \in G$, by hypothesis (4') is true. If $Q(X \rightsquigarrow_\varphi Y) \in G$. There are six cases:

Case 1: $Q(X \rightsquigarrow_\varphi Y)$ is obtained from G by reflexivity axiom. Since $Y \subseteq X$ then $Y \subseteq Z\ \forall Z \in \mathrm{NEWD}^{(k)}$. Hence (4') is true.

Case 2: $Q(X \rightsquigarrow_\varphi Y)$ is obtained from G by augmentation axiom. There exists $Q(X \rightsquigarrow_\varphi Y') \in G$, $Y = X'Y'$, $X' \subseteq X$.
By hypothesis $\exists Z \in \mathrm{NEWD}^{(k)}$ such that $Y' \subseteq Z$. Since $X' \subseteq X \subseteq Z$ then $Y \subseteq Z$. Hence (4') is true.

Case 3: $Q(X \rightsquigarrow_\varphi Y)$ is obtained from G by Transitivity 1 axiom (QFFD3). There exists $Q(X \rightsquigarrow_\varphi V) \in G$ and $V \rightsquigarrow_\varphi Y \in G$. By hypothesis $\exists Z \in \mathrm{NEWD}^{(k)}$ such that $V \subseteq Z$.
By Lemma 5.3 we have $Y \subseteq Z$. Hence (4') is true.

Case 4: $Q(U \rightsquigarrow_\varphi Y)$ is obtained from G by Transitivity 2 axiom (QFFD3'). There exists $X \rightsquigarrow_\varphi V \in G$ and $Q(V \rightsquigarrow_\varphi Y) \in G$. By Lemma 5.4 we have $Y \subseteq Z$.

Case 5: $Q(U \rightsquigarrow_\varphi Y)$ is obtained from G by QFFD4. There exists $Q(U \rightsquigarrow_{\varphi'} V) \in G$ and $\varphi' \geq \varphi$. By hypothesis we have (4') is true.

Case 6: $Q(U \rightsquigarrow_\varphi Y)$ is obtained from G by QFFD5. There exists $Q'(U \rightsquigarrow_\varphi V) \in G$ and $|Q'| \geq |Q|$. By hypothesis we have (4') is true.

The proof is complete. ∎

The minimum family of dependency sets $\overline{Dep}_F(X, Q, \varphi)$ is dependent to Q and φ. In following part, we give concept of family of dependency sets (with respect to F) that only dependent to X and F.

Definition 5.3. Let F be a set of QFFDs of scheme (R, C, α), $X \subseteq R$, Q is a quantifier, $\varphi \in [0, 1]$. *Family of dependency sets* of X (with respect to F), denoted by $Dep_F(X)$, is set of triple (Y, Q, φ) such that $Q(X \rightsquigarrow_\varphi Y) \in F^+$

$$Dep_F(X) = \{(Y, Q, \varphi) | Q(X \rightsquigarrow_\varphi Y) \in F^+\}.$$

Remark 5.2. By axioms QFFD4, QFFD5, QFFD8, we have if $(Y, Q, \varphi) \in Dep_F(X)$ and $\forall Y' \subseteq Y$, $|Q'| \leq |Q|$, $\varphi' \leq \varphi$ then $(Y', Q', \varphi') \in Dep_F(X)$.

Definition 5.4. Let F be a set of QFFDs of scheme (R, C, α), $X \subseteq R$, Q be quantifier, $\varphi \in [0, 1]$. *Minimum family of dependency sets* of X (with respect to F), denoted by $\overline{Dep}_F(X)$, is defined by

$$\overline{Dep}_F(X) = \Big\{(Y, Q, \varphi) \,|\, Y \in \underset{Q(X \rightsquigarrow_\varphi Y_i) \in F^+}{\text{max-set}} \{Y_i\},\ Q = \sup_{Q_i(X \rightsquigarrow_\varphi Y) \in F^+} \{Q_i\},\ \varphi = \sup_{Q(X \rightsquigarrow_{\varphi_i} Y) \in F^+} \{\varphi_i\}\Big\}.$$

Proposition 5.2. *$Q(X \rightsquigarrow_{\varphi} Y) \in F^+$ if and only if $\exists (Z, Q_Z, \varphi_Z) \in \overline{Dep}_F(X)$ such that $Y \subseteq Z, |Q| \leq |Q_Z|, \varphi \leq \varphi_Z$.*

Proof. ($\Rightarrow$) Since $Q(X \rightsquigarrow_{\varphi} Y) \in F^+$ then $(Y, Q, \varphi) \in Dep_F(X)$. By definition of $\overline{Dep}_F(X)$ then $\exists (Z, Q_Z, \varphi_Z) \in \overline{Dep}_F(X)$ such that $Y \subseteq Z, |Q| \leq |Q_Z|,\ \varphi \leq \varphi_Z$.
($\Leftarrow$) Assume that $(Z, Q_Z, \varphi_Z) \in \overline{Dep}_F(X)$ such that $Y \subseteq Z, |Q| \leq |Q_Z|, \varphi \leq \varphi_Z$.
Since $(Z, Q_Z, \varphi_Z) \in \overline{Dep}_F(X)$ then $(Z, Q_Z, \varphi_Z) \in Dep_F(X)$. Hence $Q_Z(X \rightsquigarrow_{\varphi_Z} Z) \in F^+$. By axioms QFFD4, QFFD5, QFFD8 we have $Q(X \rightsquigarrow_{\varphi} Y) \in F^+$. ∎

Algorithm 5.3. $\overline{Dep}_F(X)$

Input: X, F

Output: $\overline{Dep}_F(X)$

Method:

Var TEMP, OLDD, NEWD, DPLUS : Set of triple (Set of attributes, Quantifier, [0,1]);
NEWD:={(X,**FA**,1)};
Repeat
  OLDD:=NEWD;
  For each $Q(V \rightsquigarrow_{\varphi} W)$ **in** F **do**
    TEMP:={∅};
    For each (Z, Q_Z, φ_Z) **in** NEWD **do**
      If ((Q = **FA**) or (Q_Z = **FA**)) and ($V \subseteq Z$) **Then**
        TEMP:=TEMP $\cup_{\subseteq,\leq} \{(ZW, \min(Q, Q_Z), \min(\varphi, \varphi_Z)\}$
      End If
    End For
    If $V \subseteq X$ **Then**
      TEMP:=TEMP $\cup_{\subseteq,\leq} \{(XW, Q, \varphi)\}$
    End If
    NEWD:=NEWD $\cup_{\subseteq,\leq}$ TEMP;
  End For
Until NEWD = OLDD;
DPLUS:=NEWD;
Return (DPLUS), {DPLUS = $\overline{Dep}_F(X)$}

where $\cup_{\subseteq,\leq}$ is a union operator of maximum-triples, $\min(Q, Q_Z) = Q$ if $|Q| \leq |Q_Z|$, Q_Z otherwise.

Example 5.2. R and F as same as in Example 5.1. Let's compute $\overline{Dep}_F(C)$.
NEWD$^{(0)}$ = {(C,**FA**,1)}
TEMP$^{(1)}$ = {(BC,**FA**,0.8),(CD,**AMP**,0.85), (CG,**FA**,0.9)};
NEWD$^{(1)}$ = {(C,**FA**,1), (BC,**FA**,0.8), (CD,**AMP**,0.85), (CG,**FA**,0.9)};
TEMP$^{(2)}$ = {(BCD,**AMP**,0.85), (ABC,**M**,0.8), (CGA,**M**, 0.9)};
NEWD$^{(2)}$ =
{(C,**FA**,1), (BC,**FA**,0.8), (CG, **FA**,0.9), (BCD,**AMP**,0.85), (ABC,**M**,0.8), (CGA,**M**,0.9)};
TEMP$^{(3)}$ = {($BCDE$,**AMP**,0.85)};
NEWD$^{(3)}$ =
{(C,**FA**,1), (BC,**FA**,0.8), (CG,**FA**,0.9), (ABC, **M**, 0.8), (CGA,**M**,0.9), ($BCDE$,**AMP**,0.85)};
TEMP$^{(4)}$ = {∅};
NEWD$^{(4)}$ = NEWD$^{(3)}$.

$\overline{Dep}_F(C) =$
$\{(C,\mathbf{FA},1), (BC,\mathbf{FA},0.8), (CG,\mathbf{FA},0.9), (ABC,\mathbf{M},0.8), (CGA,\mathbf{M},0.9), (BCDE,\mathbf{AMP},0.85)\}$

It is similar to above results, the following results are easily obtained.

Theorem 5.4. *The time complexity of Algorithm 5.3 is $O(n.p^4)$, where n is number of attributes in R and p is number of QFFDs in F.*

Lemma 5.6. *Algorithm 5.3. terminates.*

Lemma 5.7. *For all $j \geq k$ we have $NEWD^{(j)} \cup \subseteq, \leq TEMP^{(j)} = NEWD^{(j)}$.*

Lemma 5.8. *Suppose that $Q(X \rightsquigarrow_\varphi V) \in F^+$ and $(Z, Q_z, \varphi_Z) \in NEWD^{(k)}$, $|Q| \leq |Q_Z|, \varphi \leq \varphi_Z$, and $V \subseteq Z$. If $V \rightsquigarrow_\varphi Y \in F^+$, then $\exists(W, Q_W, \varphi_W) \in NEWD^{(k)}$ such that $|Q| \leq |Q_W|$, $\varphi \leq \varphi_W$, and $YZ \subseteq W$.*

Lemma 5.9. *If $Q(U \rightsquigarrow_\varphi Y) \in F^+$, and $(Z, \mathbf{FA}, \varphi_Z) \in NEWD^{(k)}$, $U \subseteq Z$, $\varphi \leq \varphi_Z$ then $\exists(W, Q_W, \varphi_W) \in NEWD^{(k)}$ such that $|Q| \leq |Q_W|$, $\varphi \leq \varphi_W$, and $YZ \subseteq W$.*

Lemma 5.10. *$NEWD^{(k)} = \overline{Dep}_F(X)$ if and only if following conditions holds*
(i) *If $(Y, Q, \varphi) \in NEWD^{(k)}$ then $Q(X \rightsquigarrow_\varphi Y) \in F^+$.*
(ii) *If $Q(X \rightsquigarrow_\varphi Y) \in F^+$, then $\exists(Z, Q_Z, \varphi_Z) \in NEWD^{(k)}$ such that $|Q| \leq |Q_Z|$, $\varphi \leq \varphi_Z$, and $Y \subseteq Z$.*

Theorem 5.5. *Algorithm 5.3 is correct.*

6. CONCLUSION

This paper deals with concept of Fuzzy functional dependency with linguistic quantifiers. The main result of this work is some algorithms for computing minimum family of dependency sets. A futher study involving, the completeness of axioms for QFFDs, an extension to fuzzy multivalued dependency with linguistic quantifier, and its applications in knowledge discovery from fuzzy data has been on going.

REFERENCES

[1] Bhattacharjee T. K., Mazumdar A. K., Axiomatisation of fuzzy multivalued dependencies in a fuzzy relational data model, *Fuzzy Sets and Systems* **96** (1998) 343–352.

[2] Buckles B. P., Petry F. E., A fuzzy representation of data for relational databases, *Fuzzy Sets and Systems* **7** (1982) 213–226.

[3] Buckles B. P., Petry F. E., Information-theoretic characterization of fuzzy relational databases, *IEEE Trans. Sys. Man. Cybernet.* **13** (1983) 74–77.

[4] Chen G., Kerre E. E., Vandenbulcke J., A computational algorithm for the FFD transitive closure and a complete axiomatization of fuzzy functional dependence, *International Journal of Intelligent Systems* **9** (1994) 421–439.

[5] Chen G., Fuzzy functional dependency as a sort of semantic knowledge: representation, preservation and use, in: *Knowledge Management in Fuzzy Databases*, Pons O., Vila M. A., Kacprzyk J., eds., Springer-Verlag, New-York, 2000, 122–134.

[6] Cubero J. C., Vila M. A., A new definition of fuzzy functional dependency in fuzzy relational databases, *International Journal of Intelligent Systems* **9** (1994) 441–448.

[7] Dubois D., Prade H., *Possibility Theory: An Approach to Computerized Processing of Uncertainty*, Plenum Press, New York, 1988.

[8] H. Thuan, T. T. Thanh, On the functional dependencies and multivalued dependencies in fuzzy relational databases, *Journal of Computer Science and Cybernetics* **17** (2) (2001) 13–19.

[9] Jyothi S., Babu M. S., Multivalued dependencies in fuzzy relational databases and lossless join decomposition, *Fuzzy Sets and Systems* **88** (1997) 315–332.

[10] Kraft D. H., Petry F. E., Fuzzy information systems: managing uncertainly in databases and information retrieval systems, *Fuzzy Sets and Systems* **90** (1997) 183–191.

[11] L. T. Vuong, H. Thuan, A relational database extended by application of fuzzy set theory and linguistic variables, *Computers and Artifical Intelligence* **8** (2) (1989) 153–168.

[12] Nakata M., On inference rules of dependencies in fuzzy relational data models: Functional dependencies, in : *Knowledge Management in Fuzzy Databases*, Pons O., Vila M. A., Kacprzyk J., eds., Springer-Verlag, New-York, 2000, 36–66.

[13] Paredaens J., et al., *The Structure of the Relational Database Model*, Springer-Verlag, New-York, 1989.

[14] Petry F., Bosc P., *Fuzzy Databases: Principles and Applications*, Kluwer, Norwell, MA, 1996.

[15] Shenoi S., Melton A., An extended version of the fuzzy relational database model, *Information Sciences* **52** (1990) 35–52.

[16] Shenoi S., Melton A., and Fan L. T., Functional dependencies and normal forms in the fuzzy relational database model, *Information Sciences* **60** (1992) 1–28.

[17] Sözat M. I., Yazici A., A complete axiomatization for fuzzy functional and multivalued dependencies in fuzzy database relations, *Fuzzy Sets and Systems* **117** (2001) 161–181.

[18] Ullman J. D., *Principles of Database System*, Comp. Science Press, 1980.

[19] Yazici A. et al., The integrity constraints for similarity - based fuzzy relational databases, *International Journal of Intelligent Systems* **13** (1998) 641–659.

[20] Zadeh L., Fuzzy sets as a basic for theory of possibility, *Fuzzy Sets and Systems* **13** (1978) 3–28.

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Ho Thuan - National Institute of Information Technology, Hanoi.
Tran Thien Thanh - Pedagogical Institute of Quy Nhon, Binh Dinh.