Tạp chí Tin học và Điều khiển học, T. 18, S. 3 (2002), 226–230

RECOGNIZING DOMINATING CYCLES IS NP-HARD

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Abstract. We use $\omega(G)$ to denote the number of components in a given graph G. Chvátal [12] defines a graph G to be *1-tough* if $\omega(G-S) \leq |S|$ for every subset S of the vertex set V with $\omega(G-S) > 1$. Given a graph G, a cycle C is called a hamiltonian cycle if C containes all vertices of G. A cycle C is said to be *dominating* if and only if $G - C$ has no edge.

The problem of deciding the existence of a hamiltonian cycle in a given graph is known to be an NP-complete one, hence it is mostly investigated in special claseses, for example in 1-tough graph, or specially investigated for studying of dominated cycles.

In the following we show that the problem of deciding the existence of a dominating cycle in a given graph is NP-complete.

Tóm tắt. Với ký hiệu $\omega(G)$ là số thành phần liên thông của một đồ thị G cho trước. Chvátal[12] định nghĩa G là một đồ thị 1- tough nếu $\omega(G-S) \leq |S|$ cho mọi tập con S của tập đỉnh V của G với $\omega(G-S) > 1$. Cho trước một đồ thị G, một chu trình C được gọi là chu trình Hamilton nếu C đi qua tất cả các đỉnh của G. Một chu trình C được gọi là chu trình Dominating khi và chỉ khi $G - C$ không còn cạnh nào cả.

Vấn đề xác định xem sự tồn tại của chu trình Halmiton trong một đồ thị cho trước được biết là một vấn đề NP - đầy đủ, và do đó vấn đề này thường được xem xét trong các lớp đồ thị đặc biệt, chẳng hạn trong các đồ thị 1 - tough, hoặc được chuyển sang xem xét các đồ thị Dominating.

Trong phần sau đây, chúng ta chỉ ra rằng vấn đề xác định xem một đồ thị cho trước có chu trình Dominating hay không cũng vẫn còn là bài toán NP - đầy đủ.

1. INTRODUCTION

We begin with some definitions and convenient notation. We refer to [11] for undefined terminology and notation. All graphs here are finite and undirected graphs without loops and multiple edges.

The vertex set of a graph G is $V(G)$ and the edge set of G is $E(G)$. We use $\omega(G)$ to denote the number of components of G, and $\alpha(G)$ denotes the cardinality of a maximum set of independent vertices. If $v \in V(G)$ then $N_G(v)$ is the set of all vertices in $V(G)$ adjacent to v and $d_G(v) = |N_G(v)|$ is the degree of v in G. The minimum degree and the maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A vertex v of a graph G with n vertices is called a *total vertex* if $d(v) = n - 1$. Herein $\kappa(G)$ denotes respectively the vertex connectivity of a graph G. We let $n = V(G)$ throughout the paper. Following Chvátal [12] we define a graph G to be *1-tough* if $\omega(G-S) \leq |S|$ for every subset S of V with $\omega(G-S) > 1$. For $k \leq \alpha$ we denote by σ_k the minimum value of the degree sum of any k pairwise nonadjacent vertices and by $NC_k(G)$ the minimum cardinality of the neighborhood union of any k such vertices. For $k > \alpha$ we set $\sigma_k = k(n - \alpha(G))$ and $NC_k = n - \alpha(G)$. Instead of σ_1 and NC_1 we use the more common notation $\delta(G)$ and NC. If no ambiguity can arise, we some time write $d(v)$ instead of $d_G(v)$ and α instead of $\alpha(G)$, etc. Let S be a nonempty subset of $V(G)$. The subgraph of G the vertex set of which is S and whose edge set is the set of all edges in G joining two different vertices of S is called the *subgraph of G induced by S* and denoted by $G[S]$. Furthermore, we use some specific notation and terminology that does not occur in [11]. For two given graphs G_1 and G_2 with vertex sets $V(G_i)$ $(i = 1, 2)$ and edge sets $E(G_i)$ $(i = 1, 2)$, we write $G_1 \subseteq G_2$ if $V(G_1) \subseteq V(G_2)$ and $E(G_1) \subseteq E(G_2)$. If $V(G_1) \cap V(G_2) = \emptyset$, then we say that G_1 and G_2 are *disjoint*. We write $G_1 \cup G_2 \cup \ldots \cup G_s$ to denote the union of $s \geq 2$ pairwise disjoint graphs $G_1, \ldots, G_s$. If $G_1, G_2, \ldots, G_s$ are isomorphic to a graph G, then we write sG instead of $G_1 \cup G_2 \cup \ldots \cup G_s$. We denote by K_n

the complete graph on n vertices and by $\bar{G}$ the complement of a graph G. If G_i is the complete K_{r_i}, we will write $K_{r_1,r_2,\ldots,r_s}$ instead of $G_1 \cup G_2 \cup \ldots \cup G_s$.

Given a graph, we can represent it by an obvious pictorial "map" in which its vertices are represented by points and its edges by lines. Given such a map, several questions can be asked: "Is it possible to make a tour such that every road is traversed exactly once?", "Is it possible to design a tour that passes through every village exactly once and starts and ends in the same village?", "What is the longest tour in which no village is visited more than once?", etc.

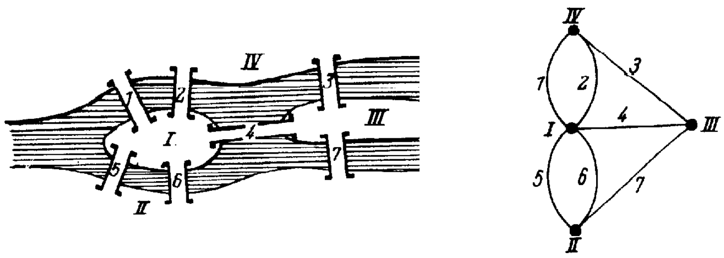

Figure 1

A tour that uses only vertices and edges of a graph G, such that every edge of G is traversed exactly once, and that returns in its initial vertex is called an *Euler tour* (Figure 1). A tour, again using only vertices and edges of G, such that every vertex is visited at most once and also returning in its initial vertex is called a *cycle* in G. Missing the last condition, it will be called a *path*. The length $\ell(C)$ of a longest cycle C in a graph G, called the *circumference* of G, is denoted by $c(G)$. A *hamiltonian cycle* of G is a cycle passing all vertices of G, and G is said to be *hamiltonian* if it contains such a cycle. Hamiltonian graphs are named after William Rowan Hamilton, although they were studied earlier by Kirkman. In 1856, Hamilton invented a mathematical game, the "icosian game", consisting of a dodecahedron each of whose twenty vertices was labeled with the name of a city. The object of the game was to travel along the edges of the dodecahedron, visiting each city exactly once and returning to the initial point (Figure 2).

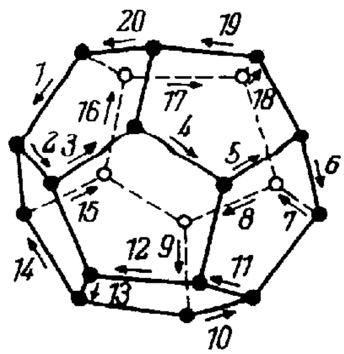

Figure 2

In fact, the beginning of both graph theory in general and of the theory of cycles in graphs, are marked by problems arising from these kinds of questions concerning the possiblities to make certain tours on a map. The problem whether a given graph contains an Euler tour was already solved by L. Euler, where an easy necessary and sufficient condition is given. In contrast, the question whether a given graph contains a hamiltonian cycle seems to be much harder to answer. Up to now, no easily verifiable conditions being both necessary and sufficient are known.

Given a graph G, a cycle C is said to be *dominating* if and only if $G - C$ has no edge (see Figure 3 for an example). The question "How difficult is it to recognize a dominating cycle in a given graph?" has remained an interesting open problem for some time in [4] and [10] ... Our purpose here is to show that the problem to decide the existence of a dominating cycle in a given graph is NP-complete. To prove this we will reduce the problem of deciding the existence of a dominating cycle in a given graph to the problem of deciding the existence of a hamiltonian cycle in graphs.

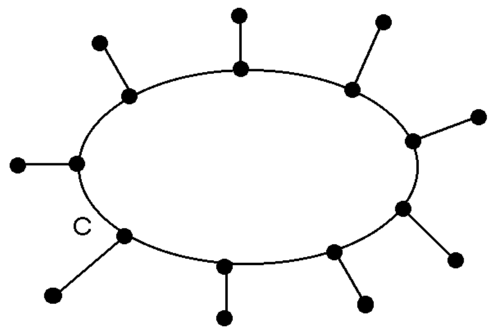

Figure 3

C. St. J. A. Nash-Williams [1] has proved the following result:

Theorem 1. *If G is a 2-connected graph of minimum degree $\delta(G)$ at least $r \geq 3$ on at most $3r-2$ vertices, then any longest cycle of G is dominating. Moreover, if $\delta(G) \geq \alpha(G)$, then G is hamiltonian.*

Nash-Williams' result stands at the very beginning of a discussion of dominating cycles. Let $\omega(G-S)$ denote the number of components of the graph $G[V(G)-S]$. A graph G is *1-tough* if $|S| \geq \omega(G-S)$ for any subset S of the vertex set $V(G)$ of G with $\omega(G-S) > 1$. Jung and Bigalke [6] strengthened Nash-Williams' result by studying 1-tough graphs G with $\delta(G) \geq \max\ \{\frac{n}{3}, \alpha(G)-1\}$. Let $\sigma_k(G)$ denote the minimum of degree sums of any k pairwise nonadjacent vertices if $k \leq \alpha(G)$, and $\sigma_k(G) = k(n-\alpha(G))$ if $k > \alpha(G)$. Instead of $\sigma_1(G)$ we use the more common notation $\delta(G)$. Bauer, Broersma, Veldman & Schmeichel [4] established lower bounds for the circumference of 1-tough graphs G with $\sigma_3 \geq n$, and proved that every longest cycle in G is a dominating cycle.

2. RESULTS

We begin by considering the following problem.

DOMINATING CYCLE

Instance: An undirected graph G.

Question: Does there exist a dominating cycle C in G?

To prove that the problem of deciding the existence of a dominating cycle in a given graph is an NP-complete one, we will reduce the following problem, which is known to be NP-complete [9].

HAMILTONIAN CYCLE

Instance: An undirected graph G.

Question: Does there exist a hamiltonian cycle C in G?

Our first goal is to establish

Theorem 2. *DOMINATING CYCLE is NP-complete.*

Proof. Clearly DOMINATING CYCLE $\in NP$, and we prove only that DOMINATING CYCLE is NP-hard. Let G be a graph with vertex ser $\{v_1, ..., v_n\}$. Construct G' from G as follows. Add to G a set $A = \{w_1, ..., w_n\}$ of independent vertices, and join v_i with w_i by an edge for $i = 1, 2, ..., n$ (Figure 4). To complete the proof, it suffices to show that G contains a hamiltonian cycle if and only if G' contains a dominating cycle.

Clearly, every hamiltonian cycle C in G is a dominating cycle in G'. Otherwise, we will prove that every dominating cycle C in G' is a hamiltonian cycle in G. Suppose that C is a dominating cycle in G' and that C avoids a vertex v_{i_0} in G, then the graph $G - C$ contains the edge (v_{i_0}, w_{i_0}), which contradicts the hypothese that C is a dominating cycle in G'. This contradiction shows that every dominating cycle in G' is a hamiltonian cycle in G. Thus, G contains a hamiltonian cycle if G' contains a dominating cycle.

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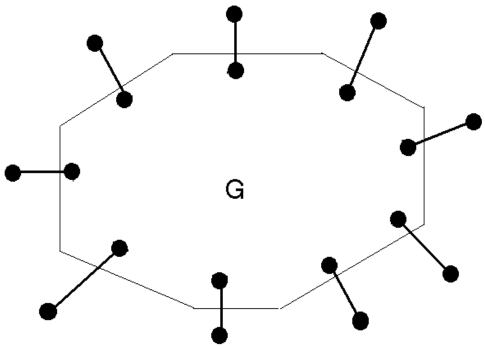

Figure 4

For what follows we will present the proof for the complexity of the problem to recognize not-1-tough graphs.

NOT-1-TOUGH

Instance: An undirected graph G.

Question: Does there exist a subset X of $V(G)$ such that $\omega(G-X) > \mid X \mid$?

With the same proof idea we will reduce the problem of recognizing 1-tough graphs to the independent set problem, which is known to be NP-complete [9].

INDEPENDENT MAJORITY

Instance: An undirected graph G.

Question: Does G contain an independent set I with $\mid I \mid \geq \mid V(G) \mid$?

The complexity of the general so called t-TOUGHNESS problem was solved in [5]. For unknown readers we present the proof for the following theorem:

Theorem 3. *NOT-1-TOUGH is NP-complete.*

Proof. It is easily to see that NOT-1-TOUGH $\in NP$, and it suffices to prove that NOT-1-TOUGH is NP-hard. Let G be a graph with vertex ser $\{v_1, ..., v_n\}$. Construct G' from G as follows. Add to G a set $A = \{w_1, ..., w_n\}$ of independent vertices, and join v_i with w_i by an edge for $i = 1, 2, ..., n$. Then add another set B of $[\frac{1}{2}(n-1)]$ vertices which induces a complete graph, and join each vertex of B to every vertex of $V(G) \cup A$. To complete the proof, we show that G contains an independent set I with $\mid I \mid \geq \frac{1}{2}n$ if and only if G' is not 1-tough.

Suppose first that G contains an independent set $I \subseteq V(G)$ with $\mid I \mid \geq \frac{1}{2}n$. Define $X' \subseteq V(G')$ by $X' = (V(G) - I) \cup B$. Note that $\mid X' \mid \leq (n - \frac{1}{2}n) + [\frac{1}{2}(n-1)] < n$. But it is easy to verify that $\omega(G' - X') = n > \mid X' \mid$, and G' is not 1-tough.

Conversely, suppose G' is not 1-tough. Then there exists $X' \subseteq V(G')$ with $\omega(G' - X') > 1$ such that $\omega(G' - X') > \mid X' \mid$. Clearly $B \subseteq X'$, or else $\omega(G' - X') = 1$. We may also assume that $X' \cap A = \emptyset$, since otherwise we have $\omega(G' - (X' - A)) \geq \omega(G' - X') > \mid X' \mid > \mid X' - A \mid$, and we could use $X' - A$ instead of X'. Let $X = X' \cap V(G)$ so that $\mid X' \mid = \mid X \mid + [\frac{1}{2}(n-1)]$. It is easily checked that $\omega(G' - X') = \mid X \mid + \omega(G - X)$. From $\omega(G' - X') > \mid X' \mid$ we obtain $\omega(G - X) > [\frac{1}{2}(n-1)]$, and so $G - X$ contains at least $\frac{1}{2}n$ components. Choosing one vertex in each component of $G - X$ yields a set of at least $\frac{1}{2}n$ independent vertices in G. ■

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Received April 26 - 2002

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