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ON SOME PROPERTIES OF ORDERING RELATION IN NON-HOMOGENEOUS HEDGE ALGEBRAS

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Abstract. In this paper, we continue to investigate basic properties of non-homogeneous hedge algebras which was introduced in [6]. One of important problems of the consideration is how to determine ordering relationships between elements in a domain of linguistic values of a linguistic variables. Therefore, some criterion for comparison of any two elements will be established. Many properties related to ordering relationships between elements will also examined.

Tóm tắt. Trong bài báo này, chúng tôi tiếp tục nghiên cứu những tính chất cơ bản của đại số gia tử không thuần nhất đã được giới thiệu trong [6]. Vấn đề cần quan tâm là làm thế nào để xác định quan hệ thứ tự giữa các phần tử trong miền giá trị của biến ngôn ngữ. Vì thế, một số tiêu chuẩn so sánh giữa hai phần tử bất kỳ và nhiều tính chất liên quan tới quan hệ thứ tự giữa các phần tử cũng được thiết lập.

1. INTRODUCTION

People use natural languages to reason, to deduce conclusions, and to make decisions. Therefore, linguistic values are used rather than numerical values in human reasoning processes. In 1973, the notion of a linguistic variable was introduced by Zadeh. That is a variable whose values are words or sentences in a natural or artificial language. For example, *Age* is a linguistic variable which can take the values *young, Very young, Possibly young, Approximately young, Possibly young or Approximately young, old, Very old, Possibly old,...*. These linguistic terms are generated from primary terms *young* and *old* by means of hedges *Very, Possibly,...* and logical connectives *and, or, not* and so on. In addition, there is an ordering relation determined by their natural meaning. For example, we can recognize that *young*<*old, Very young* <*young, Very old*>*old,...*. The notion of a linguistic variable is one of the basic tools for fuzzy logic and approximate reasoning. In fuzzy logic in Zadeh's sense, the objectives of consideration in nature are semantics of linguistic terms such as *true, Very true, Possibly true, Approximately true, Possibly true or Approximately true, false, Very false, Possibly false,...*, but their meaning are modeled by fuzzy sets. In our approach, the meaning of terms are modeled by ordered - based structure of their linguistic domain. So, one of important problems is how to determine ordering relationships between elements in this domain. The paper is aimed to establish a fundamental criteria and examine some properties related to the ordering relationship.

As we have introduced in [6], a domain of linguistic values of a linguistic variable is considered as an algebraic structure $\underline{X} = (X, G, H, \leq)$, where G is a set of primary generators, H is a set of unary operations representing linguistic hedges, X is a set of linguistic terms generated algebraically from G by using hedges in H. Therefore, each element $x \in X$ can

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be expressed in the form $x = h_n...h_1c$, where $h_1, ..., h_n \in H$, $c \in G$, $\leq$ is a semantic ordering relation on X.

The authors of [5] have to assume H is homogeneous, i.e. each operation is either positive or negative w.r.t all the operations in the same graded level of H^+ or H^-. For example, V is negative w.r.t all A, P and ML in the same graded level of H^-. In our study, we assume that H^- may contain the hedge *"Not so"* which lies on the same graded level of A, P and ML, but V is positive w.r.t *"Not so"*, which destroys the homogeneous hypothesis. So, we have to establish a new system of axioms for these algebraic structures which are called non-homogeneous hedge algebras.

The presence of the hedge N (*"Not so"*) in the new hedge algebras brings certainly new characteristics related to the elements, whose representation form contains N. We denote by S the set of the form $N \wedge (h_1 \vee ... \vee h_p)$ or its dual form. As presented in [6], by their properties, the set S was decomposed into two sets denoted by S^+ and S^- and some notions need to be introduced. For convenience in sequel, we recall here these notions and the axioms of non-homogeneous hedge algebras.

Let us consider h and k are two operators in the same graded level of LH. If $lhx > hx$ iff $lkx > kx$ for all $l \in LH$, $x \in X$, then h and k are said to be homogeneous. If $lhx > hx$ iff $lkx < kx$ for all $l \in LH$, $x \in X$ then h and k are called anti-homogeneous. By LH_i^c we denote graded level of LH^c, where LH_i^c is the lattice freely generated from i^{th} graded level H_i^c.

Assume $h, k \in LH_i^c, h \neq k$ and $\delta \in LH^*$, (δ, hx, kx) is called a translative triple if the following conditions hold:

(i) $hx < kx$ and $\delta hx < \delta kx$.

(ii) For any $y \in LH(kx)$ such that $y \ngeq \delta kx$, δhx and y are incomparable.

(iii) For any $z \in LH(hx)$ such that $z \nleq \delta hx$, δkx and z are incomparable.

Recalling the following property is necessary to easily understand some formulations of statements.

Property 1.1. ([6]) *If $h, k \in S$ then the following assertions hold:*

(i) *Both h and k belong together to either S^+ or S^- iff h and k are homogeneous.*

(ii) *Either ($h \in S^+$ and $k \in S^-$) or ($h \in S^-$ and $k \in S^+$) iff h and k are anti-homogeneous.*

Definition 1.1. ([6]) An algebra $\underline{X} = (X, G, LH, \leq)$ is said to be a non-homogeneous hedge algebra, if the following axioms hold:

(N1) Suppose $h \in LH^+, k \in LH^-$ and $x \in X$. If $l \in S$ and $hlx \neq lx$ then hlx and klx are incomparable. If $l \in LH \backslash S$ then $hlx \geq lx$ iff $klx \leq lx$.

(N2) The unit operation V of LH^+ is either positive or negative w.r.t. any operation in LH.

(N3) Semantic heredity of independent terms - If u and v are independent then $x \notin LH(v)$ for any $x \in LH(u)$ and vice versa. If $x \neq hx$ then $x \notin LH(hx)$. Furthermore, if $hx \neq kx$ then hx and kx are independent.

(N4) Comparability

(i) If hx and kx are incomparable then so are $u \in LH(hx)$ and $v \in LH(kx)$. If $a, b \in G$ and $a < b$ then $LH(a) < LH(b)$. If $hx > x$ then $LH(hx) \geq hx$, for any $h \in S^+$ and $LH(hx) \leq hx$, for any $h \in S^-$. If $hx < x$ then $LH(hx) \leq hx$, for any $h \in S^+$ and $LH(hx) \geq hx$, for any $h \in S^-$.

(ii) For any $h \neq k$ and $h, k \in LH_i^c$, where $i \in SI^c$. If $hx < kx$ then, for every $\delta \in LH^*$, (δ, hx, kx) is a translative triple in the following cases:

- $h \notin S$ and $k \notin S$ or both h and k belong together to either S^+ or S^-.
- $h \notin S, k \in S$ and $\delta hx \geq hx$.

- $h \in S, k \notin S$ and $\delta kx \leq kx$.

(N5) If $u \in LH(x)$ and suppose that $u \notin LH(hx)$, for any $h \in LH_i^c$, $i \in I^c$, then $u \geq v(u \leq v)$ for $v \in LH(hx)$, $h \in LH_i^c$, implies $u \geq h^{'}v(u \leq h^{'}v)$, for any $h^{'} \in UOS$.

Before establishing basic criteria for comparing elements in X, it is useful to recall the following statements for easy reference.

Proposition 1.1. ([6]) *$h < N$ if $h \in S^-$ and $h > N$ if $h \in S^+$.*

Property 1.2. ([6]) *For any $h, k \in LH_{i_0}^c$ such that $hx \neq x$, $kx \neq x$, where $x \in X$, the following assertion holds:*

If $h \notin S$, but $k \in S$ then $hx < hx$ implies $LH(kx) \geq kx$ and $hx > kx$ implies $LH(kx) \leq kx$.

Property 1.3. ([6]) *For any $h, k \in S$, if $hx < kx$ and h, k do not belong to the same S^+ or S^-, then $LH(hx) \leq hx < kx \leq LH(kx)$, for every $x \in X$.*

Corollary 1.1. ([6]) *For every $x \in X$, if $hx < kx$ and h, k do not belong to the same LH_i^c, where $i \in SI^c$, then $\delta hx < \delta^{'} kx$, for any $\delta, \delta^{'} \in LH^*$.*

2. BASIC COMPARISON CRITERIA

As mentioned above, truth values in fuzzy logic can be taken in a linguistic domain of the linguistic variable *truth*, which can be considered as a hedge algebra. Ordered structure of hedge algebras are defined by the meaning of linguistic hedges and primary terms, and hence the ordering relation can be called a semantic ordering relation. Therefore, an important question arises is how can we determine the ordering relationship between any two elements of a non-homogeneous algebra. The following theorem answers this question.

First of all we recall the notation $x_{(j)}$ which is defined as follows: if $x = h_n ... h_1 u$ then $x_{(j)} = h_{j-1} ... h_1 u$ for $1 \leq j \leq n$, with a convention that $h_0 = I$, the identity on X, i.e. $x_{(1)} = h_0 u = u$. In this paper, when we write LH_i^c, it is understood that $i \in SI^c$, i.e. $|H_i| > 1$, otherwise it is noticed explicitly.

We need the following lemma.

Lemma 2.1. *Suppose $h, k \in LH_i^c$. For any $x \in LH(hw)$, $y \in LH(kw)$ and $w \in X$, the following assertions hold:*

(i) *If $hw < kw$ then either $x < y$ or x and y are incomparable.*

(ii) *If $x < y$ then $hw < kw$.*

Proof

(i) Suppose that $x = \delta hw$, $y = \delta' kw$ and $hw < kw$. Let us consider the following cases:

Case 1: $h, k \notin S$ or $h, k \in S^+$ or $h, k \in S^-$. Then, (δ, hw, kw) is a translative triple and it implies $\delta hw < \delta kw$. If $\delta kw \leq \delta' kw$ then $x < y$. If $\delta' kw \ngeq \delta kw$ then y and x are incomparable. That is (i) holds.

Case 2: $h \notin S$ and $k \in S$. Then, we have $LH(kw) \geq kw$, by Property 1.2. Therefore, if $\delta hw \leq hw$ then $x = \delta hw < kw \leq y$. If $\delta hw > hw$ then, by the axiom (N4)(ii), (δ, hx, kx) is a translative triple and hence by the same argument as in Case 1, the desired results follow.

Case 3: $h \in S$ and $k \notin S$. By Property 1.2, we have $LH(hw) \leq hw$. Hence, if $\delta' kw \geq kw$ then it follows that $x < y$. And if $\delta' kw < kw$ then, by the same argument as in Case 2, (δ', hx, kx) is a translative triple and the desired results follow.

Case 4: $h, k \in S$ and h, k do not belong to the same S^+ or S^-. By Property 1.3, $LH(hw) \leq hw < kx \leq LH(kw)$ which implies that $x < y$.

(ii) Suppose that $x < y$. If $hw = kw$ then w is a fixed point. It follows that $x = y$ which is contrary to the assumption. If hw and kw are incomparable then by (N4)(i), x and y are incomparable, and we obtain a contradiction. If $hw > kw$ then by (i) we have either $x > y$ or x and y are incomparable, which is a contradiction as well. Therefore, the only one case holds is that $hw < kw$. ■

Now we establish some comparison criteria which are given in the following theorem.

Theorem 2.1. *Let $x = h_n...h_1u$ and $y = k_m...k_1u$ be two arbitrary canonical representations of x and y w.r.t u, respectively. Then there exists an index $j \leq min(m, n) + 1$ such that $h_i = k_i$, for all $i < j$ (note that if for example $j = m+1$ then it can be understood under our convention that $k_j = I$, the identity), and*

(1) *$x = y$ iff $m = n$ and $h_i = k_i$, for all $i \leq n$.*

(2) *$x < y$ iff $h_jx_{(j)} < k_jx_{(j)}$ and either h_j and k_j do not belong to the same LH_i^c or h_j, k_j belong to the same LH_i^c and one of the following conditions holds (where for short, by (*) we denote the condition $[\delta h_jx_{(j)} \leq \delta' h_jx_{(j)}$ or $\delta k_jx_{(j)} \leq \delta' k_jx_{(j)}]$ and $\delta = h_n...h_{j+1}, \delta' = k_m...k_{j+1}$):*

(i) *h_j, k_j are homogeneous and condition (*),*

(ii) *$h_j \notin S$, $k_j \in S, \delta h_jx_{(j)} \geq h_jx_{(j)}$ and condition (*),*

(iii) *$h_j \notin S$, $k_j \in S$ and $\delta h_jx_{(j)} < h_jx_{(j)}$,*

(iv) *$h_j \in S$, $k_j \notin S$, $\delta' k_jx_{(j)} \leq k_jx_{(j)}$ and condition (*),*

(v) *$h_j \in S$, $k_j \notin S$ and $\delta' k_jx_{(j)} > k_jx_{(j)}$,*

(vi) *h_j, k_j are anti-homogeneous.*

Moreover, a more strict formulation of the necessity, in which the "or" in the condition (*) is replaced with "and", denoted by (*and), is still valid.

(3) *x and y are incomparable iff either $h_jx_{(j)}, k_jx_{(j)}$ are incomparable or $h_jx_{(j)} < k_jx_{(j)}$ and h_j, k_j belong to the same LH_i^c and one of the following conditions holds:*

(i) *h_j, k_j are homogenous and $\delta k_jx_{(j)} \nleq \delta' k_jx_{(j)}$ or $\delta h_jx_{(j)} \nleq \delta' h_jx_{(j)}$, denoted by (**).*

(ii) *$h_j \notin S, k_j \in S$, $\delta h_jx_{(j)} \geq h_jx_{(j)}$ and condition (**).*

(iii) *$h_j \in S, k_j \notin S$, $\delta k_jx_{(j)} \leq k_jx_{(j)}$ and condition (**).*

Moreover, a more strict formulation of the necessity, where (**) replaced by (**and), is still valid.

Proof. Let j be the greatest index such that $h_i = k_i$, for all $i < j$. It can easily be seen that such a j alway exists. If $j < min(m, n) + 1$ then $h_j \neq k_j$. If $j = min(m, n) + 1$ and suppose that $m < n$, then, with our convention, $k_j = I$ and we also have $h_j \neq k_j$.

(1) If $m = n$ and $h_i = k_i$, for all $i \leq n$ then two canonical representations of x and y is identical, and hence $x = y$. Now, we shall prove the necessary condition of assertion (1) by contradiction. Assume that two canonical representations of x and y are different. Then there exists $j \leq min(m, n) + 1$ such that $h_j \neq k_j$ and $h_i = k_i$, for all $i < j$. If $h_jx_{(j)} = k_jx_{(j)}$ then by (iii) of Theorem 5.1 [6], $x_{(j)}$ is a fixed point, which leads to the fact that $n = m$ and $h_i = k_i$, $i \leq n$, a contrary to the assumption. Therefore, $h_jx_{(j)} \neq k_jx_{(j)}$ and we have the following three cases:

a) $h_jx_{(j)}$ and $k_jx_{(j)}$ are incomparable. Then, by (N4)(i) x and y are also incomparable, a contradiction.

b) $h_jx_{(j)} < k_jx_{(j)}$. If h_j and k_j do not belong to the same LH_i^c then by assertion (i) of Corollary 1.1 , $x < y$ holds, a contradiction. If $h_j, k_j \in LH_i^c$ then by Lemma 2.1, we have either $x < y$ or x and y are incomparable, and this is a contradiction, as well.

c) For the case $h_jx_{(j)} > k_jx_{(j)}$, by similar argument as in Case b) we also obtain a contradiction.

Thus, we have proved that if $x = y$ then their canonical representations are identical.

(2) *Sufficiency:* Suppose that $h_jx_{(j)} < k_jx_{(j)}$. Then, if h_j and k_j do not belong to the same LH_i^c then $x < y$, by Corollary 1.1. If $h_j, k_j \in LH_i^c$, then it follows from (i) Lemma 2.1, where h_j, k_j, $x_{(j)}$ play the role of h, k and w, respectively, that either $x < y$ or x and y are incomparable. Now, notice that, by Property 1.1, the hypothesis on homogeneous in (i) is equivalent to the hypothesis in the first case of (N4)(ii). Therefore, if (i) or (ii) or (iv) holds, by (N4)(ii) then $(\delta, h_jx_{(j)}, k_jx_{(j)})$ or $(\delta', h_jx_{(j)}, k_jx_{(j)})$ are translative triples and hence we get $\delta h_jx_{(j)} < \delta k_jx_{(j)}$ or $\delta' h_jx_{(j)} < \delta' k_jx_{(j)}$. These inequalities and condition (*) lead to the desired inequality that $x < y$.

Now, assume (iii) holds. Since $h_j \notin S$, $k_j \in S$, by Property 1.2, we obtain $LH(k_jx_{(j)}) \geq k_jx_{(j)}$, which together with $\delta h_jx_{(j)} < h_jx_{(j)}$ lead to $x < y$.

Similarly, we have the proof for (v).

Suppose (vi) holds. By Property 1.1, h_j and k_j do not belong together to the same S^+ or S^-. In addition, since $h_jx_{(j)} < k_jx_{(j)}$, by Property 1.3, we have $LH(h_jx_{(j)}) \leq h_jx_{(j)} < k_jx_{(j)} \leq LH(k_jx_{(j)})$, that lead to the inequalities $x = \delta h_jx_{(j)} \leq h_jx_{(j)} < k_jx_{(j)} \leq \delta' k_jx_{(j)} = y$. So, the sufficiency part of the theorem has been proved completely.

Necessity: Assuming $x < y$, by (ii) Lemma 2.1, we have $h_jx_{(j)} < k_jx_{(j)}$. In addition, if h_j and k_j belong to the same LH_i^c then one of the following cases should holds:

(2.1) h_j, k_j are homogeneous or equivalently, either $h_j, k_j \notin S$ or they both belong to the same S^+ or S^-.

(2.2) Either $h_j \notin S$, $k_j \in S$ or $h_j \in S$, $k_j \notin S$.

(2.3) $h_j, k_j \in S$ and they are anti-homogenous or equivalently, h_j and k_j do not belong to the same S^+ or S^-.

If (2.1) holds, then by (N4)(ii), $(\delta, h_jx_{(j)}, k_jx_{(j)})$ is a translative triple, for all $\delta \in LH^*$. Hence, if $\delta h_jx_{(j)} \nleq \delta' h_jx_{(j)}$ holds then x and y should be incomparable, which contradicts the hypothesis. Thus, we have $\delta h_jx_{(j)} \leq \delta' h_jx_{(j)}$. By an analogous argument we also have $\delta' k_jx_{(j)} \geq \delta k_jx_{(j)}$. Therefore, condition (i) is true.

Now, we assume that $h_j \notin S, k_j \in S$. If $\delta h_jx_{(j)} \geq h_jx_{(j)}$, then by (N4)(ii), $(\delta, h_jx_{(j)}, k_jx_{(j)})$ is a translative triple, and by the same argument as above, we obtain (ii), else we obtain (iii).

If $h_j \in S$ and $k_j \notin S$, then by the same argument as for the case $h_j \notin S$ and $k_j \in S$, we obtain conditions (iv) and (v).

Since (2.3) is just the condition (vi), the proof of necessity (2) is completed.

(3) Now, we prove point (3) of the theorem.

Sufficiency: If $h_jx_{(j)}$ and $k_jx_{(j)}$ are incomparable then so are x and y, by (N4)(i). Now, let us suppose that $h_jx_{(j)} < k_jx_{(j)}$, where $h_j, k_j \in LH_i^c$, and one of the conditions $(i) - (iii)$ holds. By (N4)(ii), $(\delta, h_jx_{(j)}, k_jx_{(j)})$ is a translative triple, for all $\delta \in LH^*$. Therefore, if (**) is true, then x and y are incomparable.

Necessity: Assume that x and y are incomparable. If h_j and k_j do not belong to the same LH^c, then $h_{j-1} \in S$. Indeed, assuming that $h_{j-1} \notin S$, we have, by the axiom (N1), either $h_jh_{j-1}w > k_jh_{j-1}w$ or $h_jh_{j-1}w < k_jh_{j-1}w$. It means that either $h_jx_{(j)} > k_jx_{(j)}$ or $h_jx_{(j)} < k_jx_{(j)}$. Since h_j and k_j do not belong to the same LH_i^c and by Corollary 1.1, x and y are comparable, a contrary to the hypothesis. Thus, $h_{j-1} \in S$ and, again by the axiom (N1), $h_jx_{(j)}$ and $k_jx_{(j)}$ are incomparable.

If $h_j, k_j \in LH^c$ and h_j and k_j do not belong to the same LH_i^c, then h_j and k_j are always comparable. Since X and LH are compatible, $h_jx_{(j)}$ and $k_jx_{(j)}$ are also comparable. By Corollary 1.1, x and y are comparable, which contradicts the hypothesis. Therefore, we can suppose that $h_j, k_j \in LH_i^c$. Then, one of the following cases occurs:

Assume that h_j, k_j are homogeneous and $h_jx_{(j)} < k_jx_{(j)}$. By (N4)(ii), we have $\delta h_jx_{(j)} < \delta k_jx_{(j)}$. If $\delta k_jx_{(j)} \leq \delta' k_jx_{(j)}$ then $x = \delta h_jx_{(j)} < \delta k_jx_{(j)} \leq \delta' k_jx_{(j)} = y$, a contradiction. Thus, $\delta k_jx_{(j)} \not\leq \delta' k_jx_{(j)}$ holds. Similarly, we have $\delta h_jx_{(j)} \not\leq \delta' h_jx_{(j)}$, i.e. (**and) is true.

Now, assume that $h_j \notin S$, $k_j \in S$ and $h_jx_{(j)} < k_jx_{(j)}$. By Property 1.2, $LH(k_jx_{(j)}) \geq k_jx_{(j)}$ which implies that $y = \delta' k_jx_{(j)} \geq k_jx_{(j)}$. Therefore, if $\delta h_jx_{(j)} < h_jx_{(j)}$ then $x < y$, a contradiction. Hence, $\delta h_jx_{(j)} \geq h_jx_{(j)}$. Then, by (N4)(ii), $\delta h_jx_{(j)} < \delta k_jx_{(j)}$. If $\delta k_jx_{(j)} \leq \delta' k_jx_{(j)}$ then $x < y$. We also obtain a contradiction and hence, $\delta k_jx_{(j)} \not\leq \delta' k_jx_{(j)}$.

For the case $h_j \in S$, $k_j \notin S$, the proof is similar.

Finally, if $h_j, k_j \in S$ and h_j and k_j are anti-homogenous, then by Property 1.3, x and y are comparable, a contradiction. Therefore this case does not occur. This concludes the proof. ∎

3. SOME ADDITIONAL PROPERTIES RELATED TO ORDERING RELATIONSHIPS BETWEEN ELEMENTS

It is observed from Theorem 2.1 that determining the ordering relationship between two elements becomes very complicated, when h_j and k_j belong to the same graded level lattice LH_i^c, that is when the considered elements belong to $LH(LH_i^c[x])$, where $LH_i^c[x] = \{hx : h \in LH_i^c\}$, $i \in SI^c$. Therefore, in this section, we shall examine ordering relationships between elements in $LH(LH_i^c[x])$.

Theorem 3.1. *For any $x \in X$ and $i \in SI^c$, if there exists a hedge $h \in LH_i^c$ such that hx is a fixed point, then so is kx, for any $k \in LH_i^c$.*

Proof. First, we prove for the case where hx and kx are comparable. If $hx = kx$ then, by (iii) Theorem 5.1 ([6]), x is a fixed point and so the assertion in the theorem is evident. Assume that $hx < kx$. We shall prove the assertion by cases as follows:

(1) Assume that $h, k \notin S$ and hx is a fixed point. By (N4)(ii), (δ, hx, kx) is a translative triple for all δ, and hence $Vhx < Vkx$ and $Lhx < Lkx$. So, if $Vkx > kx$ then Vhx and kx are incomparable and it contradicts the assumption that $Vhx = hx < kx$.

If $Vkx < kx$ then it follows from $k \notin S$ and (N1) that $Lkx > kx$. So, we conclude that Lhx and kx are incomparable and it contradicts the assumption that $Lhx = hx < kx$. Since Vkx and kx are always comparable, the only remaining case holds is that $Vkx = kx$, and so kx is a fixed point.

(2) Assume in turn that $h \notin S$ and $k \in S$. By Property 1.2, we have $LH(kx) \geq kx$, which implies $Vkx \geq kx$. Since hx is a fixed point, we can write $Vhx \geq hx$. Therefore, one of the case in (N4)(ii) is satisfied and hence (V, hx, kx) is a translative triple. So, $Vhx < Vkx$. Now, if $Vkx > kx$ would hold, then Vhx and kx should be incomparable, a contradiction. Hence $Vkx = kx$, and so kx is a fixed point.

(3) Assume that $h \in S$ and $k \notin S$. The proof for this case is similar as in (1), using (N4)(ii) and (N1) and we can conclude that kx is a fixed point.

(4) Now, assume that $h, k \in S$. By Proposition 1.1, h and N, as well as k and N are comparable. Note that $N \notin S$, therefore, by Case (3) above, if hx is a fixed point then so is Nx. In turn, again by Case (3), kx is a fixed point.

For the case $hx > kx$, the proof is similar.

Now, it remains to assume that hx and kx are incomparable. Clearly, $(h \vee k)x$ and hx are comparable and so are $(h \vee k)x$ and kx. Therefore, as proved above, if hx is a fixed point, then so is $(h \vee k)x$ and, in turn, kx is a fixed point, as well. This concludes the proof of the theorem. ∎

Theorem 3.2. *Assume that $hx < kx$ and $h, k \in LH_i^c$, where $i \in SI^c$, and $x \in X$. For all $\delta \in LH^*$, if $\delta hx < hx$ and $\delta kx > kx$, then we have*

(i) *$h'\delta hx \leq \delta hx$ iff $h'\delta kx \geq \delta kx$, for any $h' \in LH$.*

(ii) *$h'\delta hx \geq \delta hx$ iff $h'\delta kx \leq \delta kx$, for any $h' \in LH$.*

Proof

(i) We shall prove the theorem by induction on the length of δ. Suppose that $|\delta| = 1$ and $\delta = h_1$, i.e. $h_1hx < hx$ and $h_1kx > kx$. Assume that $h'h_1hx \leq h_1hx$. Hence, h' is positive w.r.t h_1, and from $h_1kx > kx$ the desired inequality $h'h_1kx \geq h_1kx$ follows. Now, assume that $h'h_1hx \geq h_1hx$ which says that h' is negative w.r.t h_1, and so, $h'h_1kx \leq h_1kx$. Since h and k play a similar role, the proof for the sufficiency is similar. It shows that assertion (i) is true for $|\delta| = 1$.

Assume that the assertion is true for all $\delta \in LH^*$ such that $|\delta| \leq n$, where $n > 1$. We shall prove that the assertion is true for $|\delta| = n + 1$.

Consider any string of hedges $h'\delta$, where $\delta = h_nh_{n-1}...h_1$, $h_i \in LH$, $i = 1, ..., n$, and assume that $h'\delta hx \leq \delta hx$. If $h_nh_{n-1}...h_1hx \leq h_{n-1}...h_1hx$, it shows that h' is positive w.r.t h_n. From the induction hypothesis it follows that $h_nh_{n-1}...h_1kx \geq h_{n-1}...h_1kx$. Since h' is positive w.r.t h_n, we have $h'\delta kx \geq \delta kx$. If $h_nh_{n-1}...h_1hx > h_{n-1}...h_1hx$ then h' is negative w.r.t h_n. By a similar argument, we obtain $h'\delta kx \geq \delta kx$. The proof for the opposite direction is similar and it concludes the proof of (i).

Since the proof for (ii) is similar, the proof of the theorem is completed. ∎

As a generalization of Theorem 3.1, we have the following result.

Theorem 3.3. *For any $x \in X$ and $h,\ k \in LH_i^c$ we have δhx is a fixed point iff δkx is a fixed point for any $\delta \in LH^*$.*

Proof. Let δhx be a fixed point. First, we prove for the case where h and k are comparable. If $hx = kx$ then the assertion is trivial. Since the proof for the case where $hx > kx$ is quite similar, let us suppose that $hx < kx$. We shall examine case by case as follows:

(1) Assume that $h, k \notin S$ and in this case, by (N4)(ii), (δ, hx, kx) is a translative triple and so we have $\delta hx < \delta kx$. Assume the contrary that δkx is not a fixed point and that $\delta = k_m...k_1$. If $k_m \notin S$ then, by (N1), there exists $k' \in LH$ such that $k'\delta kx < \delta kx$. By the definition of a translative triple, δhx and $k'\delta kx$ are incomparable. Again by property of (δ, hx, kx), we have $k'\delta hx < k'\delta kx$. Since $k'\delta hx = \delta hx$, it follows that $\delta hx < k'\delta kx$, which contradicts the fact that δhx and $k'\delta kx$ are incomparable. Now, assume that $k_m \in S$. Then, by (N4)(i), there are only two possibilities: $LH(\delta kx) \leq \delta kx$ and $LH(\delta kx) \geq \delta kx$. If $LH(\delta kx) \leq \delta kx$, we have again the inequality $k'\delta kx < \delta kx$, and by the same argument, it leads to a contrary, as well. If $LH(\delta kx) \geq \delta kx$ then we have $k'\delta kx > \delta kx$. Since $(k'\delta, hx, kx)$ is a translative triple, δkx and $k'\delta hx$ are incomparable, which contradicts the fact that $k'\delta hx = \delta hx < \delta kx$. By contradiction, it proves that δkx is a fixed point.

(2) Assuming that $h \notin S$ and $k \in S$, we have $LH(kx) \geq kx$, by Property 1.2. If $\delta hx \geq hx$ then, by (N4)(ii), (δ, hx, kx) is a translative triple and the proof for this case is similar as in (1). Assume that $\delta hx < hx$. If $\delta kx = kx$ then kx is a fixed point, and by Theorem 3.1, hx is a fixed point. This contradicts the fact that $\delta hx < hx$. Hence, we should have $\delta kx > kx$. Since δhx is a fixed point, we have $h'\delta hx = \delta hx$, for any $h' \in LH$, and we can write $h'\delta hx \leq \delta hx$. By Theorem 3.2, we have $h'\delta kx \geq \delta kx$. On the other hand, we can also write $h'\delta hx \geq \delta hx$ and by Theorem 3.2, $h'\delta kx \leq \delta kx$. Therefore, we get $h'\delta kx = \delta kx$, i.e. δkx is a fixed point.

(3) For the case $h \in S$ and $k \notin S$, the proof is similar as for the Case (2).

(4) Now, assume that $h, k \in S$. Then, if h and k are anti-homogeneous then, by Property 1.3, we have $LH(hx) \leq hx$ and $LH(kx) \geq kx$. Assuming that $\delta hx = hx$, it says that hx is a fixed

point. By Theorem 3.1, kx is a fixed point and hence so is δkx. If $\delta hx < hx$ then $\delta kx > kx$ and the proof for this case is similar to Case (2). Since the proof for the case $\delta hx > hx$ is similar, we are going to consider the case that h and k are incomparable. Since we see that operations $h \vee k$ and h, as well as $h \vee k$ and k are always comparable, if δhx is a fixed point then, as proved above, so is $\delta(h \vee k)x$ and, in its turn, it implies that δkx is a fixed point.

Since h and k play a symmetrical role, we conclude the proof. ■

A question arises that if $\delta' hx < \delta hx$ or δhx and $\delta' hx$ are incomparable, then what relationships between $\delta' kx$ and δkx occur in the case that h and k are homogenous. The following theorem deduced directly from Proposition 3.4 ([5]) answers this question.

Theorem 3.4. *For any $h, k \in LH_i^c$, for some $i \in SI^c$, and for any $x \in X$, we have the following assertions:*

(i) $\delta hx > x (\delta hx < x)$ *iff* $\delta kx > x (\delta kx < x)$, *for any* $\delta \in LH^*$.

(ii) *If h and k are homogeneous and $hx \neq kx$ then δhx and $\delta' hx$ are incomparable iff δkx and $\delta' kx$ are incomparable, for any $\delta, \delta' \in LH^*$,*

(iii) *If h and k are homogeneous then $\delta hx > \delta' hx$ iff $\delta kx > \delta' kx$, for any $\delta, \delta' \in LH^*$.*

In order explain more clearly the structure of the non-homogeneous hedge algebras we formulate some additional results as corollaries. The following one is deduced from Theorem 3.4.

Corollary 3.1. *For any $h, k \in LH_i^c$, if $hx < kx$ and one of the following conditions holds:*

(i) *h and k are homogeneous,*

(ii) *$h \notin S, k \in S$ and $\delta hx \geq hx$ and $\delta' hx \geq hx$,*

(iii) *$h \in S, k \notin S$ and $\delta kx \leq kx$ and $\delta' kx \leq kx$.*

then we have the following assertions:

(a) *δhx and $\delta' hx$ are incomparable iff δkx and $\delta' kx$ are incomparable.*

(b) *$\delta' hx < \delta hx$ iff $\delta' kx < \delta kx$.*

As a consequence of Theorem 3.3, we have the following.

Corollary 3.2. *For any $h, k \in LH_i^c$, for all $\delta, \delta' \in LH^*$, if $hx < kx$ then $\delta hx \leq hx$ and $kx \leq \delta' kx$ when one of the following conditions holds:*

(i) *$h, k \in S$ and h and k are anti-homogeneous.*

(ii) *$h \notin S, k \in S$ and $\delta hx \leq hx$.*

(iii) *$h \in S, k \notin S$ and $\delta' kx \geq kx$.*

4. CONCLUSIONS

In this paper, a comparative criterion for determining the ordering relationship between elements has been established. It can be seen that by Theorem 2.1, we have an algorithm for determining the comparability of elements. The remaining results not only serve the above purpose but also clarify the structure of the non-homogeneous hedge algebra. These results shall pave the way for us to study further important properties such as lattice property and symmetric property of this algebra.

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