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ON A STOCHASTIC LINEAR DYNAMICAL SYSTEM DRIVEN BY A VOLTERRA PROCESS

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Abstract. The aim of this note is considering a dynamical system expressed by a Langevin equation driven by a Volterra process. An Ornstein - Uhlenbeck process as the solution of this kind of equation is described and a problem of state estimation (filtering) for this dynamical system is investigated as well.

Keywords. Dynamical system; Volterra driven - Langevin equation; State estimation.

1. INTRODUCTION

It is known that turbulent flows are characterized by rapid changes of their momentum, pressure and velocity in space and time. Similar to the case of distribution of log returns in financial market, that of log of velocity in moments for turbulent flows has been paid attention from researchers. And one realized that Volterra processes or Moving average processes are useful for studying this feature. A typical example is the fractional Brownian motion

$$B_t^H = \int_0^t K(t,s)dW_s$$

where $K(t,s) = \int_{-\infty}^{\infty} \left[(t-s)_+^{H-1/2} - (-s)_+^{H-1/2}\right] dW_s,$ W is the standard Brownian motion, H is Hurst index, $0 < H < 1$ [10, 13].

Many things have been done for study of Volterra processes in various types and its applications where driving processes can be some random measure or some stochastic process such as semimartingale while the kernel $K(s,t)$ is some real deterministic function (refer to [1, 2, 3, 8] or [5, 6]).

A. Basse and J. Pedersen in [1] have studied semimartingale property of continuous Lévy driven Volterra processes. And for such Volterra processes, in a most recently works by G.D.Nunno and al. [8], a kind of semimartingale approximation has been considered with applications to construction of pathwise fractional Volterra integration.

In this note, first we make some remarks on Volterra process of the form

$$Y_t = \int_0^t h(t-s)dL_s$$

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where $h(u)$ is a deterministic kernel and L_t is a Lévy process. In Section 2, a representation of $h(u)$ is given with an illustrated example. A dynamical system expressed by an Y_t−driven Langevin equation is considered in Section 3. And a state estimation of U_t from a semimartingale observation is investigated in the last Section 4.

2. A REPRESENTATION OF THE KERNEL $h(u)$.

Remark 2.1. Suppose $h = h(u), u > 0$ is a positive real valued function. Then $h(u)$ can be represented in the form

$$h(u) = \int_0^{+\infty} e^{-ux} d\mu, x > 0,$$

where μ is some positive measure on $(0, +\infty)$ and finite on compact sets.
Indeed, by putting

$$\mu(dx) = d\mu = uh(u)dx,$$

where dx is the Lebesgue measure on $(0, +\infty)$ we can see

$$\int_0^{\infty} e^{-ux} d\mu = \int_0^{+\infty} h(u)e^{-ux} d(ux) = h(u).$$

Example. In a work by G.Di Nunno and al [8] on approximation of Lévy - driven Volterra processes, the authors consider a very interesting example on Gamma Volterra proess $\int_0^t (t-s)^{\beta} e^{-\gamma(t-s)} dL_s$. For the kernel of this process $h(u) = u^{\beta} e^{-\gamma u}$ we can put $du = u^{\beta+1} e^{-\gamma u} dx$.

Inspired by an idea of the work [4] we can make the following notice.

Remark 2.2. Consider the Volterra process

$$Y_t = \int_0^t h(t-s) dL_s.$$

Since $h(u) = h(t-s) = \int_0^{+\infty} e^{-x(t-s)} \mu(dx)$ with $\mu(dx) = uh(u)dx$, then by Fubini theorem we get

$$Y_t = \int_0^{+\infty} \int_0^t e^{-x(t-s)} dL_s \mu(dx)$$

or

$$Y_t = \int_0^{+\infty} V_t^x \mu(dx)$$

where $V_t^x = \int_0^t e^{-x(t-s)} dL_s$ is an Ornstein - Uhlenbeck process driven by the Lévy process L_t.
So we can see that Y_t is a linear functional of a family of O - U processes V_t^x.

Then we can think of a spacial approximation of Y_t. For example we can write

$$Y_t \simeq \sum_{\xi_i \in (\pi)} C_i V_t^{\xi_i}$$

where (π) is some finite partition of an interval $[a, N] \subset (0, \infty)$ with $a > 0$ small enough and N large enough and coefficients C_i depend only on the kernel h.

Consider a partition (π) of $[a, N]$ consisting of N subintervals $[x_0, x_1), [x_1, x_2), \ldots, [x_{N-1}, x_N]$, $x_0 = a, X_N = N$. And choose a point $\xi_k \in [x_{k-1}, x_k)$ for each k, $k = 1, \ldots, N$. Then we get from each t

$$Y_t = \lim_{\substack{N \to \infty \\ a \to 0}} \lim_{\max |\Delta \mu_i| \to 0} \sum Y_t^{\xi_i} \Delta \mu_i$$

provided this limit exists, whatever the way a partition (π) is made and the points ξ_k are chosen.

3. A STOCHASTIC LINEAR DYNAMICAL SYSTEM

Consider the following dynamical system driven by a Volterra process Y_t

$$dU_t = -\lambda U_t dt + \sigma dY_t, \tag{1}$$

where λ and σ are positive constants and the Volterra process $Y_t = \int_0^t h(t-s) dL_t$ is supposed to be a semimartingale with a Lévy process L_t and a deterministic and derivable function $h : \mathbb{R}^+ \to \mathbb{R}$.

3.1. Proposition 3.1

The solution of equation (1) is given by

$$U_t = U_0 e^{-\lambda t} + \sigma \int_0^t e^{-\lambda(t-s)} dY_s \tag{2}$$

and called an Ornstein - Uhlenbeck process driven by the Volterra process Y_t.

Proof. It is easy to obtain the formula (2) by the method of variation of constant from the equation $dU_t = -\lambda U_t dt$. ■

In some works on Lévy driven Volterra processes and their applications as in [8], one needs to have an expression for the differential of Y_t or U_t.

3.2. Proposition 3.2

We have $dY_t = \left(\int_0^t h'(t-s) dL_t \right) dt + h(0) dL_t$ and $dU_t = (-\lambda U + \sigma \varphi(t)) dt + \sigma h(0) dt$

Proof. Firstly, we observe that for a Volterra process Y_t of the form

$$Y_t = \int_0^t h(t, s) dL_s$$

where $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ we have

$$dY_t = \left(\int_0^t \partial_t h(t, s) dL_s \right) dt + h(t, t) dL_s.$$

Indeed we have

$$dY_t = \int_0^{t+dt} h(t+dt, s)dL_s - \int_0^t h(t,s)dL_s$$

$$= \left(\int_0^t \partial_t h(t,s)dL_s\right) dt + h(t,t)dL_t.$$

In the case where $h(t,s) = h(t-s)$ we get

$$dY_t = \left(\int_0^t h'(t-s)dL_s\right) dt + h(0)dL_t$$

or

$$Y_t = \int_0^t \int_0^u h'(t-s)dL_s du + h(0)L_t \tag{*}$$

Now the equation (1) can be rewritten as

$$dU_t = (-\lambda U_t + \sigma\varphi(t))dt + \sigma h(0)dL_t, \tag{3}$$

where $\varphi(t) := \displaystyle\int_0^t h'(t-s)dL_s.$ ■

3.3. Remark

The relation (*) can be obtained by another way as follows:
In taking the differential of $h(t-s)L_s$ we have

$$d[h(t-s)L_s] = h(t-s)dL_s - h'(t-s)L_s ds.$$

Then

$$\int_0^t h(t-s)dL_s = \int_0^t h'(t-s)L_s ds + h(t-s)L_s\Big|_0^t$$

$$= \int_0^t \left(h'(t-s)\int_0^s dL_s\right) ds + h(0)L_t$$

$$= \int_0^t \int_0^s h'(t-s)dL_s ds + h(0)L_t$$

$$= \int_0^t \left(\int_0^u h'(t-s)dL_s\right) du + h(0)L_t.$$

4. A PROBLEM OF FILTERING

In this section we consider a problem of filtering (state estimation) for the dynamical system (1). Suppose that we can not obtain directly state values of the dynamics U_t and we can estimate them only via some observation process Z_t given by

$$Z_t = \int_0^t H_s ds + V_t \tag{4}$$

where $H_s = H(U_s),\ E\displaystyle\int_0^t H_s^2 ds < \infty$ and V_t is a Brownian motion independent of Y_t.

Suppose also that the Lévy process L_t in $Y_t = \int_0^t h(t-s)dL_s$ is with bounded jumps. Then the Lévy - Ito decomposition of L_t can be written as

$$L_t = B_t + \widetilde{N}_t + \alpha t \tag{5}$$

where B_t is a Brownian motion independent of $\widetilde{N}_t = \displaystyle\int_{|x|<1} x\left(N_t(.,dx) - t\nu(dx)\right),\ N_t(.,dx)$ is a Poisson process and $\nu(dx)$ is a measure on $\mathbb{R}\backslash\{0\}$ such that $\displaystyle\int min(1,x^2)\nu(dx) < \infty$, and

$$\alpha = E\left[X_1 - \int_{|x|>1} xN_1(.,dx)\right]$$

(cf Theorem 42 of [13]). Also, $\widetilde{N_t}$ is a simimartingale.
Now we have

$$\begin{aligned} dY_t &= \varphi(t)dt + h(0)dL_t = \varphi(t)dt + h(0)d(B_t + \widetilde{N_t} + \alpha t) \\ &= (\varphi(t) + \alpha h(0))dt + h(0)(dB_t + d\widetilde{N_t}). \end{aligned}$$

The state process given by

$$dU_t = -\lambda U_t dt + \sigma dY_t$$

can be splitted as $U_t = U_t^{(1)} + U_t^{(2)} + U_t^{(3)}$ that is expressed by the three following equations:

$$dU_t^{(1)} = -\lambda U_t^{(1)} dt + \sigma h(0) dB_t, \tag{6}$$

$$dU_t^{(2)} = -\lambda U_t^{(2)} dt + \sigma h(0) d\widetilde{N_t}, \tag{7}$$

$$\frac{dU_t^{(3)}}{dt} + \lambda U_t^{(3)} = \sigma(\varphi(t) + \alpha h(0)). \tag{8}$$

We have to find the filtering $\pi_t(U)$ of the state U_t from the observation Z_t (4)

$$\pi_t(U) := E(U_t|\mathcal{F}_t^Z),$$

where $\mathcal{F}_t^Z$ is the $\sigma-$algebra generated by all $Z_s,\ s \le t$.

We will find reparately state estimations of $U_t^{(1)},\ U_t^{(2)}$ and $U_t^{(3)}$.

(i) For $U_t^{(1)}$ the innovation method can be used. Note that $U_t^{(1)}$ is an Orstein - Uhlenbeck that is the solution of the Langevin equation (6)

$$U_t^{(1)} = U_t^{(1)} e^{-\lambda t} + \sigma h(0) \int_0^t e^{-\lambda(t-s)} dB_s. \tag{10}$$

So $U_t^{(1)}$ is a continuous semimartingale, then an approach by H.Kunita [10] to filtering formula can be applied. The innovation process I_t is defined as

$$I_t = Z_t - \int_0^t \pi_s(H) ds \tag{11}$$

and the state estimation for $U_t^{(1)}$ is given by

$$\pi_t(U^{(1)}) = \pi_0(U^{(1)}) - \lambda \int_0^t \pi_s(U^{(1)}) ds + \int_0^t \left[\pi_s(HU^{(1)}) - \pi_s(H)\pi_s(U^{(1)}) \right] dI_s. \tag{12}$$

(ii) For $U_t^{(2)}$ the above mentioned method can not be applied since it is a discontinuous process. We will use a Bayesian approach that led to a generalized Kallianpur - Striebel formula introduced by P.K. Mandal and V. Mandrekar [11] as follows

$$\pi_t(U^{(2)}) = E\left(U_t^{(2)} | \mathcal{F}_t^Z\right) = \frac{\int U_t^{(2)} \exp(\int_0^t H_s dZ_z - \frac{1}{2}\int_0^t H_s^2 ds) dP_{U_t^{(2)}}}{\int \exp(\int_0^t H_s dZ_s - \frac{1}{2}\int_0^t H_s^2 ds) dP_{U_t^{(2)}}} \tag{13}$$

where $P_{U_t^{(2)}}$ is the probability measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ induced from P by $U_t^{(2)}$.

(iii) As for $U_t^{(3)}$, it is the solution of equation (8)

$$U_t^{(3)} = e^{-\lambda t} U_0^{(3)} + \sigma \int_0^t e^{-\lambda(t-s)} (\varphi(s) + \alpha h(0)) ds.$$

It is simply to get that

$$\pi_t(U_t^{(3)}) = E\left(U_t^{(3)} | \mathcal{F}_t^Z\right) = e^{-\lambda t} \pi_t(U^0) + \sigma \pi_t(K) \tag{14}$$

where

$$K := \int_0^t e^{-\lambda(t-s)} \left(\varphi(s) + \alpha h(0)\right) ds.$$

We are now in the position to state the following.

Theorem 1. *The filtering $\pi_t(U) = E\left(U_t | \mathcal{F}_t^Z\right)$ of the state U_t (1) from the observation Z_t (4) is given by*

$$\pi_t(U) = \pi_t(U^{(1)}) + \pi_t(U^{(2)}) + \pi_t(U^{(3)}) \tag{15}$$

where $\pi_t(U^{(1)}), \pi_t(U^{(2)})$ and $\pi_t(U^{(3)})$ are given by (12), (13) and (14), respectively.

5. CONCLUSION

In taking part of the process of creating mathematical model that describes the physical behavior of turbulent flows and the distribution of log - returns in finance, this note introduced some facts of Volterra processes driven by a Levy process and a filtering problem for a linear dynamical system related to these Volterra processes. Of course, further studies on this direction are needed.

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