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DIFFERENCE SCHEMES FOR WEAK SOLUTIONS OF MIXED PROBLEM FOR A CLASS OF PARABOLIC DIFFERENTIAL EQUATIONS, I

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Abstract. The approximate methods for the problems of parabolic differentail equations with non-regular data are studied by some authors. For example in [1-3] are considered the cases of data belonging to the Sobolev spaces $W_p^m(\Omega)$. The difference schemes for solutions in $H^m(\Omega)$ of the boundary problems for the elliptic equations with variable coefficients were investigated in [4, 8]. In this paper we extend the ideas introduced in [4, 8] to consider the approximate solutions of the mixed problem in the Sobolevic spaces $H^{2m,m}(\Omega)$ for the parabolic equations (see sec. 2). In section 3 it is first time this approximate problem is considered in the space of Schwartz distributions $\mathcal{D}'(\Omega)$.

Tóm tắt. Các phương pháp xấp xỉ cho bài toán đối với các phương trình loại parabol với dữ liệu không trơn đã được nhiều tác giả nghiên cứu. Chẳng hạn trong [1 - 4, 8] đã xét các trường hợp dữ kiện thuộc các không gian Sobolev $W_p^m(\Omega)$. Trong bài này chúng tôi phát triển các phương pháp trong [4, 8] để nghiên cứu nghiệm sai phân của bài toán hỗn hợp trong các không gian kiểu Sobolev $H^{2m,m}(\Omega)$. Trong Mục 3, lần đầu tiên bài toán xấp xỉ này được xét trong không gian các phân bố Schwartz $\mathcal{D}'(\Omega)$ tương ứng với các dữ kiện có độ không trơn bậc cao.

1. INTRODUCTION

Consider the mixed problem for the following class of parabolic differential equations:

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(k_i(x)\frac{\partial u}{\partial x_i}\right) = f(x,t), \quad (x,t) \in \Omega, \tag{1}$$

$$u(x,t)\big|_{t=0} = \varphi(x), \quad x \in G, \tag{2}$$

$$u = 0, \quad (x,t) \in \Gamma, \tag{3}$$

where the coefficients $k_i(x) \in C^1(\overline{G})$, $k_i(x) \geq C > 0$, $i = 1, 2$, C is a constant, G is a bounded region in R^2, $\Omega = G \times (0,T) = \{(x,t) : x \in G,\ 0 < t < T < \infty\}$, $\Gamma = \partial G \times (0,T)$.

It is known that in many applied problems the data $f(x,t)$, $\varphi(x)$ are not continuously differentiable in the classical sense. In these cases the generalized solutions (GS) are considered. Below, at first, we consider GS u of the problem (1)–(3) in the Hilbert spaces $H^{2m,m}(\Omega)$ with the corresponding test functions v defined in the spaces $\overset{\circ}{H}{}^{2m,m}(\Omega)$, m being the nonnegative integers (see [5] for the spaces $H^{2m,m}(\Omega)$).

By [6] the GS $u(x,t)$ of the problem (1)–(3) satisfies the condition (3) and the following integral equality (u is extended by zero onto $\Omega^- = R^1(t) \times R^2(x) \setminus \overline{\Omega}$):

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$$\int\limits_{\overline{\Omega}} \left\{ \frac{\partial u(\zeta)}{\partial \zeta_3} - \sum_{i=1}^{2} \frac{\partial}{\partial \zeta_i} \left[k_i(\zeta_1, \zeta_2) \frac{\partial u}{\partial \zeta_i} \right] \right\} v(\zeta) d\zeta$$
$$= \int\limits_{\Omega} \left[f(\zeta) v(\zeta) + \varphi(\zeta_1, \zeta_2) v(\zeta_2, \zeta_2, 0) \right] d\zeta, \quad (4)$$

where $u(\zeta) \equiv u(\zeta_1, \zeta_2, \zeta_3)$, $\varphi(x) = \varphi(x_1, x_2)$, $v(x, t) \equiv v(x_1, x_2, t_3)$.

2. DIFFERENCE SCHEME FOR GS

2.1. Construction of difference scheme

For simplicity of presentation, let Ω be the unit cube:
$\Omega = \{(x, t) = (x_1, x_2, t_3) : 0 < x_1, x_2, t_3 < 1\}$. Let us introduce in Ω a grid $\overline{\omega}$:

$$\overline{\omega} = \Big\{ (x_1, x_2, t_3) : x_i \equiv x_{ij_i} = j_i h_i,\ t_3 = j_3 h_3;\ j_i = 0, 1, ..., N_i,$$
$$h_i = \frac{1}{N_i},\ i = 1, 2,\ j_3 = 0, 1, ..., M,\ h_3 = \frac{1}{M} \Big\},$$

where N_i and M are positive integers. For the steplengths, assume that $C_1 \leq \dfrac{h_1}{h_2} \leq C_2$, $C_3 \leq \dfrac{h_2}{h_3} \leq C_4$ uniformly as $h_1, h_2, h_3 \to 0$, here C_l, $l = \overline{1,4}$, being positive constants.

Denote the set of interior and boundary gridpoints of Ω by ω and γ respectively, $\gamma = \overline{\omega} \setminus \omega$.

To obtain a net problem we introduce an auxiliary cubic grid covered the cube Ω and containing three families of planes which are parallel to the coordinate planes x_1Ox_2, x_2Ot_3, t_3Ox_1 with steplength distances h_1, h_2, h_3 respectively. In Ω denote by ω^* this grid consisting of the parallelepipeds with centres at the gridpoint (x, t) of the grid ω. The cell of ω^* at the gridpoint (x, t) is denoted by e:

$$e \equiv e(x, t) = \{\zeta = (\zeta_1, \zeta_2, \zeta_3) : |\zeta_i - x_i| < 0.5\, h_i\,,\ i = 1, 2,\ |\zeta_3 - t_3| < 0.5\, h_3\}\,.$$

Now, as in [8] (see Sec. 1, Chap. 3) one may take the test function v in (4) by the form

$$v(x, t) = \begin{cases} (h_1\, h_2\, h_3)^{-1} & \text{for } (x, t) \in e, \\ 0 & \text{for } (x, t) \in \overline{\Omega} \setminus e. \end{cases} \quad (5)$$

Then, by (4), the GS u of the problem (1)–(3) satisfies the following integral equality:

$$\frac{1}{h_1 h_2 h_3} \int\limits_{e} \left\{ \frac{\partial u}{\partial \zeta_3} - \sum_{i=1}^{2} \frac{\partial}{\partial \zeta_i} \left[k_i(\zeta_1, \zeta_2) \frac{\partial u}{\partial \zeta_i} \right] \right\} d\zeta = \frac{1}{h_1 h_2 h_3} \int\limits_{e} \{ f(\zeta) + \varphi(\zeta_1, \zeta_2) \} d\zeta. \quad (6)$$

Let us set

$$S_\alpha u(y) \equiv \frac{1}{h_\alpha} \int\limits_{y_\alpha - 0.5 h_\alpha}^{y_\alpha + 0.5 h_\alpha} u(y_1, ..., \zeta_\alpha, ..., y_n) d\zeta_\alpha\,,$$

$$u^{(\pm k_\alpha)} \equiv u^{(\pm k_\alpha)}(y) = u(y_1, ..., y_\alpha \pm kh_\alpha, ..., y_n),$$

$$u_{y_\alpha} \equiv u_{y_\alpha}(y) = \frac{u^{(+1_\alpha)} - u}{h_\alpha}, \quad u_{\overline{y}_\alpha} \equiv u_{\overline{y}_\alpha}(y) = \frac{u - u^{(-1_\alpha)}}{h_\alpha},$$

where $1 \le \alpha \le n, \quad k = 0,5; 1; ..., y \in R^n$, here $n = 3$.

Then, from (6) we have the following net problem for the GS $u(x,t)$ of the problem (1)–(3):

$$P^e u \equiv S_1 S_2 S_3 \frac{\partial u}{\partial \zeta_3} - \sum_{i=1}^{2} \left[S_3 S_{3-i} \left(k_i \frac{\partial u}{\partial x_i} \right)^{(-0.5_i)} \right]_{x_i} = Rf + S\varphi, \; (x,t) \in \omega, \qquad (7)$$
$$u = 0, \; (x,t) \in \gamma,$$

where $R \equiv S_1 S_2 S_3, \; S \equiv S_1 S_2$.

Now, to obtain the difference schemes of the operator $P^e u$ (7), one may approximate the mean integral operator S_i by the quadrature formula of average rectangles and the partial derivatives by difference quotients as in [8]. For instance, one has

$$S_3 S_2 \left(k_1 \frac{\partial u}{\partial x_1} \right)^{(-0.5_1)}$$
$$= \frac{1}{h_3 h_2} \int\limits_{t_3-0.5h_3}^{t_3+0.5h_3} \int\limits_{x_2-0.5h_2}^{x_2+0.5h_2} k_1(x_1 - 0.5h_1, z_2) \frac{\partial u}{\partial x_1}(x_1 - 0.5h_1, z_2, z_3) dz_2 dz_3$$
$$\approx k_1^{(-0.5_1)} u_{\overline{x}_1}.$$

Then, one has the following difference scheme of the problem (1)–(3)

$$P_h^e y = y_{\overline{t}_3} - \sum_{i=1}^{2} \left(k_i^{(-0.5_i)} y_{\overline{x}_i} \right)_{x_i} = q(x_1, x_2, t_3), \; (x,t) \in \omega, \qquad (8)$$
$$y(x,t) = 0, \; (x,t) \in \gamma,$$

where $q = Rf + S\varphi$.

The difference schemes of the form (8) are investigated by many authors (see, e.g., [9]).

The scheme (8) may be written in the form:

$$P_h^e y = S_1 S_2 S_3 y_{\overline{t}_3} - \sum_{i=1}^{2} \left(k_i^{(-0.5_i)} y_{\overline{x}_i} \right)_{x_i} = q(x_1, x_2, t_3), \; (x,t) \in \omega, \qquad (9)$$
$$y(x,t) = 0, \; (x,t) \in \gamma,$$

2.2. Estimation of the convergence rate

Consider now the convergence of the approximate solution y to the GS u of the form (4), (5) of the problem (1)–(3). For this purpose we estimate the method error $z \equiv y - u$ of the scheme (9). From (9) one has

$$Ly \equiv \sum_{i=1}^{2} \left(k_i^{(-0.5_i)} y_{\overline{x}_i} \right)_{x_i} = S_1 S_2 S_3 y_{\overline{t}_3} - q \equiv p, \quad (x,t) \in \omega.$$

Hence, by (7) and (9),

$$Lz = p - Lu \equiv -\Psi(x,t),\ (x,t) \in \omega,$$
$$z(x,t) = 0,\ (x,t) \in \gamma,$$

where

$$\Psi = \sum_{i=1}^{2} \left(k_i^{(-0.5_i)} u_{\overline{x}_i}\right)_{x_i} - \sum_{i=1}^{2} \left[S_3 S_{3-i} \left(k_i \frac{\partial u}{\partial x_i}\right)^{(-0.5_i)}\right] + S_1 S_2 S_3 \left(\frac{\partial u}{\partial \zeta_3} - y_{\overline{t}_3}\right).$$

Thus,

$$-\sum_{i=1}^{2} \left(k_i^{(-0.5_i)} z_{\overline{x}_i}\right)_{x_i} = \sum_{i=1}^{2} \left(\eta_i\right)_{x_i} + \lambda_0,\ (x,t) \in \omega, \qquad (10)$$
$$z(x,t) = 0,\ (x,t) \in \gamma,$$

where

$$\eta_i = k_i^{(-0.5_i)} u_{\overline{x}_i} = S_3 S_{3-i} \left(k_i \frac{\partial u}{\partial x_i}\right)^{(-0.5_i)},\quad \lambda_0 = S_2 S_2 S_3 \left(\frac{\partial u}{\partial \zeta_3} - y_{\overline{t}_3}\right). \qquad (11)$$

Now, to obtain the error estimation, consider the space H of grid functions u on $\overline{\omega}$ and let H_0 be its subset of the functions satisfying the condition $u(x,t) = 0$ as $(x,t) \in \gamma$.

Let $a(x,t)$, $b(x,t) \in H_0$ or H. Introduce the following scalar products and corresponding grid norms:

$$(a,b) = \sum_{(x,t)\in\omega} a(x,t) b(x,t) h_1 h_2 h_3\,,$$
$$(a,b]_1^{j_0} = \sum_{j=0}^{M} \sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2-1} a(i_1h_1, i_2h_2, j_3h_3) b(i_1h_1, i_2h_2, j_3h_3) h_1 h_2 h_3\,,$$
$$(a,b]_2^{j_0} = \sum_{j=0}^{M} \sum_{i_1=1}^{N_1-1} \sum_{i_2=1}^{N_2} a(i_1h_1, i_2h_2, j_3h_3) b(i_1h_1, i_2h_2, j_3h_3) h_1 h_2 h_3\,,$$
$$\|a\|^2 \equiv \|a\|_{0,\omega}^2 = (a,a),\quad \|a]|_{i0}^2 = (a,a]_i^{j_0},\quad i = 1,2.$$

Let us scalar multiply both sides of (10) with $z(x,t)$:

$$-\sum_{i=1}^{2} \left(\left(k_i^{(-0.5_i)} z_{\overline{x}_i}\right)_{x_i}, z\right) = \sum_{i=1}^{2} \left(\eta_{i_{x_i}}, z\right) + (\lambda_0, z).$$

Then, by the same way as in Sec. 2.2 [4] one has

$$\|z\|_{1,\omega} \le C\left(\|\eta_1]|_{10} + \|\eta_2]|_{20} + \|\lambda_0\|\right), \qquad (12)$$

where the constant C is independent of h and z $\left(|h|^2 = h_1^2 + h_2^2 + h_3^2\right)$,

$$\|z\|_{1,\omega} \equiv \|z\|_{0,\omega}^2 + \|\nabla z\|_{0,\omega}^2\,,\quad \|\nabla z\|_{0,\omega}^2 \equiv \sum_{i=1}^{2} \|z_{\overline{x}_i}]|_{i0}^2\,.$$

To estimate the terms in the right-hand side of (12), we introduce in the space $H^{2m,m})(\Omega)$ following seminorms and norms:

$$|u|_{2m,\Omega} = \Big[\sum_{|\alpha|+2\beta=2m} \int_\Omega \Big|D_x^\alpha D_t^\beta u\Big|^2 dxdt\Big]^{1/2},$$

$$\|u\|_{2m,\Omega} = \Big[\sum_{|\alpha|+2\beta\le 2m} \int_\Omega \Big|D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} D_t^\beta\Big| dxdt\Big]^{1/2},$$

where $|\alpha| = \alpha_1 + \alpha_2$, α_i and β being the nonnegative integers.

Now, consider the functional $\eta_1(x,t)$ defined by (11)

$$\eta_1(x,t) = k_1(x_1 - 0.5h_1, x_2)u_{\overline{x}_1} - \frac{1}{h_3}\int_{t_3-0.5h_3}^{t_3+0.5h_3} \times$$
$$\Big\{\frac{1}{h_2}\int\limits_{x_2-0.5h_2}^{x_2+0.5h_2} k_1(x_1-0.5h_1,\zeta_2)\frac{\partial u}{\partial\zeta_1}(x_1-0.5h_1,\zeta_2,\zeta_3)d\zeta_2\Big\}d\zeta_3.$$

The expression of $\eta_i(x,t)$ is anologous to the one of η_i in Sec. 2 [4], then in the same way as we did for the estimation (26) in that section, one has

$$\|\eta_i\|_{i0} \le C|h|\,\|u\|_{2,\Omega}, \quad i = 1, 2. \tag{13}$$

For the term λ_0, by (11) one has

$$\lambda_0 = S_1S_2S_3\left(\frac{\partial u}{\partial\zeta_3} - u_{\overline{t}_3}\right) + S_1S_2S_3\left(u_{\overline{t}_3} - y_{\overline{t}_3}\right) \equiv \lambda_0^1 + \lambda_0^2. \tag{14}$$

By the Cauchy–Buniakovskij inequality one has

$$|\lambda_0^1| \le (h_1h_2h_3)^{-1/2}\Big\{\int_e\Big[\frac{\partial u(\zeta)}{\partial\zeta_3} - u_{\overline{t}_3}(x,t)\Big]^2 d\zeta\Big\}^{1/2}.$$

One has

$$\int_e\Big[\frac{\partial u}{\partial\zeta_3} - u_{\overline{t}_3}(x,t)\Big]^2 d\zeta = \int_e\Big\{h_3^{-1}\int\limits_{t_3-h_3}^{t_3}\Big[\frac{\partial u(\zeta)}{\partial\zeta_3} - \frac{\partial u(x_1,x_2,\alpha)}{\partial\alpha}\Big]d\alpha\Big\}^2 d\zeta.$$

Then,

$$|\lambda_0^1| \le C(h_1h_2h_3)^{-1/2}|h|\left(|u|_{2,e} + |u|_{2,e^3}\right),$$

where

$$e^3 \equiv e^3(x,t) = \{\zeta = (\zeta_1,\zeta_2,\zeta_3) : |\zeta_i - x_i| < 0.5h_i,\ i = 1,2;\ t_3 - h_3 < \zeta_3 < t_3\}.$$

Hence,

$$\|\lambda_0^1\| = \Big[\sum_{(x,t)\in\omega}|\lambda_0^1|^2h_1h_2h_3\Big]^{1/2} \le C|h|\,|u|_{2,\Omega}. \tag{15}$$

From (14) and (15) it follows

$$\|\lambda_0\| \leq C|h|\,\|u\|_{2,\Omega}. \tag{16}$$

Combining (12), (13) and (16) yelds

$$\|z\|_{1,\omega} = \|y-u\|_{1,\omega} \leq C|h|\,\|u\|_{2,\Omega}. \tag{17}$$

Further, for the problem (1)–(3) one has the following a priori estimate (see [5, Sec. 2, Chap. 6]):

$$\|u\|_{2,\Omega} \leq C\big(\|\varphi\|_{H^1(G)} + \|f\|_{L^2(\Omega)}\big),$$

where the constant C is independent of φ and f.

Finally, from the last inequality and (17) one has the following

Theorem 1. *Let the given functions $f(x,t) \in L_2(\Omega)$, $\varphi(x) \in H^1(G)$ and $k_i(x) \in W^1_\infty(G) \cap C(\overline{G})$, $i = 1, 2$. Then the solution y of the difference scheme (8) converges to the GS (6) $u(x,t)$ ($u \in H^{2,1}(\Omega)$) of the problem (1)–(3) in the grid norm $\|\cdot\|_{1,\omega}$ with the rate $O(|h|)$, such that one has the following error estimation*

$$\|y-u\|_{1,\omega} \leq C|h|\,\|u\|_{2,\Omega},$$

where the constant C is independent of h and $u(x,t)$.

3. DIFFERENCE SCHEMES FOR WEAK SOLUTION

Now consider the GS $u(x,t)$ of the problem (1)–(3) in the form (4), where the test functions have the form

$$v(x,t) = \begin{cases} (4\pi h_1^l h_2^l h_3^l)^{-3/2} \exp\Big\{-\frac{x_1^2+x_2^2+t_3^2}{4h_1^l h_2^l h_3^l}\Big\}, & (x,t) \in e, \\ 0 \qquad , & (x,t) \in \overline{\Omega} \setminus e, \end{cases} \tag{18}$$

where l is a positive integer.

Then, by (4) the GS $u(x,t)$ of the problem (1)–(3) satisfies the following equality:

$$(h_1h_2h_3)^{-1} \int_e \left\{ \frac{\partial u(\zeta)}{\partial \zeta_3} - \sum_{i=1}^{2} \frac{\partial}{\partial \zeta_i}\left[k_i(\zeta_1,\zeta_2)\frac{\partial u}{\partial \zeta_i}\right]\right\} \alpha(\zeta)d\zeta =$$
$$(h_1h_2h_3)^{-1} \int_e \{f(\zeta)\alpha(\zeta) + \varphi(\zeta_1,\zeta_2)\alpha(\zeta_1,\zeta_2,0)\}\,d\zeta, \tag{19}$$

where $\alpha(\zeta) = h_1h_2h_3\,v(\zeta)$.

Thus, one has the following net problem for the GS $u(x,t)$:

$$\begin{aligned} P^e u \equiv S_1S_2S_3\,\alpha\frac{\partial u}{\partial \zeta_3} - \sum_{i=1}^{2}\left[S_3S_{3-i}\left(\alpha k_i \frac{\partial u}{\partial x_i}\right)^{(-0.5_i)}\right]_{x_i} \\ + S_1S_2S_3\sum_{i=1}^{2} k_i(\zeta_1,\zeta_2)\frac{\partial \alpha}{\partial \zeta_i}\frac{\partial u}{\partial \zeta_i} = \widetilde{R}f + \widetilde{S}\varphi \equiv \tilde{q},\ (x,t)\in\omega, \\ u(x,t) = 0,\ (x,t)\in\gamma, \end{aligned} \tag{20}$$

where $\widetilde{R}f = S_1S_2S_3\,\alpha(\zeta)f(\zeta),\ \widetilde{S}\varphi = S_1S_2\left[\alpha(\zeta_1,\zeta_2,0)\varphi(\zeta_1,\zeta_2)\right].$

3.1. Difference schemes

Arguing as in the proof of the form (8), Sec. 2.1, we obtain the following difference scheme of the net problem (20):

$$\begin{aligned} {}^1P_h^e\breve{y} \equiv S_1S_2S_3\,\alpha(\zeta)\breve{y}_{\bar{t}_3} - \sum_{i=1}^{2}\left(a_i\breve{y}_{\bar{x}_i}\right)_{x_i} + S_1S_2S_3\sum_{i=1}^{2}k_i(x)\alpha_{\bar{x}_i}\breve{y}_{\bar{x}_i}& \\ = \tilde{q}(x,t),\ (x,t)\in\omega,& \\ \breve{y}(x,t) = 0,\ (x,t)\in\gamma,& \end{aligned} \tag{21}$$

where $a_i = \alpha^{(-0.5_i)}k_i^{(-0.5_i)}(x)$.

Furthermore, as in [4] (see the forms (9) and (12) in Sec. 2.1) one has an other difference approximation for the problem (20):

$$\begin{aligned} {}^2P_h^e\hat{y} \equiv S_1S_2S_3\,\hat{y}_{\bar{t}_3} - \sum_{i=1}^{2}\left(b_i\hat{y}_{\bar{x}_i}\right)_{x_i} = \tilde{q},\ (x,t)\in\omega,& \\ \hat{y}(x,t) = 0,\ (x,t)\in\gamma,& \end{aligned} \tag{22}$$

where $b_i(x) = k_i^{(-0.5_i)}(x)$,

$$\tilde{q}(x,t) = S_1S_2S_3\frac{\partial u}{\partial\zeta_3} - \sum_{i=1}^{2}\left[S_3S_{3-i}\left(\alpha k_i\frac{\partial u}{\partial x_i}\right)^{(-0.5_i)}\right]_{x_i} + S_1S_2S_3\sum_{i=1}^{2}k_i(\zeta_1,\zeta_2)\frac{\partial\alpha}{\partial\zeta_i}\frac{\partial u}{\partial\zeta_i}. \tag{23}$$

3.2. Estimation of the convergence rate

a. Consider first the scheme (22). We see that the scheme (22) has the form (9), then by (17) one has

$$\|\hat{y}-u\|_{1,\omega} \le C|h|\,\|u\|_{2,\Omega}, \tag{24}$$

where $\hat{y}$ being the solution of the scheme (22), u being the GS of the problem (1)–(3).

Now consider the following mixed scheme:

$$\begin{aligned} My \equiv \frac{1}{2}\left({}^1P_h^e + {}^2P_h^e\right)y = \tilde{q},\ (x,t)\in\omega,& \\ y(x,t) = 0,\ (x,t)\in\gamma,& \end{aligned}$$

where $y = \dfrac{1}{2}\left(\breve{y}+\hat{y}\right)$, $\breve{y}$ and $\hat{y}$ being defined by (21) and (22) respectively.

Then, by (21) and (22),

$$\begin{aligned} My = \frac{1}{2}S_1S_2S_3\left[1+\alpha(\zeta)\right]y_{\bar{t}_3} - \frac{1}{2}\sum_{i=1}^{2}\left[(a_i+b_i)y_{\bar{x}_i}\right]_{x_i}& \\ + \frac{1}{2}S_1S_2S_3\sum_{i=1}^{2}k_i\alpha_{\bar{x}_i}y_{\bar{x}_i} = \tilde{q},\ (x,t)\in\omega,& \\ y(x,t) = 0,\ (x,t)\in\gamma.& \end{aligned} \tag{25}$$

Note that it can be verified that

$$\lim_{h_1,h_2,h_3\to 0}\int_{R^3} g(\zeta)v(\zeta)d\zeta = g(x,t),$$

where the function $g(\zeta)$ is summable in e and continuous at the gridpoint (x,t).

Hence, if h_1, h_2 and h_3 are sufficiently small, one may write (25) in the form:

$$\begin{gathered} y_{\bar{t}_3} - \frac{1}{2}\sum_{i=1}^{2}\left[k_i^{(-0.5_i)}\left(1+\alpha^{(-0.5_i)}\right)y_{\bar{x}_i}\right]_{x_i} + \frac{1}{2}\sum_{i=1}^{2}k_i\alpha_{\bar{x}_i}y_{\bar{x}_i} = \tilde{q},\ (x,t)\in\omega, \\ y(x,t)=0,\ (x,t)\in\gamma. \end{gathered} \tag{26}$$

By [9] it should be noted that there exists uniquely a solution of the difference problem (26) and, then, of the scheme (21).

Consider now (25), one has

$$\begin{aligned} M_0 y \equiv \sum_{i=1}^{2}[(a_i+b_i)y_{\bar{x}_i}]_{x_i} &= S_1S_2S_3[1+\alpha(\zeta)]y_{\bar{t}_3} \\ &+ S_1S_2S_3\sum_{i=1}^{2}k_i\alpha_{\bar{x}_i}y_{\bar{x}_i} - 2\tilde{q} \equiv \tilde{p},\ (x,t)\in\omega. \end{aligned} \tag{27}$$

Let $z \equiv y-u$, where u being the GS of the problem (1)–(3) in the form (4), (18). By (27), (23) and (20),

$$\begin{gathered} M_0 z = \tilde{p} - M_0 u \equiv -\psi(x,t),\ (x,t)\in\omega, \\ z(x,t)=0,\ (x,t)\in\gamma, \end{gathered} \tag{28}$$

where

$$\begin{aligned} &\psi = -\sum_{i=1}^{2}[(a_i+b_i)z_{\bar{x}_i}]_{x_i} = \sum_{i=1}^{2}[\eta_i+\mu_i]_{x_i} + \lambda_0+\lambda_1+\beta, \\ &\eta_i \equiv a_i u_{\bar{x}_i} - S_3S_{3-i}\left(\alpha k_i\frac{\partial u}{\partial x_i}\right)^{(-0.5_i)},\ \ \mu_i \equiv b_i u_{\bar{x}_i} - S_3S_{3-i}\left(\alpha k_i\frac{\partial u}{\partial x_i}\right)^{(-0.5_i)}, \\ &\lambda_0 \equiv S_1S_2S_3\left(\frac{\partial u}{\partial\zeta_3}-y_{\bar{t}_3}\right),\ \ \lambda_1 \equiv S_1S_2S_3\alpha\left(\frac{\partial u}{\partial\zeta_3}-y_{\bar{t}_3}\right), \\ &\beta \equiv S_1S_2S_3\sum_{i=1}^{2}k_i\left(2\frac{\partial\alpha}{\partial\zeta_i}\frac{\partial u}{\partial\zeta_i}-\alpha_{\bar{x}_i}y_{\bar{x}_i}\right). \end{aligned} \tag{29}$$

From (28), (29) one has the following inequality analogous to (12):

$$\|z\|_{1,\omega} \le C\left(\sum_{i=1}^{2}[\|\eta_i\|_{i0}+\|\mu_i\|_{i0}] + \|\lambda_0\| + \|\lambda_1\| + \|\beta\|\right).$$

The terms in the right hand side of the last inequality are estimated by exactly the same manners as they were done above in Sec. 2 and as in [4] (see Sec. 2). Then one has

$$\|z\|_{1,\omega} = \frac{1}{2}\left\|\breve{y}+\hat{y}-2u\right\|_{1,\omega} \le C|h|\,\|u\|_{2,\Omega}. \tag{30}$$

Finally, by (24) and (30) we get the estimation of method error for the difference scheme (21):

$$\left\|\breve{y} - u\right\|_{1,\omega} \le C|h|\, \|u\|_{2,\Omega}. \tag{31}$$

Remark. By a manner analogous to the proof of the inequality (31), one may verify that this estimation is also valid if, in the form of the GS $u(x,t)$ (4), $v(x,t)$ is any test function in the Schwartz basic space $\mathcal{D}(\Omega')$, $\Omega' \Subset \Omega$.

b. The estimation (31) is obtained with the assumption $f \in L_2(\Omega)$ and $\varphi \in H^1(G)$, we now show that the result may be generalized to the equations with right hand side $f \in \mathcal{D}'(\Omega)$ and the initial condition $\varphi \in \mathcal{D}'(G)$, here $\mathcal{D}'(G)$ being the space of Schwartz distributions on G [7].

Indeed, by our assumption and by the theorem on local structure of distributions and its corollary, there exist the functions $g(x,t) \in L_2(e)$, $\psi(x) \in H^1(e_0)$ and the nonnegative integers k_1, k_2 such that

$$f = D_x^{k_1} D_t^{k_1} g(x,t), \quad \varphi \in D_x^{k_2}\psi, \tag{32}$$

where $D_x^k \equiv \dfrac{\partial^{2k}}{\partial x_1^{k_1}\partial x_2^{k_2}}$, the open set $e \Subset \Omega \subset R^2(x) \times R^1(t)$, $e_0 \equiv \{(x,t) : (x,t) \in e,\ t = 0\}$.

Let $v(x,t) \in \mathcal{D}(e)$. By [6], the GS u of the problem (1)–(3) satisfies the following equality:

$$\left\langle \frac{\partial \tilde{u}}{\partial t} - \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(k_i \frac{\partial \tilde{u}}{\partial x_i}\right), v \right\rangle = \left\langle \left[\tilde{f} + \tilde{\varphi}(x) \times \delta(t)\right], v \right\rangle, \tag{33}$$

where $\tilde{u}$, $\tilde{f}$ and $\tilde{\varphi}$ are the extended functions of u, f and φ by zero onto $\Omega^- = R^1(t) \times R^2(x) \setminus \overline{\Omega}$ and $\overline{G^-} = R^2(x) \setminus G$ respectively, $\langle \tilde{u}, v\rangle$ denotes the value of a functional $\tilde{u}$ on the basic function v.

By (32) one may write (33) as

$$\int_e \left\{ \frac{\partial u}{\partial \zeta} - \sum_{i=1}^{2} \frac{\partial}{\partial \zeta_i}\left[k_i(\zeta_1,\zeta_2)\frac{\partial u}{\partial \zeta_i}\right]\right\} v(\zeta)d\zeta = \widetilde{R}g + \widetilde{S}\psi, \tag{34}$$

where

$$\widetilde{R}g = \int_e g(\zeta)v_1(\zeta)d\zeta,; \ \widetilde{S}\psi = \int_e \psi(\zeta_1,\zeta_2)v_2(\zeta_1,\zeta_2)d\zeta,$$
$$v_1(x,t) \equiv (-1)^{k_1} D_x^{k_1} D_t^{k_1} v(x,t), \quad v_2(x) = D_x^{k_2} v(x,0).$$

We see that $v_1(x,t)$ and $v_2(x)$ are also the test functions. Thus, the equation (34) has the form (4), (20). Hence, one may repeat the procedure used above for the difference schemes (21), (22) and obtains the following result analogous to (31):

Theorem 2. *Let in the problem* (1)–(3) *the data* $f \in \mathcal{D}'(\Omega)$, $\varphi \in \mathcal{D}'(G)$ *and* $k_i(x) \in W^1_\infty(G)$, $i = 1,2$. *Then, there exists uniquely a solution of the difference scheme* (21) *and this solution* $\breve{y}$ *converges to the GS* (33), (34) $u(x,t)$ *of the problem* (1)–(3) *with the rate* $O(|h|)$ *such that one has the error estimation*

$$\left\|\breve{y} - u\right\|_{1,\omega} \le C|h|\, \|u\|_{2,\Omega'},$$

where $\Omega' \Subset \Omega$.

Remark

1°. For the sake of simplicity, the homogeneous condition (3) was considered. In the case of nonhomogeneous condition, the theorems 1, 2 may be proved quite analogously.

2°. In the part II of this publication we will consider the difference schemes of the problem (1)–(3) in a region of arbitrary form.

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