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NONLINEAR EQUATIONS OF HAMMERSTEIN TYPE

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Abstract. The aim of this paper is to present an algorithm to determine a regularized solution for the operator equation of Hammerstein type $x + F_2F_1(x) = f$ under the non-monotone perturbations F_1^h and F_2^h of the operators F_1 and F_2, respectively. Galerkin approximations and convergence of Galerkin approximations of this regularized solution are considered in combination with regularizations.

Tóm tắt. Bài báo giới thiệu một thuật toán tìm nghiệm hiệu chỉnh cho phương trình toán tử loại Hammerstein $x + F_2F_1(x) = f$, với các nhiễu F_1^h và F_2^h của F_1 và F_2 tương ứng là không đơn điệu. Xấp xỉ Galerin và tốc độ hội tụ của xấp xỉ Galarin cho nghiệm hiệu chỉnh xét gộp với quá trình xấp xỉ hữu hạn chiều.

1. INTRODUCTION

Let X be a real Banach space and X^* be its dual which are uniformly convex. For the sake of simplicity the norms of X and X^* will be denoted by one symbol $\|.\|$. We write $\langle x^*, x\rangle$ instead of $x^*(x)$ for $x^* \in X^*$ and $x \in X$. Let $F_1 : X \to X^*$ and $F_2 : X^* \to X$ be monotone, bounded (i.e., image of any bounded subset is bounded) and continuous operators.

The main aim of this paper is to generalize the method in [4] for solving the operator equation of Hammerstein type

$$x + F_2F_1(x) = f, \; f \in X. \tag{1.1}$$

The approximate solution is defined by the solution x_α of the operator equation

$$x + F_{2\alpha}F_{1\alpha}(x) = f, \tag{1.2}$$

where $F_{1\alpha} = F_1 + \alpha U_1$, U_1 is the standard dual mapping of X (see [12], p. 311), i.e.,

$$\langle U_1(x), x\rangle = \|U_1(x)\|\|x\| = \|x\|^2, \quad \forall\, x \in X,$$

$F_{2\alpha} = F_2 + \alpha U_2$, U_2 is the standard dual mapping of X^*, and $\alpha > 0$ is a small parameter. For every $\alpha > 0$, equation (1.2) has a unique solution x_α, and the sequence $\{x_\alpha\}$ converges to a solution of (1.1), as $\alpha \to 0$. Moreover, this solution x_α, for every $\alpha > 0$, depends continuously on f. Let P_n be a linear projection from X onto its finite-dimensional subspace X_n such that $X_n \subset X_{n+1}$, $P_nx \to x$, as $n \to \infty$ for every $x \in X$, and P_n^* is the dual of P_n with $\|P_n\| \le \tilde{c} =$ constant for all n. Then, the finite-dimensional problem

$$x + F_{2\alpha n}F_{1\alpha n}(x) = f_n, \; x \in X_n,$$

where $F_{2\alpha n} = P_nF_{2\alpha}P_n^*$, $F_{1\alpha n} = P_n^*F_{1\alpha}P_n$, $f_n = P_nf$, has a unique solution $x_{\alpha n}$, and the sequence $\{x_{\alpha n}\}$ converges to x_α, as $n \to \infty$, without additional conditions on F_i, $i = 1, 2$. The convergence rates for the sequences $\{x_\alpha\}$ and $\{x_{\alpha n}\}$ are given in our recent paper [6]. Usually, instead of F_2 and F_1 we know only their continuous approximations F_2^h and F_1^h, respectively, such that

$$\|F_2^h(x^*) - F_2(x^*)\| \le h, \quad \forall x^* \in X^*,$$
$$\|F_1^h(x) - F_1(x)\| \le h, \quad \forall x \in X,$$

then the regularized solution is constructed by the solution of the approximate equation

$$x + F_{2\alpha}^h F_{1\alpha}^h(x) = f, \tag{1.3}$$

where $F_{2\alpha}^h = F_2^h + \alpha U_2,\ F_{1\alpha}^h = F_1^h + \alpha U_1$, if F_i^h are both monotone (see [5]). When one of F_i^h is not monotone, equation (1.3), perhaps, does not have solution. So, we have to determine an element in X as regularized solution for (1.1). In this paper, a new approach of determining such element is presented, the finite-dimensional approximations of this element and their convergence in combination with regularization are given.

Below, the symbols $\rightharpoonup$ and $\to$ denote weak convergence and convergence in norm, respectively.

From now on, we suppose that one of $F_i^h,\ i = 1,\ 2$ or both of them are not monotone.

2. REGULARIZED SOLUTION

An element x_ω, (ω depends on $\alpha,\ h$ and ε) is called a regularized solution of (1.1), if there exists an element $x_\omega^* \in X^*$ such that $z_\omega = [x_\omega, x_\omega^*]$, x_ω^* satisfies the following variational inequality

$$\begin{aligned} &\langle F_{1\alpha}^h(x_\omega) - x_\omega^*, x - x_\omega\rangle + \langle F_{2\alpha}^h(x_\omega^*) + x_\omega - f, x^* - x_\omega^*\rangle \ge \\ &-\varepsilon\Big(\|x - x_\omega\|^2 + \|x^* - x_\omega^*\|^2\Big)^{1/2},\quad \forall x \in X,\ x^* \in X^*,\ \varepsilon > \sqrt{2}h. \end{aligned} \tag{2.1}$$

Lemma 2.1. *The set S_ω of solutions of (2.1) is not empty.*

Proof. Let $x_\alpha^* = F_{1\alpha}(x_\alpha)$ and $z_\alpha = [x_\alpha, x_\alpha^*]$. Then, z_α is a solution of the following operator equation

$$\mathcal{F}(z) + \alpha J(z) = \overline{f}, \tag{2.2}$$

where

$$\begin{aligned} \mathcal{F}(z) &= [F_1(x) - x^*, F_2(x^*) + x],\\ J(z) &= [U_1(x), U_2(x^*)],\\ \overline{f} &= [0, f],\ z = [x, x^*] \in Z, \end{aligned}$$

and $Z = X \times X^*$ is the product of the spaces X and X^* with the norm $\|z\|^2 = \|x\|^2 + \|x^*\|^2$. Because z_α is a solution of (2.2), we have

$$\langle \mathcal{F}(z_\alpha) + \alpha J(z_\alpha) - \overline{f}, z - z_\alpha\rangle = 0,\ \forall z \in Z.$$

It is easy to see that $\|\mathcal{F}^h(z) - \mathcal{F}(z)\| \le \sqrt{2}h \le \varepsilon,\ \forall z \in Z$, where $\mathcal{F}^h(z) = [F_1^h(x) - x^*, x + F_2^h(x^*)]$. Therefore,

$$\begin{aligned} &\langle \mathcal{F}^h(z_\alpha) + \alpha J(z_\alpha) - \overline{f}, z - z_\alpha\rangle = \langle \mathcal{F}^h(z_\alpha) - \mathcal{F}(z_\alpha), z - z_\alpha\rangle \\ &+ \langle \mathcal{F}(z_\alpha) + \alpha J(z_\alpha) - \overline{f}, z - z_\alpha\rangle \ge -\sqrt{2}h\|z - z_\alpha\| \ge -\varepsilon\|z - z_\alpha\|, \end{aligned} \tag{2.3}$$

i.e., z_α satisfies (2.1). Consequently, S_ω contains x_α. This means that $S_\omega \ne \emptyset$.

Lemma 2.2. *If* $(\varepsilon + \sqrt{2}h)/\alpha$ *is bounded, then the set* S_ω *is bounded, too.*

Proof. We can rewrite (2.1) in the form (see [5], [10])

$$\langle \mathcal{F}^h(z_\omega) + \alpha J(z_\omega) - \overline{f}, z - z_\omega \rangle \geq -\varepsilon \|z - z_\omega\|, \ \forall z \in Z. \tag{2.3}$$

From the monotone property of $\mathcal{F}$ in Z it follows that

$$\langle \mathcal{F}(z_\omega) - \overline{f}, z_\omega - z \rangle \geq 0, \ \forall z \in S_1 = S_0 \times F_1(S_0),$$

where S_0 denotes the set of solutions of (1.1) that we assume to be nonempty. Hence,

$$\langle \mathcal{F}^h(z_\omega) + \alpha J(z_\omega) - \overline{f}, z_\omega - z \rangle + \sqrt{2}h\|z_\omega - z\| \geq \alpha \langle J(z_\omega), z_\omega - z \rangle.$$

By (2.3), we have

$$\langle J(z_\omega), z_\omega - z \rangle \leq \frac{\varepsilon + \sqrt{2}h}{\alpha} \|z_\omega - z\|, \ \forall z \in S_1. \tag{2.4}$$

Because of the boundedness of $(\varepsilon + \sqrt{2}h)/\alpha$, from the last inequality follows the boundness of the set $\{z_\omega\}$ in Z. Thus, $\{x_\omega\}$ is bounded in X.

Theorem 2.3. *Assume that* α, $(\varepsilon + \sqrt{2}h)/\alpha \to 0$. *Then, there exists a subsequence of the set* $\{x_\omega\}$ *converging to a solution of (1.1). Moreover, all the limit points of* $\{x_\omega\}$ *belong to* S_0.

Proof. If $(\varepsilon + \sqrt{2}h)/\alpha \to 0$, then $(\varepsilon + \sqrt{2}h)$ is bounded. Therefore, by Lemma 2.2, the sequence $\{z_\omega\}$, $z_\omega = [x_\omega, x_\omega^*]$, is bounded. Without loss of generality, assume that

$$x_\omega \rightharpoonup x_1, \quad \text{and} \ x_\omega^* \rightharpoonup x_1^*.$$

Consequently, $z_\omega \rightharpoonup z_1 := [x_1, x_1^*]$. As $\mathcal{F}$ is monotone in Z, we have

$$\begin{aligned}\langle \mathcal{F}(z) - \overline{f}, z - z_\omega \rangle \geq \langle \mathcal{F}(z_\omega) - \overline{f}, z - z_\omega \rangle &= \langle \mathcal{F}(z_\omega) - \mathcal{F}^h(z_\omega), z - z_\omega \rangle \\ &+ \langle \mathcal{F}^h(z_\omega) + \alpha J(z_\omega) - \overline{f}, z - z_\omega \rangle + \alpha \langle J(z_\omega), z_\omega - z \rangle.\end{aligned}$$

Therefore, from (2.3) it implies that

$$\langle \mathcal{F}(z) - \overline{f}, z - z_\omega \rangle \geq -(\varepsilon + \sqrt{2}h)\|z - z_\omega\| - \alpha \langle J(z_\omega), z - z_\omega \rangle.$$

After passing α, h, ε $(\varepsilon > \sqrt{2}h) \to 0$ in the last inequality, we obtain

$$\langle \mathcal{F}(z) - \overline{f}, z - z_1 \rangle \geq 0, \quad \forall z \in Z.$$

Thus, $z_1 \in S_1$, i.e., x_1 is a solution of (1.1). On the other hand, from (2.4) and the property of the standard dual mapping J of Z it follows that

$$0 \leq \Big(\|z\| - \|z_\omega\| \Big)^2 \leq \frac{\varepsilon + \sqrt{2}h}{\alpha}\|z - z_\omega\| + \langle J(z), z - z_\omega \rangle. \tag{2.5}$$

Hence, $\langle J(z), z - z_1 \rangle \geq 0$. Since z, $z_1 \in S_1$ and S_1 is a convex closed subset in Z (see [4]), replacing z by $tz + (1-t)z_1$ in the last inequality, dividing the result by t, and then tending $t \to 0$, we derive $\langle J(z_1), z - z_1 \rangle \geq 0$ (see [1]). Hence,

$$\|z_1\| \leq \|z\|, \quad \forall z \in S_1,$$

$S_1 \neq \emptyset$, because $S_0 \neq \emptyset$. Therefore, the sequence $\{z_\omega\}$ converges weakly to z_1. Moreover, from (2.5), by replacing z by z_1, we also derive that $\|z_\omega\| \to \|z_1\|$. Finally, we have $x_\omega \rightharpoonup z_1$, and $\|x_\omega\| \to \|x_1\|$ (see [4]). As X is uniformly convex, then $x_\omega \to x_1$.

3. GALERKIN APPROXIMATIONS

Consider the following variational inequality: find an element $x_{\omega n}$ such that $z_{\omega n} = [x_{\omega n}, x^*_{\omega n}]$, $x^*_{\omega n}$ is an element of X^*_n, satisfying the inequality

$$\begin{aligned} &\langle F^{hn}_{1\alpha}(x_{\omega n}) - x^*_{\omega n}, x_n - x_{\omega n}\rangle + \langle F^{hn}_{2\alpha}(x^*_{\omega n}) + x_{\omega n} - f_n, x^*_n - x^*_{\omega n}\rangle \geq \\ &- \varepsilon\Big(\|x_n - x_{\omega n}\|^2 + \|x^*_n - x^*_{\omega n}\|^2\Big)^{1/2}, \ \forall x_n \in X_n,\ x^*_n \in X^*_n,\ \varepsilon > \sqrt{2}h, \end{aligned} \tag{3.1}$$

where

$$F^{hn}_{1\alpha}(x_n) = (P^*_n F^h_1 P_n + \alpha P^*_n U_1 P_n)(x_n),$$
$$F^{hn}_{2\alpha}(x^*_n) = (P_n F^h_2 P^*_n + \alpha P_n U_1 P^*_n)(x^*_n).$$

The existence of $x_{\omega n}$ in (3.1) is proved in the same way as in Section 2 for x_ω.

Theorem 3.1. *For every fixed $\alpha > 0$, $h > 0$, and $\varepsilon > \sqrt{2}h$ the set $\{x_{\omega n}\}$ has a limit point being a solution of (2.1), and all the limit points of $\{x_{\omega n}\}$ are the solutions of (2.1).*

Proof. From (3.1) we have

$$\langle \mathcal{F}^{hn}(z_{\omega n}) + \alpha J^n(z_{\omega n}) - \overline{f}_n, z_n - z_{\omega n}\rangle \geq -\varepsilon\|z_n - z_{\omega n}\|, \ \forall z_n \in Z_n, \tag{3.2}$$

Therefore,

$$\begin{aligned} \alpha\Big(\|z_n\| - \|z_{\omega n}\|\Big)^2 &\leq \alpha\langle J^n(z_n) - J^n(z_{\omega n}), z_n - z_{\omega n}\rangle \\ &\leq \alpha\langle J^n(z_n), z_n - z_{\omega n}\rangle + \varepsilon\|z_n - z_{\omega n}\| + \langle \mathcal{F}^h(z_{\omega n}) - \overline{f}_n, z_n - z_{\omega n}\rangle \\ &\leq \alpha\langle J^n(z_n), z_n - z_{\omega n}\rangle + (\varepsilon + \sqrt{2}h)\|z_n - z_{\omega n}\| \\ &+ \langle \mathcal{F}^n(z_n) - \overline{f}_n, z_n - z_{\omega n}\rangle, \end{aligned} \tag{3.3}$$

where $\mathcal{F}^n(z_n) = [P^*_n F_1 P_n(x_n) - x^*_n, P_n F_2 P^*_n(x^*_n) + x_n]$. Since $\alpha > 0$ is fixed, then from the last inequality it implies that the sequence $\{z_{\omega n}\}$ is bounded. Without loss of generality, assume that

$$z_{\omega n} \rightharpoonup z'_\omega := [x'_\omega, x^{*\prime}_\omega], \quad \text{as} \quad n \to +\infty.$$

From (3.3), the continuous property of $\mathcal{F}^h$ and $z_n \to z$, we obtain (2.4) with $z_\omega = z'_\omega$. This means that x'_ω is a solution of (2.1). On the other hand, by replacing z_n in (3.5) by $z'_n = [P_n x'_\omega, P^*_n x^{*\prime}_\omega]$, we can see that $\|z_{\omega n}\| \to \|z'_\omega\|$, as $n \to +\infty$. Since X is uniformly convex, the sequence $x_{\omega n} \to x'_\omega$, as $n \to +\infty$ (see [2]). Clearly, any convergent subsequence of $\{x_{\omega n}\}$ converges to a solution of (2.1).

We now establish when the sequence $\{x_{\omega n}\}$ converges to a solution of (1.1), as α, h, $\varepsilon(\varepsilon \geq \sqrt{2}h) \to 0$, and $n \to \infty$.

Theorem 3.2. *Assume that the following conditions hold:*
(i) *F_1 is Fréchet differentiable at some neighbourhood $\mathcal{O}_0$ of S_0, and F_2 is Fréchet differentiable at some neighbourhood $\mathcal{Q}$ of $F_1(S_0)$,*
(ii) *there exists a constant $\tilde{L} > 0$ such that*

$$\|F'_1(x) - F'_1(y)\| \leq \tilde{L}\|x - y\|, \ \forall\ x \in S_0,\ y \in \mathcal{O}_0,$$
$$\|F'_2(x^*) - F'_2(y^*)\| \leq \tilde{L}\|x^* - y^*\|, \ \forall\ x^* \in F_1(S_0),\ y^* \in \mathcal{Q}_0,$$

(iii) $\gamma_x = \gamma_x(n) \to 0$ *such that* $\gamma_x/\alpha \to 0,\ \forall x \in S_0,$ *as* $n \to +\infty$, *where* $\gamma_x(n)$ *is defined by*

$$\gamma_x(n) = max\ \{\|(I - P_n)x\|, \|(I^* - P_n^*)F_1(x)\|\},$$

where I *and* I^* *denote the identity operators in* X *and* X^*, *respectively. Then, the sequence* $\{x_{\omega n}\}$ *converges to a solution of (1.1).*

Proof. From (3.2) and the property of the standard dual mapping J of the space Z, it follows that for every fixed $z \in S_1,\ (x \in S_0,\ x^* \in F_1(S_0)),\ z_n = [P_n x, P_n^* x^*]$

$$\alpha(\|z_n\| - \|z_{\omega n}\|)^2 \leq \alpha\big\langle J(z_n),\ z_n - z_{\omega n}\big\rangle + \varepsilon\|z_n - z_{\omega n}\| + \big\langle \mathcal{F}^h(z_{\omega n}) - \mathcal{F}(z_{\omega n}) + \mathcal{F}(z_{\omega n}) - \mathcal{F}(z_n) + \mathcal{F}(z_n) - \mathcal{F}(z),\ z_n - z_{\omega n}\big\rangle.$$

Since

$$\|\mathcal{F}^h(z_{\omega n}) - \mathcal{F}(z_{\omega n})\| \leq \sqrt{2}h \leq \varepsilon,$$
$$\big\langle \mathcal{F}(z_{\omega n}) - \mathcal{F}(z_n),\ z_n - z_{\omega n}\big\rangle \leq 0,$$

we have

$$\alpha(\|z_n\| - \|z_{\omega n}\|)^2 \leq \Big(2\varepsilon + \|\mathcal{F}(z_n) - \mathcal{F}(z)\|\Big)\|z_{\omega n} - z_n\| + \alpha\big\langle J(z_n),\ z_n - z_{\omega n}\big\rangle. \tag{3.4}$$

It is easy to verify that

$$\|\mathcal{F}(z_n) - \mathcal{F}(z)\|^2 \leq 2\Big(\|z_n - z\|^2 + \|F_1(x_n) - F_1(x)\|^2 + \|F_2(x_n^*) - F_2(x^*)\|^2\Big).$$

Since for sufficiently large n, $x_n \in \mathcal{O}_0,\ x_n^* \in \mathcal{Q}_0$, we can write

$$F_1(x_n) = F_1(x) + F_1'(x)(x_n - x) + r_n,$$
$$F_2(x_n^*) = F_2(x^*) + F_2'(x^*)(x_n^* - x^*) + \tilde{r}_n,$$

with $\|r_n\| \leq \frac{\tilde{L}}{2}\|(I - P_n)x\|^2$ and $\|\tilde{r}_n\| \leq \frac{\tilde{L}}{2!}\|(I^* - P_n^*)x^*\|^2$. Therefore, from (3.4) and condition (ii) of the Theorem it is easy to see that

$$\alpha(\|z_n\| - \|z_{\omega n}\|)^2 \leq \Big(2\varepsilon + \|F_1'(x)(I - P_n)x\| + \|F_2'(x^*)(I^* - P_n^*)x^*\| + \frac{\tilde{L}}{2}\|(I - P_n)x\|^2 + \frac{\tilde{L}}{2}\|(I^* - P_n^*)x^*\|^2\Big)\|z_n - z_{\omega n}\| + \alpha\big\langle J(z_n),\ z_n - z_{\omega n}\big\rangle,\ \forall z \in S_1.$$

Obviously, from this inequality and condition (iii) we obtain the boundedness of the sequence $\{z_{\omega n}\}$. Without loss of generality, suppose that $z_{\omega n} \rightharpoonup z_1' := [x_1', x_1'^*] \in Z$, as $n \to +\infty$ and $\alpha,\ \varepsilon \to 0$. By virtue of (2.1) we have

$$\big\langle \mathcal{F}(z_n) - \overline{f},\ z_n - z_{\omega n}\big\rangle + \alpha\big\langle J(z_n),\ z_n - z_{\omega n}\big\rangle + (\varepsilon + \sqrt{2}h)\|z_n - z_{\omega n}\| \geq 0.$$

After passing $n \to +\infty$ and $\alpha,\ \varepsilon(\varepsilon \geq \sqrt{2}h) \to 0$ in this inequality, the continuity of $\mathcal{F}$ and the weak convergence of the sequence $\{z_{\omega n}\}$ give us

$$\big\langle \mathcal{F}(z) - \overline{f},\ z - z_1\big\rangle \geq 0,\ \forall z \in Z.$$

Again, by Minty's lemma $z_1' \in S_1$, i.e., $x_1' \in S_0$. By replacing z_n in (3.4) by $z_{1n} = [P_n x_1', P_n^* x_1'^*]$ and tending $n \to \infty$, ε, $\alpha \to 0$, we see that $\{z_{\omega n}\}$ converges to z_1'. Consequently, the sequence $\{x_{\omega n}\}$ converges strongly to x_1', a solution of (1.1).

As in the proof of Theorem 2.3 we have $\|z_1'\| \le \|z\|$, $\forall z \in S_1$. Since X and X^* are uniformly convex, they are strictly convex. Therefore, Z is strictly convex (see [4]). Thus, z_1' is a unique element in S_1 having minimal norm. Hence, the sequence $\{z_{\omega n}\}$ converges to z_1'. Consequently, the sequence $\{x_{\omega n}\}$ converges to x_1', a solution of (1.1).

Remark. Many problems of nonlinear deformation of circular and annular elastic membranes are described in the form (1.1) (see [9, 14, 15]). Therefore, the results presented in this paper can be applied to solve them.

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