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CONVERGENCE RATES IN REGULARIZATION FOR ILL – POSED MIXED VARIATIONAL INEQUALITIES

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Abstract. In this paper the convergence rates of the regularized solution for ill-posed inverse-strongly monotone mixed variational inequalities in Banach spaces are obtained on the base of choosing the regularization parameter a priory as well as a posteriori by the generalized discrepancy principle.

Tóm tắt. Bài báo nghiên cứu tốc độ hội tụ của nghiệm hiệu chỉnh cho bất đẳng thức biến phân hỗn hợp không chính quy với toán tử ngược đơn điệu mạnh trong không gian Banach dựa trên việc chọn tham số hiệu chỉnh trước hoặc sau bằng nguyên lý độ lệch suy rộng.

1. INTRODUCTION

Let X be a real reflexive Banach space having the E-property: the weak convergence and convergence of norms of any sequence in X follow its strong convergence, and X^*, the dual space of X, be strictly convex. For the sake of simplicity, the norms of X and X^* are denoted by the symbol $\|.\|$. We write $\langle x^*, x\rangle$ instead of $x^*(x)$ for $x^* \in X^*$ and $x \in X$. Let A be a hemi-continuous and monotone operator from X into X^*, and $\varphi(x)$ be an weakly lower semicontinuous functional on X.

For a given $f \in X^*$, consider the mixed variational inequality: find an element $x_0 \in X$ such that

$$\langle A(x_0) - f, x - x_0\rangle + \varphi(x) - \varphi(x_0) \geqslant 0, \quad \forall x \in X. \tag{1.1}$$

Note that, if φ is the indicator function of a closed convex set K in X, that is,

$$\varphi(x) = I_K(x) = \begin{cases} 0, & \text{if} \quad x \in K \\ +\infty, & \text{otherwise,} \end{cases}$$

then problem (1.1) is equivalent to that of finding $x_0 \in K$ such that

$$\langle A(x_0) - f, x - x_0\rangle \geqslant 0, \quad \forall x \in K. \tag{1.2}$$

When K is the whole space X, the variational inequality (1.2) has the form of operator equation

$$A(x) = f \tag{1.3}$$

which belongs to a class of ill-posed problems involving monotone operator.

To solve ill-posed problems (1.2) or (1.3) there are a lot of investigations (see [1–3,8,9,11,16, 18,23–26] and references therein). Meantimes, for ill-posed problem (1.1) there are several works (see [6,10,17]). To regularize (1.1), in those papers, one uses the following mixed variational inequality (see [10,17]).

$$\langle A_h(x_\alpha^\tau) + \alpha U^s(x_\alpha^\tau - x_*) - f_\delta, x - x_\alpha^\tau\rangle + \varphi_\varepsilon(x) - \varphi_\varepsilon(x_\alpha^\tau) \geqslant 0, \quad \forall x \in X, \tag{1.4}$$

where A_h are also monotone operators from X into X^* and approximate A in the sense

$$\|A_h(x) - A(x)\| \leqslant hg(\|x\|), \quad h \to 0, \tag{1.5}$$

with a nonegative continuous and bounded (image of bounded set is bounded) function $g(t)$, U^s is the generalized duality mapping of X, i.e., U^s is the mapping from X onto X^* satisfying the condition (see [3])

$$\langle U^s(x), x\rangle = \|x\|^s, \quad \|U^s(x)\| = \|x\|^{s-1}, \quad s \geqslant 2, \tag{1.6}$$

f_δ are the approximations of $f : \|f_\delta - f\| \leqslant \delta, \delta \to 0, \varphi_\varepsilon$ are the functionals on X having the same properties as φ, and

$$\begin{aligned} |\varphi(x) - \varphi_\varepsilon(x)| &\leqslant \varepsilon d(\|x\|), \quad \varepsilon \to 0, \\ |\varphi_\varepsilon(x) - \varphi_\varepsilon(y)| &\leqslant C_0\|x - y\|, \quad \forall x, y \in X, \end{aligned} \tag{1.7}$$

where C_0 is some positive constant, $d(t)$ has the same properties as $g(t)$, $\tau = (h, \delta, \varepsilon)$, and x_* is some element in X playing the role of a criterion selection. By the choice of x_*, we can influence which solution we want to approximate. Assume that x_* is not a solution of (1.1), i.e., there exists an element $x_1 \in X$ such that

$$\langle A(x_*) - f, x_1 - x_*\rangle + \varphi(x_1) - \varphi(x_*) < 0. \tag{1.8}$$

In [10], [17] it is showed the existence and uniqueness of the solution x_α^τ for every $\alpha > 0$ and for arbitrary $A_h, f_\delta, \varphi_\varepsilon$. And, the regularized solution x_α^τ converges to $x_0 \in S_0$, the set of solutions of (1.1) which is assumed to be nonempty, with

$$\|x_0 - x_*\| = \min_{x\in S_0} \|x - x_*\|,$$

if $(h + \delta + \varepsilon)/\alpha, \alpha \to 0$. Up to now, the problem of choosing the value of the regularization parameter α depending on τ, i.e. $\alpha = \alpha(h, \delta, \varepsilon)$, is still opened. Moreover, the convergence rate for the regularized solution x_α^τ does not have been studied yet. The purpose of the paper is to answer two above questions. On the one hand, we show that the parameter α can be chosen by the generalized discrepancy principle, i.e., $\alpha = \alpha(h, \delta, \varepsilon)$ can be solved the equation

$$\rho(\alpha) = (h + \delta + \varepsilon)^p\alpha^{-q}, \quad p, q > 0, \tag{1.9}$$

where $\rho(\alpha) = \alpha\|x_\alpha^\tau - x_*\|^{s-1}$. Note that the generalized discrepancy principle for regularization parameter choice is presented in [14] for the ill-posed operator equation (1.3) when A is a linear and bounded operator in Hilbert space. Recently, it is considered and applied to estimate convergence rates of the regularized solution for equation (1.3) involving an m-accretive (in general nonlinear) operator (see [22]).

On the other hand, to estimate the value $\|x_\alpha^\tau - x_0\|$ we assume that A is inverse-strongly monotone operator, i.e., A possesses the property

$$\langle A(x) - A(y), x - y\rangle \geqslant m_A\|A(x) - A(y)\|^2, \quad \forall x, y \in X, \tag{1.10}$$

where m_A is some positive constant. Note that the operator A with property (1.10) has been introduced independently and/or used several authors (see [7,12,27]), and has been given different names (e.g., the Dunn property, co-coercivity, etc.). Without (1.10) and the monotone property of A_h, the convergence rates of the regularized solution for (1.3) is considered in [20] and [21]. The convergence rates of regularized solutions for (1.2) in Hilbert space are studied in [18] when $A_h \equiv A$.

Later, the symbols $\rightharpoonup$ and $\to$ denote weak convergence and convergence in norm, respectively, and the notation $a \sim b$ is meant that $a = O(b)$ and $b = O(a)$.

Concerning the mapping U^s, assume that

$$\langle U^s(x) - U^s(y), x - y\rangle \geqslant m_s\|x - y\|^s, \tag{1.11}$$

where m_s is some positive constant depending the properties of the space X. It is well-known that when $X \equiv H$, the Hilbert space, $m_s = 1, \quad s = 2$, and when $X = L_p$ or W_p, $m_s = p - 1, s = 2$ for the case $1 < p < 2$. In the case $p > 2$, $m_s = 2^{2-p}/p$ and $s = p$ (see [3]).

2. MAIN RESULT

To obtain the result on the convergence rate for $\{x^\tau_{\alpha(h,\delta,\varepsilon)}\}$ as in [14] we need the following lemmas.

Lemma 1. *For each $p, q, h, \delta, \varepsilon > 0$, there exists at least a value α such that* (1.9) *holds.*

Proof. Let $\alpha_1, \alpha_2 \geqslant \alpha_0$ are arbitrary ($\alpha_0 > 0$). From (1.4) we can obtain

$$\alpha_1\langle U^s(x^\tau_{\alpha_1} - x_*), x^\tau_{\alpha_2} - x^\tau_{\alpha_1}\rangle + \alpha_2\langle U^s(x^\tau_{\alpha_2} - x_*), x^\tau_{\alpha_1} - x^\tau_{\alpha_2}\rangle \geqslant 0 \tag{2.1}$$

or

$$\alpha_1\langle U^s(x^\tau_{\alpha_1} - x_*) - U^s(x^\tau_{\alpha_2} - x_*), x^\tau_{\alpha_1} - x^\tau_{\alpha_2}\rangle \leqslant (\alpha_2 - \alpha_1)\langle U^s(x^\tau_{\alpha_2} - x_*), x^\tau_{\alpha_1} - x^\tau_{\alpha_2}\rangle.$$

Thus, by virtue of (1.6) and (1.11) we have got

$$m_s\|x^\tau_{\alpha_1} - x^\tau_{\alpha_2}\|^{s-1} \leqslant \frac{|\alpha_1 - \alpha_2|}{\alpha_0}\|x^\tau_{\alpha_2} - x_*\|^{s-1}.$$

Obviously, $x^\tau_{\alpha_1} \to x^\tau_{\alpha_2}$, as $\alpha_1 \to \alpha_2$. It means that the function $\|x^\tau_\alpha - x_*\|$ is continuous on $[\alpha_0, +\infty)$. Therefore, $\rho(\alpha)$ is continuous, too.

From (1.6) and (2.1) it follows

$$(\alpha_1\|x^\tau_{\alpha_1} - x_*\|^{s-1} - \alpha_2\|x^\tau_{\alpha_2} - x_*\|^{s-1})(\|x^\tau_{\alpha_1} - x_*\| - \|x^\tau_{\alpha_2} - x_*\|) \leqslant 0.$$

Consequently, the function $\|x^\tau_\alpha - x_*\|$ is not increasing and $\rho(\alpha)$ is nondecreasing.

We show that for every fixed τ we have $\rho(\alpha) > 0$ when $\alpha > 0$. Indeed, if $\alpha_1 > 0$ and $\rho(\alpha_1) = 0$, then $x^\tau_{\alpha_1} = x_*$. Therefore, from(1.4) it implies that

$$\langle A_h(x_*) - f_\delta, x - x_*\rangle + \varphi_\varepsilon(x) - \varphi_\varepsilon(x_*) \geqslant 0, \quad \forall x \in X$$

for every fixed $h, \delta, \varepsilon > 0$. Now, by virtue of (1.5) and (1.7) after passing h, δ, ε to zero in the last inequality, we see that x_* is a solution of (1.1) that contradicts our assumption (1.8).

Since $0 < \rho(\alpha) \leqslant \rho(\alpha_1)$ for $0 < \alpha \leqslant \alpha_1$, then

$$\lim_{\alpha\to+0} \alpha^q\rho(\alpha) = 0.$$

Similarly, $0 < \rho(\alpha_1) \leqslant \rho(\alpha)$ for $0 < \alpha_1 \leqslant \alpha$. Therefore,

$$\lim_{\alpha\to+\infty} \alpha^q\rho(\alpha) = +\infty.$$

Hence, the result follows from the intermediate value theorem.

Lemma 2. $\lim_{h,\delta,\varepsilon\to 0}\alpha(h,\delta,\varepsilon)=0$.

Proof. Let $h_n,\delta_n,\varepsilon_n\to 0$ and $\alpha_n=\alpha(h_n,\delta_n,\varepsilon_n)\to\infty$ as $n\to\infty$. From (1.4) it follows

$$\begin{aligned}\langle A_{h_n}(x_{\alpha_n}^{\tau_n})+\alpha_n U^s(x_{\alpha_n}^{\tau_n}-x_*)-f_{\delta_n},x-x_{\alpha_n}^{\tau_n}\rangle\geqslant\\ \varphi_{\varepsilon_n}(x_{\alpha_n}^{\tau_n})-\varphi_{\varepsilon_n}(x),\quad\forall x\in X.\end{aligned}\tag{2.2}$$

Taking $x=x_*$ in this inequality and using the properties of φ_ε , A_h and U^s, we can deduce that

$$\|x_{\alpha_n}^{\tau_n}-x_*\|^{s-1}\leqslant(\|A_{h_n}(x_*)-f_{\delta_n}\|+C_0)/\alpha_n.$$

Hence, $x_{\alpha_n}^{\tau_n}\to x_*$, as $n\to+\infty$. On the other hand, by using the properties of A_{h_n} and U^s we can write (2.2) in the form

$$\begin{aligned}\langle A_{h_n}(x)-f_{\delta_n},x-x_{\alpha_n}^{\tau_n}\rangle+\varphi_{\varepsilon_n}(x)-\varphi_{\varepsilon_n}(x_{\alpha_n}^{\tau_n})&\geqslant\alpha_n\langle U^s(x_{\alpha_n}^{\tau_n}-x_*),x_{\alpha_n}^{\tau_n}-x\rangle\\ &\geqslant-\alpha_n\|x_{\alpha_n}^{\tau_n}-x_*\|^{s-1}\|x_{\alpha_n}^{\tau_n}-x\|=-\rho(\alpha_n)\|x_{\alpha_n}^{\tau_n}-x\|\\ &=-(h_n+\delta_n+\varepsilon_n)^p\alpha_n^{-q}\|x_{\alpha_n}^{\tau_n}-x\|.\end{aligned}$$

After passing n to ∞ in the last inequality, on the base of (1.5), (1.7), $f_{\delta_n}\to f$ and $x_{\alpha_n}^{\tau_n}\to x_*$ as $n\to\infty$ we obtain

$$\langle A(x)-f,x-x_*\rangle+\varphi(x)-\varphi(x_*)\geqslant 0,\quad\forall x\in X,\tag{2.3}$$

which is equivalent to

$$\langle A(x_*)-f,x-x_*\rangle+\varphi(x)-\varphi(x_*)\geqslant 0,\quad\forall x\in X,$$

(see [13]). It contradicts (1.8).

Thus, $\alpha(h,\delta,\varepsilon)$ remains bounded as $h,\delta,\varepsilon\to 0$. Let $h_n,\delta_n,\varepsilon_n\to 0$ as $n\to\infty$, and meantime $\alpha_n\to c>0$. Since $\alpha_n^{1+q}\|x_{\alpha_n}^{\tau_n}-x_*\|^{s-1}=(h_n+\delta_n+\varepsilon_n)^p$, we have $\|x_{\alpha_n}^{\tau_n}-x_*\|\to 0$, as $n\to\infty$. Again, $x_*\in S_0$.

Hence, $\lim_{h,\delta,\varepsilon\to 0}\alpha(h,\delta,\varepsilon)=0$.

Lemma 3. *If* $0<p<q$, *then* $\lim_{h,\delta,\varepsilon\to 0}(h+\delta+\varepsilon)/\alpha(h,\delta,\varepsilon)=0$.

Proof. Obviously,

$$\begin{aligned}\left[\frac{h+\delta+\varepsilon}{\alpha(h,\delta,\varepsilon)}\right]^p&=[(h+\delta+\varepsilon)^p\alpha^{-q}(h,\delta,\varepsilon)]\alpha^{q-p}(h,\delta,\varepsilon)\\ &=\rho(\alpha(h,\delta,\varepsilon))\alpha^{q-p}(h,\delta,\varepsilon).\end{aligned}$$

As in the proof of lemma 2, we have

$$\left[\frac{h+\delta+\varepsilon}{\alpha(h,\delta,\varepsilon)}\right]^p\leqslant(\|A_h(x_*)-f_\delta\|+C_0)\alpha^{q-p}(h,\delta,\varepsilon)\to 0$$

as $h,\delta,\varepsilon\to 0$, since $q>p$. Therefore,

$$\lim_{h,\delta,\varepsilon\to 0}\left[\frac{h+\delta+\varepsilon}{\alpha(h,\delta,\varepsilon)}\right]^p=0.$$

The lemma is proved. ■

Lemma 4. *Let $0 < p < q$. Then, there exist constants $C_1, C_2 > 0$ such that, for sufficiently small $h, \delta, \varepsilon > 0$, the relation*

$$C_1 \leqslant (h+\delta+\varepsilon)^p \alpha^{-1-q}(h,\delta,\varepsilon) \leqslant C_2$$

holds.

Proof. Clearly,

$$(h+\delta+\varepsilon)^p \alpha^{-1-q}(h,\delta,\varepsilon) = \alpha^{-1}(h,\delta,\varepsilon)\rho(\alpha(h,\delta,\varepsilon)) = \|x^\tau_{\alpha(h,\delta,\varepsilon)} - x_*\|^{s-1}.$$

By Lemma 2 the sequence $\{x^\tau_{\alpha(h,\delta,\varepsilon)}\}$ converges to x_0, as $\tau = (h,\delta,\varepsilon) \to 0$. Therefore, there exists a positive constant C_2 in the lemma.

On the other hand, as X is reflexive and $\{x^\tau_{\alpha(h,\delta,\varepsilon)}\}$ is bounded, there exists a subsequence of the sequence $\{x^\tau_{\alpha(h,\delta,\varepsilon)}\}$ that converges weakly to some element $\tilde{x}_*$ in X. Without loss of generality, denote the subsequence again by $\{x^\tau_{\alpha(h,\delta,\varepsilon)}\}$. Then,

$$\|\tilde{x}_* - x_*\| \leqslant \liminf \|x^\tau_{\alpha(h,\delta,\varepsilon)} - x_*\|.$$

We can conclude that $\tilde{x}_* \neq x_*$. Indeed, if $\tilde{x}_* = x_*$, then from the monotone hemi-continuous property of A_h, U^s and (1.4) it follows

$$\langle A_h(x) + \alpha(h,\delta,\varepsilon)U^s(x - x_*) - f_\delta, x - x^\tau_\alpha\rangle + \varphi_\varepsilon(x) - \varphi_\varepsilon(x^\tau_\alpha) \geqslant 0, \quad \forall x \in X.$$

After passing h, δ and ε in the last inequality to zero we obtain (2.3). It contradicts (1.8). Thus, any weak cluster point of $\{x^\tau_{\alpha(h,\delta,\varepsilon)}\}$ is different from x_*. Therefore, there exists a constant C_1 in the lemma. ■

Theorem 1. *Assume that the following conditions hold:*

(i) A is an inverse-strongly monotone operator from X into X^* with the property

$$\|A(x) - A(x_0) - A'(x_0)(x - x_0)\| \leqslant \tilde{\tau}\|A(x) - A(x_0)\|, \quad \forall x \in X,$$

where $A'(x)$ denotes the Frchet derivative of A at x, and $\tilde{\tau}$ is some positive constant;

(ii) There exists an element $z \in X$ such that $A'(x_0)^* z = U^s(x_0 - x_*)$;

(iii) The parameter α is chosen by (1.9).

Then, we have

$$\|x^\tau_{\alpha(h,\delta,\varepsilon)} - x_0\| = O((h+\delta+\varepsilon)^{\mu_1}), \quad \mu_1 = \frac{1}{1+q}\min\left\{\frac{1+q-p}{s}, \frac{p}{2s}\right\}.$$

Proof. From (1.1), (1.4), (1.5) and (1.7) it follows

$$\begin{aligned}&\langle A(x^\tau_\alpha) - A(x_0), x^\tau_\alpha - x_0\rangle + \alpha\langle U^s(x^\tau_\alpha - x_*) - U^s(x_0 - x_*), x^\tau_\alpha - x_0\rangle \\ &\leqslant \alpha\langle U^s(x_0 - x_*), x_0 - x^\tau_\alpha\rangle + \langle A_h(x^\tau_\alpha) - A(x^\tau_\alpha), x_0 - x^\tau_\alpha\rangle \\ &+\langle f_\delta - f, x^\tau_\alpha - x_0\rangle + \varepsilon[d(\|x_0\|) + d(\|x^\tau_\alpha\|)]. \qquad (2.4)\end{aligned}$$

Hence, from (1.10), (2.4) and the monotone property of U^s we can write

$$\begin{aligned}\|A(x^\tau_\alpha) - A(x_0)\|^2 \leqslant m_A^{-1}\Big\{&[hg(\|x^\tau_\alpha\|) + \delta + \alpha\|x_0 - x_*\|^{s-1}]\|x^\tau_\alpha - x_0\| \\ &+ \varepsilon[d(\|x_0\|) + d(\|x^\tau_\alpha\|)]\Big\}.\end{aligned}$$

Further, from (1.4), (1.7), (1.11), (2.4) and the monotone property of A which is followed from (1.10) we have

$$\begin{aligned} m_s\|x_\alpha^\tau - x_0\|^s &\leqslant \langle U^s(x_\alpha^\tau - x_*) - U^s(x_0 - x_*), x_\alpha^\tau - x_0\rangle \\ &\leqslant \frac{hg(\|x_\alpha^\tau\|) + \delta}{\alpha}\|x_\alpha^\tau - x_0\| + \langle U^s(x_0 - x_*), x_0 - x_\alpha^\tau\rangle \\ &\quad + \frac{\varepsilon}{\alpha}[d(\|x_0\|) + d(\|x_\alpha^\tau\|)]. \end{aligned} \tag{2.5}$$

When $\alpha = \alpha(h, \delta, \varepsilon)$ is chosen by (1.9), from lemma 3 it follows the boundedness of $\{x_\alpha^\tau\}$. Then,

$$\|A(x_\alpha^\tau) - A(x_0)\| \leqslant O(\sqrt{h + \delta + \varepsilon + \alpha}).$$

By virtue of conditions (i), (ii) and (2.4) we obtain

$$\begin{aligned} \langle U^s(x_0 - x_*), x_0 - x_\alpha^\tau\rangle &= \langle z, A'(x_0)(x_0 - x_\alpha^\tau)\rangle \\ &\leqslant \|z\|(\tilde{\tau} + 1)\|A(x_\alpha^\tau) - A(x_0)\| \\ &\leqslant \|z\|(\tilde{\tau} + 1)O(\sqrt{h + \delta + \varepsilon + \alpha}). \end{aligned}$$

Now, (2.5) has the form

$$\begin{aligned} m_s\|x_\alpha^\tau - x_0\|^s &\leqslant \frac{hg(\|x_\alpha^\tau\|) + \delta}{\alpha}\|x_\alpha^\tau - x_0\| + O(\sqrt{h + \delta + \varepsilon + \alpha}) \\ &\quad + \frac{\varepsilon}{\alpha}[d(\|x_0\|) + d(\|x_\alpha^\tau\|)]. \end{aligned}$$

Further, from lemma 4 it implies that

$$\alpha(h, \delta, \varepsilon) \leqslant C_1^{-1/(1+q)}(h + \delta + \varepsilon)^{p/(1+q)}$$

and

$$\begin{aligned} \frac{h + \delta + \varepsilon}{\alpha(h, \delta, \varepsilon)} &\leqslant C_2(h + \delta + \varepsilon)^{1-p}\alpha^q(h, \delta, \varepsilon) \\ &\leqslant C_2C_1^{-q/(1+q)}(h + \delta + \varepsilon)^{1-p}(h + \delta + \varepsilon)^{pq/(1+q)} \\ &\leqslant C_2C_1^{-q/(1+q)}(h + \delta + \varepsilon)^{1-p/(1+q)}. \end{aligned}$$

In final, we have

$$\begin{aligned} m_s\|x_{\alpha(h,\delta,\varepsilon)}^\tau - x_0\|^s &\leqslant \max\{1, \tilde{C}_0\}C_2C_1^{-q/(1+q)}(h + \delta + \varepsilon)^{1-p/(1+q)} \\ &\quad \times \|x_{\alpha(h,\delta,\varepsilon)}^\tau - x_0\| + O(\sqrt{h + \delta + \varepsilon + \alpha(h, \delta, \varepsilon)}) \\ &\quad + O\big((h + \delta + \varepsilon)^{1-p/(1+q)}\big) \\ &\leqslant O\big((h + \delta + \varepsilon)^{1-p/(1+q)}\big)\|x_{\alpha(h,\delta,\varepsilon)}^\tau - x_0\| \\ &\quad + O\big((h + \delta + \varepsilon)^{p/2(1+q)}\big) + O\big((h + \delta + \varepsilon)^{1-p/(1+q)}\big). \end{aligned}$$

where the constant $\tilde{C}_0 \geqslant g(\|x_{\alpha(h,\delta,\varepsilon)}^\tau\|)$.

Using the implication

$$a, b, c \geqslant 0, s > t, a^s \leqslant ba^t + c \Longrightarrow a^s = O(b^{s/(s-t)} + c)$$

we obtain

$$\|x_{\alpha(h,\delta,\varepsilon)}^\tau - x_0\| = O\big((h + \delta + \varepsilon)^{\mu_1}\big).$$

Theorem is proved.

Remark. If α is chosen a priory such that $\alpha \sim (h+\delta+\varepsilon)^p, 0<p<1$, then from (2.5) we obtain the inequality

$$m_s\|x^\tau_{\alpha(h,\delta,\varepsilon)} - x_0\|^s \leqslant O\big((h+\delta+\varepsilon)^{1-p}\big)\|x^\tau_{\alpha(h,\delta,\varepsilon)} - x_0\| + O\big((h+\delta+\varepsilon)^{p/2}\big) + O\big((h+\delta+\varepsilon)^{1-p}\big).$$

Therefore,

$$\|x^\tau_{\alpha(h,\delta,\varepsilon)} - x_0\| = O((h+\delta+\varepsilon)^{\mu_2}), \quad \mu_2 = \min\Big\{\frac{1-p}{s}, \frac{p}{2s}\Big\}.$$

3. EXAMPLE

Let Ω be a bounded domain in R^n, $n \geqslant 1$, with a Lipshitz continuous boundary Γ, $\xi \to g(\xi^2)\xi$ is a function defining the filtration law. We assume that $g(\xi^2)\xi = g_0(\xi^2)\xi + g_1(\xi^2)\xi$ and the following conditions are satisfied:

$$g_i(\xi^2)\xi \geqslant 0 \quad for \quad \xi \geqslant 0, \quad g_i(\xi^2)\xi = 0 \quad for \quad \xi \leqslant \beta, \quad i = 0,1, \tag{3.1}$$

$g_0(\xi^2)\xi$ is continuous and

$$(g_0(\xi^2)\xi)'_\xi > 0 \quad for \quad \xi \geqslant \beta, \tag{3.2}$$

$\beta \geqslant 0$ is a limiting gradient, there exist $c_0, c_1, c_2 > 0$, $p > 1$ such that

$$c_1(\xi-\beta)^{p-1} \leqslant g_0(\xi^2)\xi \leqslant c_2(\xi-\beta)^{p-1} \quad for \quad \xi \geqslant \beta, \tag{3.3}$$

$$(g_0(\xi^2)\xi - g_0(\eta^2)\eta)/(\xi-\eta) \leqslant c_0(1+\xi+\eta)^{p-2} \qquad \forall \xi,\eta \in R \quad > 0, \tag{3.4}$$

$$g_1(\xi^2)\xi = \vartheta \quad for \quad \xi > \beta. \tag{3.5}$$

Furthermore let $\Gamma = \Gamma_0 \cup \Gamma_1$, $mes\Gamma_0 > 0$, $mes\Gamma_1 > 0$, $X = \{x \in W^1_p(\Omega) : x(t) = 0, t \in \Gamma_0\}$, $K = \{x \in X : x(t) \geqslant 0, t \in \Gamma_1\}$ (Γ_1 is a semipermeable part of the boundary), $A : X \to X^*$ is an operator generated by the form

$$\langle Ax, y\rangle = \int_\Omega g_0(|\bigtriangledown x|^2)(\bigtriangledown x, \bigtriangledown y)dt, \qquad x, y \in X, \tag{3.6}$$

where $(.,.)$ and $|\,.\,|$ are the inner production and norm in R^n respectively.

We also define the functional $\varphi : X \to R$ by the relation

$$\varphi(x) = \int_\Omega \int_0^{|\bigtriangledown x|} g_1(\xi^2)\xi d\xi dt = \vartheta \int_\Omega (|\bigtriangledown x| - \beta)^+ dt, \quad a^+ = (|a| + a)/2.$$

We call the function $x_0 \in K$ the solution of the seepage problem of an incompressible liquid which is the solution of the variational inequality (1.1) (see [4,15]).

From the definition of the functional φ, the second of the conditions (3.1), and equality (3.5) we have (see [5] and references therein) the functional φ satisfy the condition

$$|\varphi(x) - \varphi(x_0)| \leqslant \vartheta \|\bigtriangledown(x - x_0)\|_{L^1}.$$

it follows that the functional φ is Lipshitz continuous.

We now define the regularized function (see [5])

$$g_{1\varepsilon}(\xi^2)\xi = \begin{cases} 0 & , \ \xi \leqslant \beta - \varepsilon \\ \vartheta(\xi - \beta + \varepsilon)/\varepsilon & , \ \beta - \varepsilon \leqslant \xi \leqslant \beta \\ \vartheta & , \ \xi \geqslant \beta \end{cases}$$

This function generates the corresponding regularized functional φ_ε. In [5] it is showed that the functional φ_ε satisfies the first of the conditions (1.7) with $d(\|x\|) = 2mes\Omega$ and the operator A is monotone.

When $p = 2$ the space X is a Hilbert space, in [19] inequality (1.10) is determined for the operator A defined in (3.6).

Thus, for the seepage problems under consideration the conditions of the previous sections of this paper are satisfied and, consequently, Theorems 1 hold.

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