NOVEL APPROACH OF ROBUST $H_{\infty}$ TRACKING CONTROL FOR UNCERTAIN FUZZY DESCRIPTOR SYSTEMS USING FIXED LYAPUNOV FUNCTION

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Abstract. This paper proposes a novel uncertain fuzzy descriptor system which is an extension from standard T-S fuzzy system. A fixed Lyapunov function-based approach is considered and controller design for this rich class of fuzzy descriptor systems is formulated as a problem of solving a set of LMIs. The design conditions for the descriptor fuzzy system are more complicated than the standard state-space-based systems. However, the descriptor fuzzy system-based approach has the advantage of possessing fewer number of matrix inequality conditions for certain special cases. Hence, it is suitable for complex systems represented in descriptor form which is often observed in highly nonlinear mechanical systems.

Keywords. Descriptor fuzzy system; Lyapunov function; Uncertain nonlinear mechanical systems; Robust $H_{\infty}$ tracking control; LMI matrix inequality.

1. INTRODUCTION

Nowadays fuzzy logic-based control has proven to be a successful approach for controlling uncertain nonlinear systems [1, 2, 3, 4, 5]. The fuzzy-model proposed by Takagi and Sugeno [6], known as the T-S fuzzy model, is becoming a popular type of fuzzy model representation. Up to now there have been numerous successful applications of the T-S fuzzy model-based approach in uncertain nonlinear control systems. Linear matrix inequality (LMI)-based T-S fuzzy control is an important and successful approach used in uncertain nonlinear control. Up to now adequate studies are available that discusses linear matrix inequality (LMI)-based T-S fuzzy control system design using the fixed Lyapunov function [7, 8, 9, 10]. Although LMI-based approach gained popularity and great success, conservatism is still dominant in fixed quadratic Lyapunov function-based approach due to the limited choice of Lyapunov function [11].

In the robust control approaches discussed in [12], a T-S fuzzy model is employed, where its consequent parts are described via linear state-space systems. The description system improved from a standard state-space form successfully describes a wider class of systems and then can be used in certain mechanical and electrical systems. Then the T-S fuzzy model will be a special case of the descriptor fuzzy model. The advantage of choosing the
descriptor representation over the state-space model is that the amount of LMI ineq-
ualities for designing the controller can be reduced for certain problems [13]. Compared with
the standard state-space based system representation, descriptor representation holds more
complicated structure and hence the controller design is also more complex [14].

Up to now, considerable work has been done involving stability control, H stabiliza-
tion and model following control for fuzzy descriptor systems [13]. The necessity for such
control techniques is principally improved via the increasingly experimental interest for a
generalized system descriptor taking the intrinsically physical structure into consideration.
Furthermore, the conventional state-space system problem can be considered as a special
case of descriptor systems and then is able to be efficiently resolved by applying descriptive
system computational methods [15].

Recently, numerous results obtained for robust $H_\infty$ stabilization with parametric Lyapu-
nov function have been presented in reviewing the results from literature for fixed Lyapunov
function based on robust $H_\infty$ stabilization for fuzzy descriptor systems [16, 17, 18, 19]. Zhi
et al. (2018) in [16] proposed a new robust $H_\infty$ control for T-S fuzzy descriptor systems with
state and input time-varying delays. Xue et al. in [17] introduced a robust sliding mode
proposed a robust $H_\infty$ stabilization for T-S fuzzy descriptor systems with time-varying de-
lays and memory sampled-data control. Nasiri et al. in [19] introduced a new method for
reducing conservatism in an $H_\infty$ robust state-feedback control design of T-S fuzzy descriptor
systems.

A model following control is considered in [13] and observer using $H$ tracking control
problem is introduced in [14]. For a state feedback $H_\infty$ tracking control problem, this
proposed approach yielded the conditions in terms of bilinear matrix inequalities (BMI)
usually resolved by a two-step process. Based on this approach, the sufficient condition for
implementing a state-feedback controller cannot be framed as LMIs.

Based on results abovementioned, this paper innovatively proposes an LMI formulation
with respect to design conditions using fixed Lyapunov function for a model reference tra-
jectory tracking problem responding to $H_\infty$ performance criteria. Next these results are
combined with the concepts presented in [15] and parametric Lyapunov function-based de-
sign for controlling using uncertain descriptor fuzzy systems is proposed here.

The rest of this paper is structured as follows. Section 2 introduces the T-S fuzzy descrip-
tor system and constant Lyapunov function-based stability conditions. Section 3 presents the
performance of $H$ trajectory tracking control for the T-S fuzzy descriptor system. Section 4
proposes the novel T-S fuzzy descriptor for uncertain nonlinear system. Section 5 presents
and analyses the simulation of proposed robust $H_\infty$ tracking control implementation with
fixed Lyapunov function using T-S fuzzy descriptor system. Finally, Section 6 includes the
conclusions.

2. PROPOSED T-S FUZZY DESCRIPTOR SYSTEM

This paper starts with introduction to T-S fuzzy model and then $H$ tracking control
problem is formulated. The T-S fuzzy model initially introduced by Takagi and Sugeno [6]
describes the dynamics of an uncertain nonlinear plant based on fuzzy IF-THEN laws. Let
us investigate the descriptor fuzzy model of a nonlinear system in the form as follows.
Plant law $k - i$: IF $z_1^i(t)$ is $N_{k_1}^{e}$, ..., $z_p^i(t)$ is $N_{k_p}^{e}$ and $z(t)$ is $N_{1...p}(t)$ is $N_{ip}$ THEN

$$
E_k \dot{x}(t) = A_i x(t) + B_i u(t),
$$

$$
y(t) = C_i x(t), \quad i = 1, 2, ..., r, \quad k = 1, 2, ..., r_e,
$$

where $z_1(t), ..., z_p(t)$ represent premise variables, $p$ represents the amount of premise variables, $N_{kj}^{e}$ ($j = 1...p$), $N_{ij}$ ($j = 1...p$) are the fuzzy sets and $r$ represents the number of laws. Furthermore, $x(t) \in R^{n \times 1}$ represents the state vector, $y(t) \in R^{m \times 1}$ is the input vector, $A_i \in R^{n \times n}$, $B_i \in R^{n \times m}$, $C_i \in R^{m \times n}$, $E_k \in R^{n \times n}$ are constant real matrices. The necessary assumptions are that rank($N_{kj}^{e}$) represents the uncertainties and are bounded, i.e., $\| A_i \| < \delta_i$, where $\| . \|$ denotes spectral norm and $\delta_i$ represents positive value. Other specific constraints can be consulted in [14]. From input $x(t)$ and output $u(t)$, the eventual output of the fuzzy descriptor system is determined as follows

$$
\sum_{k=1}^{r_e} \mu_k(z(t)) E_k \dot{x}(t) = \sum_{i=1}^{r} \mu_i(z(t)) \{ A_i x(t) + B_i u(t) + D_i w(t) \},
$$

$$
y(t) = \sum_{k=1}^{r} \mu_i(z(t)) C_i x(t),
$$

where

$$
\mu_i(z(t)) = \frac{\zeta_i(z(t))}{\sum_{j=1}^{p} \zeta_j(z(t))}, \quad \zeta_i(z(t)) = \prod_{j=1}^{p} N_{ij}(z_j(t)),
$$

$$
\mu_k(z(t)) = \frac{\zeta_k(z^e(t))}{\sum_{j=1}^{r_e} \zeta_j(z^e(t))}, \quad \zeta_k(z^e(t)) = \prod_{j=1}^{r_e} N_{kj}(z_j^e(t)),
$$

and $N_{ij}(z_j(t))$, $N_{kj}^e(z_j^e(t))$ are the degrees of membership of $z_j(t)$ and $z_j^e(t)$ in the fuzzy set $N_{ij}$ and $N_{kj}^e$, respectively. Here $\sum_{i=1}^{r} \mu_i(z(t)) = 1$ and $\sum_{k=1}^{r_e} \mu_k(z(t)) = 1$. We investigate a referential model described as [20]

$$
\dot{x}_r(t) = A_r x_r(t) + D_r r(t),
$$

with $x_r(t)$ represents the reference state, $A_r$ represents specific asymptotically stable matrix, $r(t)$ represents a bounded referential input.

The trajectoryal tracking error is defined as

$$
e(t) = x(t) - x_r(t).
$$

We investigate the $H_{\infty}$ tracking performance with respect to the tracking error $e(t)$ as [21]

$$
\int_{0}^{t_f} e^T(t) Q e(t) dt \leq \rho^2 \int_{0}^{t_f} \omega^T(t) \omega(t) dt,
$$

where $Q$ represents a positive definite weight matrix, $t_f$ represents the finished time of control and $\rho$ represents the preset disturbance alleviation level.
Let us consider the Parallel Distributed Compensation (PDC) provided from fuzzy controller [12] as

\[ u(t) = \sum_{i=1}^{r} \sum_{k=1}^{r_e} \mu_i \mu_k (K_{1jk} e(t) + K_{2jk} x_r(t)), \]  

(6)

where \( K_{1jk}, K_{2jk} \) are the controller gains. Then the proposed fuzzy controller is to be designed with the feedback gains \( K_{1jk}, K_{2jk} \) \((j = 1, ..., r, k = 1, ..., r_e)\) such that the resulting closed-loop fuzzy system is asymptotically stable and also satisfies the \( H_\infty \) performance criterion given in (5).

Combining (2) and (3), the enhanced fuzzy system is to be described as

\[ E^* \dot{x}^*(t) = \sum_{i=1}^{r} \sum_{k=1}^{r_e} \mu_i \mu_k (A_{ik}^* x^*(t) + B_i^* u(t) + D_i^* \omega^*(t)), \]  

(7)

where,

\[ x^*(t) = \begin{bmatrix} e(t) \\ x_r(t) \\ \ell(t) \end{bmatrix}, \quad \omega^*(t) = \begin{bmatrix} \omega(t) \\ r(t) \end{bmatrix}, \quad E^* = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \]

\[ A_{ik}^* = \begin{bmatrix} 0 & 0 & I \\ 0 & A_r & 0 \\ A_i (A_i - E_k A_r) & -E_k \end{bmatrix}, \quad B_i^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad D_i^* = \begin{bmatrix} 0 & D_r \\ 0 & D_i -E_k D_r \end{bmatrix}. \]

3. \( H_\infty \) Trajectory Tracking Control

For the enhanced fuzzy system proposed in (7), the performance of the \( H_\infty \) trajectory tracking control is demonstrated in the following theorem.

**Theorem 1.** Let us investigate the fuzzy descriptor system (2) with respect to the control rule (6). In case it obtains the matrices \( X_{11}, X_{21}, X_{22}, X_{31}, X_{32}, X_{33} \) and \( W_{1jk}, W_{2jk} \) \((j = 1, ..., r, k = 1, ..., r_e)\) in order to satisfy the following matrix inequalities

\[ S = S^T > 0, \]
\[ \phi_{iik} < 0, \quad i = 1, 2, ..., r, \quad k = 1, 2, ..., r_e, \]
\[ \frac{1}{r-1} \phi_{ijk} + \frac{1}{2} (\phi_{ijk} + \phi_{jik}) < 0, \quad i \neq j \leq r, \quad k = 1, 2, ..., r_e, \]  

with

\[ S = \begin{bmatrix} X_{11} & X_{21}^T \\ X_{21} & X_{22} \end{bmatrix}, \]

\[ \phi_{iik} = \begin{bmatrix} H_{11}^1 & * & * & * & * \\ H_{21} & H_{22} & * & * & * \\ H_{31} & H_{32} & H_{33} & * & * \\ 0 & 0 & D_i^T & -\rho^2 I & * \\ 0 & D_r^T & -D_r^T E_k & 0 & -\rho^2 I \\ X_{11} & X_{21}^T & 0 & 0 & -Q^{-1} \end{bmatrix}. \]
Let us consider a candidate of Lyapunov function

\[ V(t) = x^* T(t) E x^* T X^{-1} x^* (t), \]  \hspace{1cm} (11)

with \( X = \begin{bmatrix} X_{11} & X_{21}^T & 0 \\ X_{21} & X_{22} & 0 \\ X_{31} & X_{32} & X_{33} \end{bmatrix} \) and \( E x^* T X^{-1} = X^{-T} E^* \geq 0. \)

If the inequalities in (9) and (10) are satisfied then

\[ \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{k=1}^{r_e} \mu_i \mu_j \mu_k \phi_{ijk} < 0. \]  \hspace{1cm} (12)

The above inequality can be written as

\[ \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{k=1}^{r_e} \mu_i \mu_j \mu_k \left( X^T \Omega_{ijk} X + X^T Q^* X D_{i}^T \rho^2 I \right) < 0, \]  \hspace{1cm} (13)

with \( \Omega_{ijk} = (A_{ik}^* + B_{i}^* K_{jk}^*)^T X^{-1} + X^{-1} (A_{ik}^* + B_{i}^* K_{jk}^*) ; \) and \( Q^* = \text{diag}\{Q, 0, 0\}. \)

Pre-multiplying and post multiplying the above inequality by block diag\( [X^{-T}, 0] \) and block diag\( [X^{-1}, 0] \), the following parameterized matrix inequality is obtained

\[ \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{k=1}^{r_e} \mu_i \mu_j \mu_k \left( \Omega_{ijk} + Q^* D_{i}^T X^{-1} \rho^2 I \right) < 0. \]  \hspace{1cm} (14)

Let us consider the candidate of Lyapunov function (11)

\[ V(t) = x^* T(t) E x^* T X^{-1} x^* (t). \]  \hspace{1cm} (15)

Let \( K_{ik}^* = [K_{1ik} \ K_{2ik} \ 0] \). Then from the derivative of the Lyapunov function, it gives

\[ \dot{V}(t) + x^* T(t) Q^* x^* (t) - \rho^2 \omega^* T(t) \omega^* (t) = \]

\[ \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{k=1}^{r_e} \mu_i \mu_j \mu_k \{ x^* T(t) ((A_{ik}^* + B_{i}^* K_{jk}^*)^T X^{-1} + X^{-1} (A_{ik}^* + B_{i}^* K_{jk}^*) + Q^*) x^* (t) \} \]

\[ + x^* T(t) X^{-T} D_{i}^T \omega^* (t) + \omega^* T(t) D_{i}^T X^{-1} x^* (t) - \rho^2 \omega^* T(t) \omega^* (t) \]  \hspace{1cm} (16)
\[ \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{k=1}^{r} \mu_i \mu_j \mu_k \begin{bmatrix} x^*T(t) & \omega^*T(t) \end{bmatrix} \begin{bmatrix} \Omega_{ijk} + Q^* & \ast \\
abla_i{T}X^{-1} & -\rho^2 I \end{bmatrix} \begin{bmatrix} x^*(t) \\
\omega^*(t) \end{bmatrix}, \] (17)

where \( x^*(t), \omega^*(t) \) are matrices and have been defined in Eq. (7); \( x^*T(t), \omega^*(t) \) are transposed matrices of \( x^*(t), \omega^*(t) \).

From (17) and (14), the following inequality is obtained

\[ \dot{V}(t) + x^*T(t)Q^*x^*(t) - \rho^2 \omega^*T(t)\omega^*(t) < 0. \] (18)

Integrating the above inequality from 0 to \( \infty \) on both sides, it yields

\[ V(\infty) - V(0) + \int_0^{\infty} (x^*T(t)Q^*x^*(t) - \rho^2 \omega^*T(t)\omega^*(t))dt < 0. \] (19)

With zero initial condition, \( V(0) = 0 \) and hence

\[ \int_0^{\infty} x^*T(t)Q^*x^*(t)dt < \int_0^{\infty} \rho^2 \omega^*T(t)\omega^*(t)dt, \] (20)

\[ \int_0^{\infty} e^*T(t)Q^*e^*(t)dt < \int_0^{\infty} \rho^2 \omega^*T(t)\omega^*(t)dt. \] (21)

Eventually the proof is complete.

3.1. Stability analysis

Let us consider (18). If \( w^*(t) = 0 \), then \( \dot{V}(t) < 0 \), which implies that the closed loop system seems asymptotically stable.

3.2. Common \( B \) matrix case

In this subsection, the case related to common \( B \) matrix is considered, where \( B_i = B \ (i = 1, 2, ..., r) \). The LMI conditions for designing the controller are given via the theorem as follows.

**Theorem 2.** Let us investigate the fuzzy descriptor system (2) with respect to the control rule (6). In case it obtains some matrices \( X_{11}, X_{21}, X_{22}, X_{31}, X_{32}, X_{33} \) and \( W_{1i}, W_{2i} \) \((i = 1, ..., r, \ k = 1, ..., r^e)\) as to satisfied the matrix inequalities as follows,

\[ S = S^T > 0, \] (22)

\[
\begin{bmatrix}
M_{11} & * & * & * & * \\
M_{21} & M_{22} & * & * & * \\
M_{31} & M_{32} & M_{33} & * & * \\
0 & 0 & D_i^T & -\rho^2 I & * \\
0 & D_i^T & -D_i^T E_i^T & 0 & -\rho^2 I \\
x_{11} & X_{21}^T & 0 & 0 & 0 & -Q^{-1}
\end{bmatrix}
\] < 0, \( i = 1, ..., r, \ k = 1, ..., r^e \),
(recall that ‘*’ represents the transposed elements in symmetric positions).

\[
S = \begin{bmatrix} X_{11} & X_{21}^T \\ X_{21} & X_{22} \end{bmatrix},
\]

\[
M_{11} = X_{31}^T + X_{31}, \quad M_{21} = X_{32}^T + A_rX_{21}, \quad M_{22} = A_rX_{22} + X_{22}^T A_r, \quad M_{31} = X_{33}^T + A_iX_{11} + (A_i - E_kA_r)X_{21} - E_kX_{31} + BW_{1ik}, \quad M_{32} = A_iX_{21}^T + (A_i - E_kA_r)X_{22} - E_kX_{32} + BW_{2ik}, \quad M_{33} = -X_{33}^TE_k^T - E_k^TX_{33}.
\]

Then the closed loop system with the controller gain matrices \([K_{1ik}, K_{2ik}] = [W_{1ik}, W_{2ik}], [X_{11}X_{21}^T; X_{21}X_{22}]^{-1}\) satisfy the given \(H_\infty\) performance criteria.

In this case, the LMI conditions for controller design are simpler and number of LMI conditions is also less than that of the general case.

### 3.3. Simulation results

Let us consider the simple uncertain nonlinear system introduced in [13] with some external disturbance. The system is represented by

\[
(1 + a \cos(\theta(t)))\dot{\theta}(t) = -b\dot{\theta}^2(t) + c\theta(t) + du(t) + 0.1\omega(t),
\]

with \(a = 0.2, b = 1, c = -1, d = 10, w(t) = \sin(5t)\) and the range of \(\dot{\theta}(t)\) is \(|\dot{\theta}(t)| < \phi, \phi = 4\). The newly proposed descriptor fuzzy model is improved from [13] as follows

\[
\sum_{k=1}^{2} \mu_k^e(z(t))E_k\dot{x}(t) = \sum_{k=1}^{2} \mu_k^c(z(t))\{A_i x(t) + B_i u(t) + D_i \omega(t)\},
\]

\[
y(t) = \sum_{k=1}^{2} \mu_k^e(z(t))C_i x(t),
\]

with \(x(t) = [x_1(t), x_2(t)]^T = [\theta(t), \dot{\theta}(t)]^T\). The parameters of the constant matrices are as

\[
E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 + a \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 - a \end{bmatrix},
\]

\[
A_1 = \begin{bmatrix} 0 & 1 \\ c & -b\phi^2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ c & 0 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 0 \\ d \end{bmatrix}, \quad D_i = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \quad i = 1, 2,
\]

\[
\mu_1(x_2(t)) = \frac{x_2^2(t)}{2}, \quad \mu_2(x_2(t)) = 1 - \frac{x_2^2(t)}{2},
\]

\[
\mu_1^c(x_1(t)) = \frac{1 + \cos(x_1(t))}{2}, \quad \mu_2^c(x_1(t)) = \frac{1 - \cos(x_1(t))}{2}.
\]

Then the referential model and referential input were considered as follows

\[
\begin{bmatrix} \dot{x}_{r_1} \\ \dot{x}_{r_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_{r_1} \\ x_{r_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 2\sin\left(\frac{\chi}{2}\right) \end{bmatrix}.
\]
Figure 1. Trajectorial results of state variables $x(t)$ (dashed line) and the referencing trajectories $x_r(t)$ (solid line)

The $H_{\infty}$ tracking controller is implemented based on the LMI requirements in Theorem 2. With $Q = 0.1I$ and $\rho^2 = 0.01$, the parameters of Lyapunov function and the feedback gain matrices $K_{1kk}, K_{2kk}$ obtained are given below

State and reference trajectories $x(t)$ and $x_r(t)$ with the initial condition $x(0) = [0.5 \ 0]^T$ and $x_r(0) = [0 \ 0]^T$ are presented in Fig. 1.
4. NOVEL T-S FUZZY DESCRIPTOR FOR UNCERTAIN NONLINEAR SYSTEM

This section starts with introduction to uncertain T-S descriptor fuzzy model and then the robust $H_{\infty}$ tracking control requirement is formulated.

The continuous T-S fuzzy model [6] denotes nonlinear system dynamics based on fuzzy IF-THEN laws. It is possible to present the newly proposed descriptor fuzzy model of an uncertain nonlinear system presented as follows.

Plant law:

$$\text{IF } z_1^e(t) = N_{k_1}^e, ..., z_p^e(t) = N_{k_p}^e \text{ and } z_1(t) = N_{i_1}, ..., z_p(t) = N_{i_p} \text{ THEN}$$

$$(E_k(\theta) + \Delta E_k(t))\dot{x}(t) = (A_i(\theta) + \Delta A_i(t))x(t) + (B_i(\theta) + \Delta B_i(t))u(t) + D_i\omega(t),$$

$$y(t) = C_i x(t), \quad i = 1, 2, ..., r, \quad k = 1, 2, ..., r^e,$$

where, $A_i(\theta) = A_{i0} + \sum_{l=1}^{L} \theta_l(t) A_{il}$, $B_i(\theta) = B_{i0} + \sum_{l=1}^{L} \theta_l(t) B_{il}$, $E_k(\theta) = E_{k0} + \sum_{l=1}^{L} \theta_l(t) E_{kl}$, $z_1(t), ..., z_p(t)$ are premise variables, $p$ is the number of premise variables, $N_{kj}^e$ ($j = 1, p^k$), $N_{ij}$ ($j = 1, p$) are the fuzzy sets and $r$ represents the amount of laws. For simplicity $\theta(t)$ is denoted as $\theta$. Here, $x(t) \in R^{n \times 1}$ is the state vector, $y(t)$ is the controlled output and $u(t)$ is the input vector. $A_{i0} \in R^{n \times n}$, $A_{il} \in R^{n \times n}$, $B_{i0} \in R^{n \times m}$, $B_{il} \in R^{n \times m}$, $E_{k0} \in R^{n \times n}$, $E_{kl} \in R^{n \times n}$, $C_i \in R^{n \times n}$ are constant real matrices, $\theta_l(t)$ represents time varying parametric uncertainties; $\Delta A_i(t)$, $\Delta B_i(t)$ and $\Delta E_k(t)$ are time-varied matrices of dimensions available, which represent modelling errors. The necessary assumptions prove that rank$(E_k) \leq n$; $\Delta A_i \in R^{n \times n}$, $\Delta B_i \in R^{n \times m}$, $\Delta E_i \in R^{n \times m}$ represent the uncertainties and are bounded, i.e., $\|\Delta A_i\| < \delta_i$, $\|\Delta B_i\| < \beta_i$, $\|\Delta E_i\| < \phi_i$ where $\|\cdot\|$ denotes spectral norm and $\delta_i, \beta_i, \phi_i$ represent any positive values. Other specific constraints can be consulted in [14].

From input $x(t)$ and output $u(t)$, the eventual state-space output of the proposed fuzzy system is described as

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i(z(t)) \{(A_i + \Delta A_i(t))x(t) + (B_i + \Delta B_i(t))u(t)\},$$

$$\sum_{k=1}^{r^e} \mu_k^e(E_k(\theta) + \Delta E_k(t))\dot{x}(t) = \sum_{i=1}^{r} \mu_i \{(A_i(\theta) + \Delta A_i(t))x(t) + (B_i(\theta) + \Delta B_i(t))u(t) + D_i\omega(t)\},$$

$$y(t) = \sum_{i=1}^{r} \mu_i C_i x(t), \quad (26)$$

with $\mu_i = \frac{\zeta_i(z(t))}{\sum_{j=1}^{p} \zeta_j(z(t))}$, $\zeta_i(z(t)) = \prod_{j=1}^{p} N_{ij}(z_j(t))$,

$$\mu_k^e = \frac{\zeta_k^e(z^e(t))}{\sum_{j=1}^{r^e} \zeta_j^e(z^e(t))}$$,

$N_{ij}(z_j(t))$ and $N_{kj}^e(z_j^e(t))$ represent the degrees of membership of $z_j(t)$ and $z_j^e(t)$ in the fuzzy set $N_{ij}$ and $N_{kj}^e$, respectively. Here $\sum_{i=1}^{r} \mu_i(z(t)) = 1$ and $\sum_{k=1}^{r^e} \mu_k^e(z(t)) = 1$. For simplicity, $\mu_k^e(z(t))$ and $\mu_i(z(t))$ were represented as $\mu_k^e(z(t))$ and $\mu_i(z(t))$ respectively.
The uncertain matrices $\Delta A_i(t)$, $\Delta B_i(t)$ and $\Delta E_k(t)$ were assigned to be norm-limited and improved from [2] as follows

$$[\Delta A_i(t) \quad \Delta B_i(t)] = \sum_{l=1}^{L_a} M_{il}^a \Delta^a_{il}(t) [N_{i1l}^a \quad N_{i2l}^a],$$

$$\Delta E_k(t) = \sum_{l=1}^{L_a} M_{kl}^e \Delta^e_{kl}(t) N_{kl}^e,$$  \hspace{1cm} (27)

with $M_{il}^a, M_{kl}^e, N_{i1l}^a, N_{i2l}^a$ and $N_{kl}^e$ represent actual constant matrices with dimension available and $\Delta^a_{il}(t), \Delta^e_{kl}(t)$ represent time-varied equations, satisfying $|\Delta^a_{il}(t)| < 1, |\Delta^e_{kl}(t)| < 1, \forall t > 0$.

Let us consider a reference model and the $H_{\infty}$ performance measure as given in Section 2 with the Parallel Distributed Compensation (PDC) fuzzy controller improved from [12],

$$u(t) = \sum_{i=1}^{r} \sum_{k=1}^{r^e} \mu_i \mu_k^e (K_{1ik} \epsilon(t) + K_{2ik} x(t)),$$  \hspace{1cm} (28)

where $K_{1ik}$ and $K_{2ik}$ are the controller gains. Newly proposed fuzzy controller is implemented with the feedback gains $K_{1ik}$ and $K_{2ik} (i = 1, ..., r, k = 1, ..., r^e)$ such that the resulting closed-loop system ensures asymptotically stable and responds the $H_{\infty}$ performance given in (5).

Combining (26) and (3) and relating to the control rule (28), the augmented fuzzy descriptor system is to be expressed as

$$E^* \dot{x}^*(t) = \sum_{i=1}^{r} \sum_{j=1}^{r^e} \sum_{k=1}^{r^e} \mu_i \mu_j \mu_k^e \{(A^*_{ik}(\theta) + \Delta A^*_{ik}(t) + (B^*_{ik}(\theta) + \Delta B^*_{ik}(t)) K^*_{jk}) x^*(t) + D^*_{ik} \omega^*(t)\},$$

where $x^*(t) = \begin{bmatrix} e(t) \\ x_r(t) \\ \dot{\epsilon}(t) \end{bmatrix}$, $\omega^*(t) = \begin{bmatrix} \omega(t) \\ r(t) \end{bmatrix}$, $A^*_{ik}(\theta) = \begin{bmatrix} 0 & 0 & I \\ 0 & A_r & 0 \\ A_i(\theta) & A_i(\theta) - E_k(\theta) A_r & -E_k(\theta) \end{bmatrix}$, $\Delta A^*_{ik}(t) = \begin{bmatrix} \Delta A_i(t) & \Delta A_i(t) - \Delta E_k(t) A_r & -\Delta E_k(t) \end{bmatrix}$, $B^*_{ik}(\theta) = \begin{bmatrix} 0 \\ 0 \\ B_i(\theta) \end{bmatrix}$, $\Delta B^*_{ik}(t) = \begin{bmatrix} 0 \\ 0 \\ \Delta B_i(t) \end{bmatrix}$, $K^*_{jk} = \begin{bmatrix} K_{1jk} & K_{2jk} & 0 \end{bmatrix}$, $D^*_{jk} = \begin{bmatrix} 0 & 0 & D_r \\ 0 & D_r & 0 \\ D_i & -E_k(\theta) D_r \end{bmatrix}$, $E^* = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

$$[\Delta A^*_{ik}(t) \quad \Delta B^*_{ik}(t)] = \sum_{l=1}^{L_a} M_{il}^a \Delta^a_{il}(t) [N_{i1l}^a \quad N_{i2l}^a] + \sum_{l=1}^{L_a} M_{kl}^e \Delta^e_{kl}(t) N_{kl}^e,$$  \hspace{1cm} (30)
The proposed fuzzy descriptor system (29) affinely depends on the parametric vector. As in [21] and [22], both lower/upper bounds of the uncertain coefficient and their rates of variation are assumed to be known. Specifically:

1. Each parameter $\theta_l$ ranges within the known lower $\theta_l^l$ and upper $\theta_l^u$ bounds, i.e.,

$$\theta_l \in [\theta_l^l, \theta_l^u].$$

2. The speed of variation $\dot{\theta}_l$ is precisely calculated at all times and satisfies

$$\dot{\theta}_l \in [\nu_l, \bar{\theta}_l],$$

where $\nu_l$ and $\bar{\theta}_l$ represent known lower/upper bounds of $\dot{\theta}_l$, respectively.

With these assumptions, the parameter vector $\theta_l$ takes values within the hyper-rectangle called parameter box and the rate vector $\dot{\theta}_l$ takes values in another hyper-rectangle called rate box. It is denoted as,

$$V : = \{ (\nu_1, \nu_2, ..., \nu_L)^T : \nu_l \in \{ \theta_l^l, \theta_l^u \} \},$$

$$W : = \{ (\omega_1, \omega_2, ..., \omega_L)^T : \omega_l \in \{ \nu_l, \bar{\theta}_l \} \},$$

which are the set of $2^L$ vertices of the parameter box and the rate box, respectively.

5. PROPOSED ROBUST $H_\infty$ TRACKING CONTROL IMPLEMENTATION WITH FIXED LYAPUNOV FUNCTION

In this section, Lyapunov function-based robust $H_\infty$ tracking controller design for proposed fuzzy descriptor system is presented. First it investigates the fixed Lyapunov function described as,

$$V(t) = x^*^T(t)E^*^T X^{-1} x^*(t)$$

with

$$X = \begin{bmatrix} X_{11} & X_{21}^T & 0 \\ X_{21} & X_{22} & 0 \\ X_{31} & X_{32} & X_{33} \end{bmatrix}, \quad E^*^T X^{-1} = X^{-T} E^* \geq 0.$$

**Theorem 3.** Let us consider the fuzzy descriptor system (29) and the control rule (28). In case it obtains certain matrices $X$ as defined in (35) and $W_{jk}$ $(j = 1, ..., r, k = 1, ..., r^e)$ as to satisfy the matrix inequalities presented as follows,

$$S^T = S > 0,$$

$$\phi_{iik}^*(\nu) < 0, \\forall \nu \in V, \; i = 1, 2, ..., r, \; k = 1, 2, ..., r^e,$$

$$\frac{1}{r-1} \phi_{iik}^*(\nu) + \frac{1}{2} (\phi_{ijk}^*(\nu) + \phi_{jik}^*(\nu)) < 0, \\forall \nu \in V, \; 1 \leq i \neq j \leq r, \; k = 1, 2, ..., r^e,$$

$$M_{il}^a = \begin{bmatrix} 0 & 0 \\ 0 & M_{il}^a \end{bmatrix}, \; M_{kl}^{c_e} = \begin{bmatrix} 0 & 0 \\ M_{kl}^{c_e} & 0 \end{bmatrix},$$

$$N_{il}^w = \begin{bmatrix} 0 & 0 \\ N_{il}^w & N_{il}^w \end{bmatrix}, \; N_{2l}^a = N_{2l}^a, \; N_{kl}^c = \begin{bmatrix} 0 & N_{kl}^c \end{bmatrix}.$$
rewritten as follows
holds for the range of defined in the parameter box improved from \([23]\). Hence, If the above inequality is satisfied in the vertices of the parameter box \(V\)

Proof. If (37) and (38) are satisfied, the following parameterized inequality is obtained

\[
\begin{bmatrix}
\phi_{ijk}^e(\nu) \\
\end{bmatrix} = \begin{bmatrix}
A_{ijk}^{31} & \cdots & A_{ijk}^{61}
\end{bmatrix},
\]

\[
A_{3i}^{31} = \begin{bmatrix}
a_i M_{i1}^{qT} \\
\vdots \\
\alpha_{iL_n} M_{iL_n}^{qT}
\end{bmatrix}, \\
A_{4i}^{41} = \begin{bmatrix}
(N_{i11}^a X + N_{i21}^a W_{j1}) \\
\vdots \\
(N_{iL_n1}^a X + N_{iL_n2}^a W_{j1})
\end{bmatrix},
\]

\[
A_{5i}^{61} = \begin{bmatrix}
N_{k1}^a X \\
\vdots \\
N_{kL_n}^a X
\end{bmatrix}, \quad \gamma = X \begin{bmatrix}
I & 0 & 0
\end{bmatrix},
\]

\[
A_{ijk}^{11}(\nu) = X^T A_{ik}^* T(\nu) + W_{jk}^* T(\nu) + B_{ik}^* T(\nu) W_{jk}^*.
\]

\[
\epsilon_i^a = \text{diag}(\epsilon_{i1}^a, \ldots, \epsilon_{iL_n}^a), \quad \epsilon_k^e = \text{diag}(\epsilon_{k1}^e, \ldots, \epsilon_{kL_n}^e)
\]

and \(W_{jk}^* = K_{jk}^* X\), then the closed loop system ensures asymptotically stable and satisfies the given \(H_\infty\) performance criteria.

If the above inequality is satisfied in the vertices of the parameter box \(V\), then the inequality holds for the range of defined in the parameter box improved from \([23]\). Hence,

\[
\sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{k=1}^{r^e} \mu_i \mu_j \mu_k^e \phi_{ijk}^e(\nu) < 0, \quad \forall \nu \in V.
\]

Based on (27), using the Schur complement Lemma and the inequality \(Y^T Z + Z^T Y \leq Y^T Y + Z^T Z\) improved from \([24]\), the matrices related to \(\Delta A_i(t), \Delta B_i(t)\) and \(\Delta E_k(t)\) can be rewritten as follows

\[
\sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{k=1}^{r^e} \mu_i \mu_j \mu_k^e \Upsilon^e_{ijk}(t, \theta) < 0
\]

with \(\Upsilon^e_{ijk}(t, \theta) = \begin{bmatrix}
\Omega^e_{ijk}(t, \theta) & * & * \\
D_{i1}^T & -\rho^2 I & * \\
\gamma & 0 & -Q^{-1}
\end{bmatrix}, W_{jk}^* T(\theta), A_{ik}^* T(\theta), B_{ik}^* T(\theta), \Delta A_{ik}^* T(t), \Delta B_{ik}^* T(t)
\]

have been defined in (30),
\[ \Omega_{i,jk}(t, \theta) = X^T A_{ik}^*(t) + W_{jk}^* B_{i}^*(t) + X^T \Delta A_{ik}^*(t) + W_{jk}^* \Delta B_{i}^*(t) + \Delta B_{i}^*(t) W_{jk}. \]

Then from the derivative of \( V(t) \), it gives,

\[
\dot{V}(t) + x^T(t) Q x(t) - \rho^2 \omega^T(t) \omega(t) = 
\sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{k=1}^{r^*} \mu_i \mu_j \mu_k \left\{ x^T(t) \left( X^T \Omega_{i,jk}^*(t, \theta) X^{-1} + Q^* \right) x(t) \right\} 
+ x^T(t) X^{-T} D_j^* X^{-1} x(t) - \rho^2 \omega^T(t) \omega(t)
\]

\[
= \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{k=1}^{r^*} \mu_i \mu_j \mu_k \left\{ x^T(t) \omega^T(t) \right\} \left[ X^{-T} \Omega_{i,jk}^*(t, \theta) X^{-1} + Q^* \right] \left[ x(t) \omega(t) \right].
\]

From (43) and (45), the following inequality can be obtained,

\[
\dot{V}(t) + x^T(t) Q x(t) - \rho^2 \omega^T(t) \omega(t) < 0.
\]

Integrating the above inequality from 0 to \( \infty \), it gives,

\[
V(\infty) - V(0) + \int_{0}^{\infty} (x^T(t) Q x(t) - \rho^2 \omega^T(t) \omega(t)) dt < 0.
\]

With zero initial condition, \( V(0) = 0 \) and hence

\[
\int_{0}^{\infty} x^T(t) Q x(t) dt < \int_{0}^{\infty} \rho^2 \omega^T(t) \omega(t) dt,
\]

\[
\int_{0}^{\infty} e^T(t) Q e(t) dt < \int_{0}^{\infty} \rho^2 \omega^T(t) \omega(t) dt.
\]

Thus the proof is completed.
Figure 2. Set-up diagram of a two-joint robot arm

Table 1. Premise variables for the fuzzy rules - two-joint robot arm

<table>
<thead>
<tr>
<th>Rules $i$</th>
<th>$N_{i1}$</th>
<th>$N_{i2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Negative</td>
<td>Negative</td>
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<tr>
<td>2</td>
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<td>Zero</td>
</tr>
<tr>
<td>3</td>
<td>Negative</td>
<td>Positive</td>
</tr>
<tr>
<td>4</td>
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<td>Negative</td>
</tr>
<tr>
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<td>7</td>
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<td>Negative</td>
</tr>
<tr>
<td>8</td>
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<td>Zero</td>
</tr>
<tr>
<td>9</td>
<td>Positive</td>
<td>Positive</td>
</tr>
</tbody>
</table>

6. SIMULATION RESULTS

Let us consider the two-joint robot arm (see Fig. 2). The dynamics of the two-joint robotic manipulator [20] is expressed as,

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau \quad (50)$$

with

$$M(q) = \begin{bmatrix} (m_1 + m_2)l_1^2 & \frac{m_2l_1l_2(s_1s_2 + c_1c_2)}{m_2l_2^2} \\ \frac{m_2l_1l_2(s_1s_2 + c_1c_2)}{m_2l_1^2} & m_2l_2^2 \end{bmatrix},$$

$$C(q, \dot{q}) = m_2l_1l_2(c_1s_2 - s_1c_2) \begin{bmatrix} 0 & -\dot{q}_2 \\ -\dot{q}_1 & 0 \end{bmatrix},$$

$$G(q) = \begin{bmatrix} -(m_1 + m_2)l_1g s_1 \\ m_2l_2gs_2 \end{bmatrix}.$$

The nominal parameters of the system are the link masses $m_1 = m_2 = 1$ kg, link lengths $l_1 = l_2 = 1$ m and the gravitational acceleration $g_r = 9.81$ m/s$^2$. In this example, structural uncertainties in masses are considered and the perturbation is assumed to be within ±5% from their nominal value.

The operating domain is considered as $x_1(t) \in [-\pi/3, \pi/3]$, $x_3(t) \in [-\pi/3, \pi/3]$, $x_2(t) \in [-5, 5]$, $x_4(t) \in [-5, 5]$ and the input $u_1(t) \in [-25, 25]$ and $u_2(t) \in [-15, 15]$. 
Equidistant triangular membership functions with centers $-\pi/3$, 0 and $\pi/3$ are assumed for both of $x_1(t)$ and $x_3(t)$. With the uncertainties in masses $m_1$ and $m_2$, the uncertainties in the fuzzy model can be derived as $\theta_1(t) \in [-0.05, 0.05]$ and $\theta_2(t) \in [-0.05, 0.05]$.

**Table 2.** Coefficients of matrices $A_i(\theta)$ and $B_i(\theta)$ - Two-joint manipulator

<table>
<thead>
<tr>
<th>$i$</th>
<th>$a_{021}$</th>
<th>$a_{023}$</th>
<th>$a_{041}$</th>
<th>$a_{043}$</th>
<th>$a_{121}$</th>
<th>$a_{123}$</th>
<th>$a_{221}$</th>
<th>$a_{223}$</th>
<th>$a_{241}$</th>
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<td>2.488</td>
<td>2.01</td>
<td>-0.085</td>
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<td>-1.91</td>
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<td>-0.881</td>
<td>0.196</td>
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<td>0.196</td>
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<tr>
<td>9</td>
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<td>-6.08</td>
<td>-0.069</td>
<td>5.564</td>
<td>-5.65</td>
<td>5.564</td>
<td>-0.069</td>
<td>-6.08</td>
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**Table 3.** Parameters of matrices $\delta A_i$ - Two-joint manipulator

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<tr>
<th>$i$</th>
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<th>$\delta a_{023}$</th>
<th>$\delta a_{024}$</th>
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<th>$\delta a_{122}$</th>
<th>$\delta a_{123}$</th>
<th>$\delta a_{124}$</th>
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<td>1.588</td>
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**Table 4.** Parameters of matrices $E_i(\theta)$ and $\Delta E_i(t)$ - Two-joint manipulator

<table>
<thead>
<tr>
<th>$i$</th>
<th>$e_{240_i}$, $e_{i=420h}$</th>
<th>$e_{224_i}$, $e_{i=242}$</th>
<th>$\Delta e_{24_i}$, $\Delta e_{i=42}$</th>
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<tr>
<td>3</td>
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<td>0.386</td>
<td>0.386</td>
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</table>

The fuzzy rules are considered as follows.
Plant law $i$: IF $x_1$ is $N_{1i}$ and $x_3$ is $N_{3i}$ THEN

$$(E_i(\theta) + \Delta E_i(t))\dot{x}(t) = (A_i(\theta) + \Delta A_i(t))x(t) + Bu(t) + D_i w(t),$$

$$y(t) = C_i x(t), \ i = 1, ..., 9,$$
\[ A_{i0}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a_{21i0} & 0 & a_{23i0} & 0 \\ 0 & 0 & 0 & 1 \\ a_{41i0} & 0 & a_{43i0} & 0 \end{bmatrix}, \quad A_{i1}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a_{21i1} & 0 & a_{23i1} & 0 \\ 0 & 0 & 0 & 1 \\ a_{41i1} & 0 & a_{43i1} & 0 \end{bmatrix}, \]

\[ B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad C_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_\theta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 & e_{24\theta} \\ 0 & 0 & 1 & 0 \\ 0 & e_{42\theta} & 0 & 1 \end{bmatrix}, \]

\[ \Delta A_i(t) = \begin{bmatrix} \Delta a_{21i}(t) & \Delta a_{22i}(t) & \Delta a_{23i}(t) & \Delta a_{24i}(t) \\ 0 & 0 & 0 & 0 \\ \Delta a_{41i}(t) & \Delta a_{42i}(t) & \Delta a_{43i}(t) & \Delta a_{44i}(t) \end{bmatrix}, \quad D_i = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \]

\[ E_{i1}(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad E_{i2}(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & e_{24i}(t) \\ 0 & 0 & 0 & 0 \\ 0 & e_{42i}(t) & 0 & 1 \end{bmatrix}, \]

\[ \Delta E_i(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta e_{24i}(t) \\ 0 & 0 & 0 & 0 \\ 0 & \Delta e_{42i}(t) & 0 & 0 \end{bmatrix}. \]

The fuzzy sets \( N_{i1} \) and \( N_{i2} \) for rules \( i = 1, \ldots, 9 \) are shown in Table 1. The parameters of the proposed fuzzy model are successfully computed via the linear programming method discussed in Section 2. The values of \( A_i(\theta) \) and \( B_i(\theta) \) coefficients are described in Table 2.

The parameters of \( \Delta A_i(t) \) are shown in Table 3. For \( E_i(\theta) \) and \( \Delta E_i(t) \), the parameters are shown in Table 4.

Let us investigate the referential model as follows:

\[ \dot{x}_r(t) = A_r x_r(t) + r(t) \quad (51) \]

where

\[ A_r = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -6 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -6 & -5 \end{bmatrix}, \]

and

\[ r(t) = \begin{bmatrix} 0 & 7 \sin(t) & 0 & 7 \cos(t) \end{bmatrix}^T. \]

The \( H_\infty \) tracking controller design problem is considered with the above referential model given by (51). In this benchmark test, the fuzzy descriptor model satisfies the condition \( \mu_i = \mu_i^* \) and \( r = r_e \). Let us assume the mass of the links as \( m_1 + \Delta m_1 = 1 + 0.05 \sin(2t) \) and \( m_2 + \Delta m_2 = 1 + 0.05 \cos(2t) \). Here \( 0.05 \sin(2t) \) and \( 0.05 \cos(2t) \) represent the uncertainties.
**Figure 3.** Trajectorial results with zooming tracking error $e(t)$ of state variables $x(t)$ (dashed line & dotted line for $\rho^2 = 0.001$ and $\rho^2 = 0.01$) and the referential trajectories $x_r(t)$ (solid line).

**Figure 4.** Trajectorial results with zooming tracking error $e(t)$ of state variables $x(t)$ (dashed line & dotted line for $\rho^2 = 0.001$ and $\rho^2 = 0.01$) and the referential trajectories $x_r(t)$ (solid line).
The external disturbances (e.g., cogging torque in the actuator) are assumed to be $w_1(t) = 0.4 \cos(10t) \cos(2t) + 0.2 \exp(-t) \sin(4t)$ and $w_2(t) = 0.3 \sin(5t) + 0.25 \exp(-2t)$.

With $Q = 0.01I$, the $H_\infty$ tracking controller is designed for different values by using the proposed Algorithm 1. With zero initial condition, the simulation results are presented in Fig. 3, Fig. 4, Fig. 5 for $\rho^2 = 0.001$ and $\rho^2 = 0.01$. In Fig. 3, the trajectorial results of $x(t)$ and the referential trajectories $x_r(t)$ for $\rho = 0.001$ and $\rho = 0.01$ are shown. The tracking error plots for these two values of $\rho^2$ are presented in Fig. 3, Fig. 4. The control inputs $u(t)$ are plotted in Fig. 5.

7. CONCLUSIONS

This paper proposes a T-S fuzzy model-based reference trajectory controller satisfying $H_\infty$ performance criterion for uncertain fuzzy descriptor systems. Sufficient conditions for
controller design satisfying the given $H_\infty$ performance criterion are formulated via LMI matrix inequalities. The proposed fuzzy descriptor system approach yields lesser number of inequality conditions than those obtained using the standard state-space approach. It is convincingly shown that, by the newly proposed design approach, the required tracking controller can be successfully implemented by resolving a set of inequalities and the specified $H_\infty$ disturbance attenuation level can be obtained. In order to demonstrate the effectiveness of the novel fuzzy controller approach, tracking control benchmark test of a two-joint robot arm under external disturbances is investigated and the simulation results show that the proposed fuzzy system robustly and precisely track the referential trajectory.

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