Journal of Computer Science and Cybernetics, V.34, N.3 (2018), 233–248
DOI 10.15625/1813-9663/34/3/13164

ON THE TESTING MULTI-VALUED MARTINGALE DIFFERENCE HYPOTHESIS

LUC TRI TUYEN

Institute of Information Technology, Vietnam Academy of Science and Technology
tuyenlt@ioit.ac.vn

Abstract. This paper presents a definition of Multi-Valued Martingale Difference (MVMD) based on Castaing representation of a multi-valued martingale that consists of martingale difference selections. Then testing the Multi-Valued Martingale Difference Hypothesis (MVMDH) is studied. Testing the Martingale Difference Hypothesis (MDH) earlier was based on linear measures, and later is developed in two directions to consider the existing nonlinearity in economic and financial data. First, the classical approaches have been modified by taking into account the possible nonlinear dependence. Second, the use of more sophisticated statistical tools such as those based on the generalized spectral analysis. In this paper, both these developments in MDH are modified for MVMDH and are applied to exchange rate data and returns of stock market data.

Keywords. Martingale difference hypothesis; Multi-values martingale difference; Generalized spectral analysis; Exchange rates.

1. INTRODUCTION

Given a time series, one may want to know whether observations in this series are independent. If they are independent, there is no way to predict the next value. The problem of testing the independence of a time series received much attention in the literature. The first approaches should be mentioned are the tests using Portmanteau statistics based on autocorrelation functions [4]. Skaug and Tjϕstheim [40] later proposed a test for serial pairwise independence based on empirical distribution function. Kendall's study [20] found that the weekly changes in a wide variety of financial prices could not be predicted from either past changes in the series or from past changes in other price series. The publication of the papers by Roberts [39] and Osborne [36] develops the proposition that it is not absolute price changes but the logarithmic price changes which are independent of each other, this implies that prices are generated as Brownian motion. However, it was proved that Brownian motion is a martingale [35].

Historically, martingale hypothesis has been widely studied and applied in economics [12, 37]. One of the main reasons is efficient market hypothesis (EMH) [15, 32]. The efficient market hypothesis is a theory in financial economics that states that asset prices fully reflect all available information. A direct implication is that it is impossible to "beat the market" consistently on a risk-adjusted basis since market prices should only react to new information. Discrete-time martingale is a discrete-time stochastic process (i.e., a sequence of random variables) $X_1, X_2, X_3, ...$ that satisfies for any time n,

$$\begin{cases} E\left(|X_n|\right) < \infty \\ E\left(X_{n+1}|X_n, X_{n-1}, ..., X_1\right) = X_n. \end{cases} \quad \text{a.s.} \tag{1}$$

Therefore, an implication of EMH is that asset price follows a martingale [2, 34]. It means that the best prediction of tomorrow's asset price is the today's. Then the asset returns which is defined by $R_t = \dfrac{X_t - X_{t-1}}{X_{t-1}}$ follows a martingale difference sequence (MDS) [13]. A martingale difference series relative to a given filtration $\mathcal{F}_t$ is a time series Y_t such that, for every t

$$\begin{array}{l} \text{(i) } Y_t \text{ is } \mathcal{F}_t\text{-measurable,} \\ \text{(ii) } E(Y_t|\mathcal{F}_{t-1}) = 0. \end{array} \tag{2}$$

From the definitions (1) and (2), the sequence X_t follows a martingale then the asset returns R_t follows a martingale difference sequence. For some technical reasons, instead of testing whether a sequence follows a martingale, it is more common to test whether its returns follow a martingale difference sequence.

There have been many studies in the literature concentrated on tests of the martingale difference hypothesis in the general form

$$E(Y_t|\mathcal{F}_{t-1}) = \mu \text{ a.s.} \tag{3}$$

for some unknown $\mu \in \mathbb{R}$. Since the procedure that has been widely used proposed by Lo and MacKinlay [29] using a variance ratio (VR) test, this undergone many improvements for testing market efficiency and return predictability [6, 7, 8, 21, 42]. Another currently popular test for predictability is Ljung and Box [28] and later generalized by Lobato, Nankervis and Savin [30, 31] and Escanciano and Lobato [12]. These two approaches are designed to test lack of serial correlation but not necessarily the MDH. Durlauf [11] and Deo [9] proposed the spectral shape tests which are powerful in testing for lack of correlations but may not be able to capture nonlinear non-martingales with zero correlations. Alternative tests designed to detect nonlinear dependence have been proposed by Hong [16, 17] and Kuan and Lee [24]. All the above tests are MDH tests that are convenient to deal with real-valued asset returns and test whether the asset returns follow a MDS.

From the works by Kendall [19] and Matheron [33], random variables are no longer limited to real values that are extended to set-valued (or multi-valued). In many real problems we are faced with random experiments whose outcomes are not numbers but are expressed in inexact linguistic terms. For instance, when we are questioned about the asset returns of a stock price, there are a group of individuals chosen to answer in linguistic terms such as 'good', 'very good', 'very very good', 'bad', 'very bad', 'very very bad'. These outcomes can be described by fuzzy set which was introduced by Zadeh [44]. A possible way to handle this type of 'data' is using the concepts of fuzzy set-valued random variable introduced by Puri and Ralescu [38]. Over past 40 years there were many important works for set-valued and fuzzy set-valued random variables related to strong law of large numbers [5, 14, 41], center limit theorems [1, 26, 27] and convergence of martingales [22, 25] in plenty of areas such as in media, imaging, and data processing. Currently there is no proposal in the literature which works on set-valued (milti-valued) martingale difference sequences and tests the null hypothesis for that concept on real-life data.

The aim of this paper is to define multi-valued Martingale difference, and then test the MDH for some asset returns in the cases of exchange rate and stock market index. The remaining of this paper is organized as follows: Section 2 reviews the preliminary definitions and previous tests for MDH both in linear and nonlinear measures of dependence. Section 3 presents a definition of multi-valued Martingale difference and its MDH test for the same data in Section 2. Finally, Section 4 concludes the results of the tests and proposes several future works.

2. PRELIMINARIES

As mentioned in the section introduction, instead of testing the Martingale hypothesis it is more convenient to test the MDH. A vast empirical and theoretical literature on MDH are also mentioned along with basic definitions of Martingale and MDS given by (1) and (3). In order to make a brief part of this literature, we start by considering the following definitions. Let $I_t = \{Y_t, Y_{t-1}, ...\}$ be the information set at time t and $\mathcal{F}_t$ be the σ-field generated by I_t. A MDS implies that Y_t linear unpredictable given any linear or nonlinear transformation of the past $w\left(I_{t-1}\right)$. That is

$$H0: \ E[Y_t|I_{t-1}] = \mu, \text{ a.s. } \mu \in \mathbb{R} \Leftrightarrow E[(Y_t - \mu)w(I_{t-1})] = 0, \tag{4}$$

for all $\mathcal{F}_{t-1}$-measurable weighting function $w(.)$ (providing the moment exists).

There are two classifications on testing the MDH according to the functions $w(.)$ are linear or nonlinear. Moreover, both classifications include the tests for a finite number of lags or not according to whether they assume that $w(I_{t-1}) = w(Y_{t-1}, ..., Y_{t-P})$ for some $P \geq 1$ or not. The following subsections demonstrate some available methods for testing the MDH by applying them to exchange rate returns and stock index returns.

The data for examination consists of five daily exchange rate returns on the Canadian Dollar (CAN), the Sterling Pound GBP (£), Euro (EUR), the Japanese Yen YEN (¥) and the Vietnamese Dong (VND) against the US Dollar. These methods test the MDH on returns of five stock market indexes including VN-Index VNI(Vietnam Stock Index), S&P500, DJIA (Dow Jones Industrial Average), FTSE(Financial Times Stock Exchange 100 Index) and HSI (Hong Kong Hang Seng Index). The data is all taken from January 1, 2014 to December 31, 2017 with more than 1000 observations. All the data are obtained from https://vn.investing.com/.

Figure 1 plots the daily returns of the exchange rates from January 1, 2014 to December 31, 2017. One can see that the returns of VND/USD (VND) are different from the others and unusual to occur big changes. Figure 2 plots the daily returns of the stock market index in the same period to the exchange rates. It is not easy to find the differences between these returns from the image.

2.1. Tests based on linear measures of dependence

This test employs a linear function $w(.)$ and the simplest approach is to set $w(I_{t-1}) = Y_{t-j}$ for some $j \geq 1$. Hence, the time series is uncorrelated is a necessary condition for the MDH to hold. That is

$$\gamma_j = Cov(Y_t, Y_{t-j}) = E\left[(Y_t - \mu)Y_{t-j}\right] = 0, \ \forall j \geq 1. \tag{5}$$

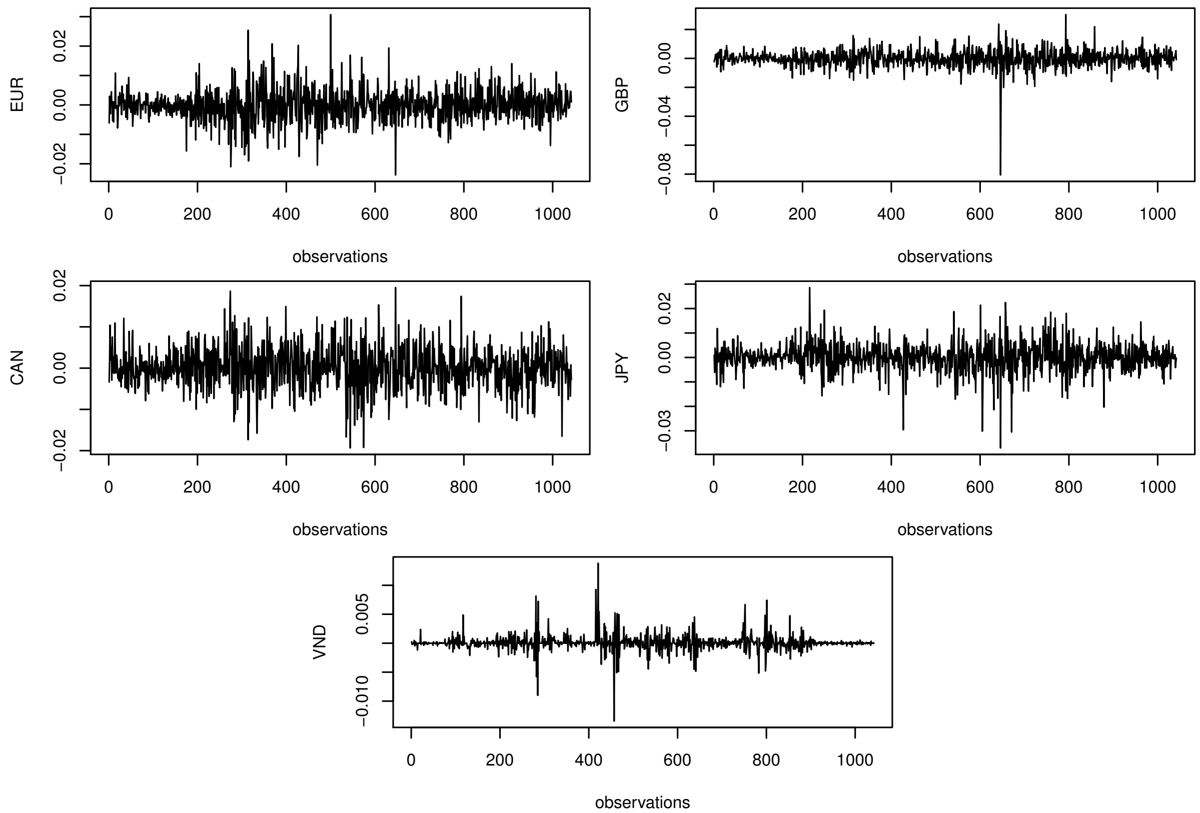

Figure 1. Daily returns of the Euro (Euro), Canadian Dollar (Can), the Sterling Pound (Pound), the Japanese Yen (Yen) and the Vietnamdong (VND) against the US Dollar.

In the case of finite number of lags, the most popular test is Box-Pierce [3] Portmanteau Q_p test if the first p autocorrelations of a series are zero (i.e. $j = 1, 2, ..., p$). Given a series of observations $\{Y_t\}_{t=1}^n$ then the sample autocovariance can be estimated

$$\widehat{\gamma}_j = (n-j)^{-1} \sum_{t=1+j}^{n} (Y_t - \overline{Y})(Y_{t-j} - \overline{Y}), \tag{6}$$

where $\overline{Y}$ is sample mean. The $j-$th order autocorrelation now is denoted by $\widehat{\rho}_j = \widehat{\gamma}_j / \widehat{\gamma}_0$. The Q_p statistic now is

$$Q_p = n \sum_{j=1}^{p} \widehat{\rho}_j^2. \tag{7}$$

Its modification is implemented by Ljung and Box [28] as follows

$$LB_p = n(n+2) \sum_{j=1}^{p} (n-j)^{-1} \widehat{\rho}_j^2. \tag{8}$$

In the infinite lag case, Escanciano and Lobato [12] modified the Box-Pierce statistic using an adaptive Neyman test in the form

$$N_n = Q_{\widetilde{p}}, \tag{9}$$

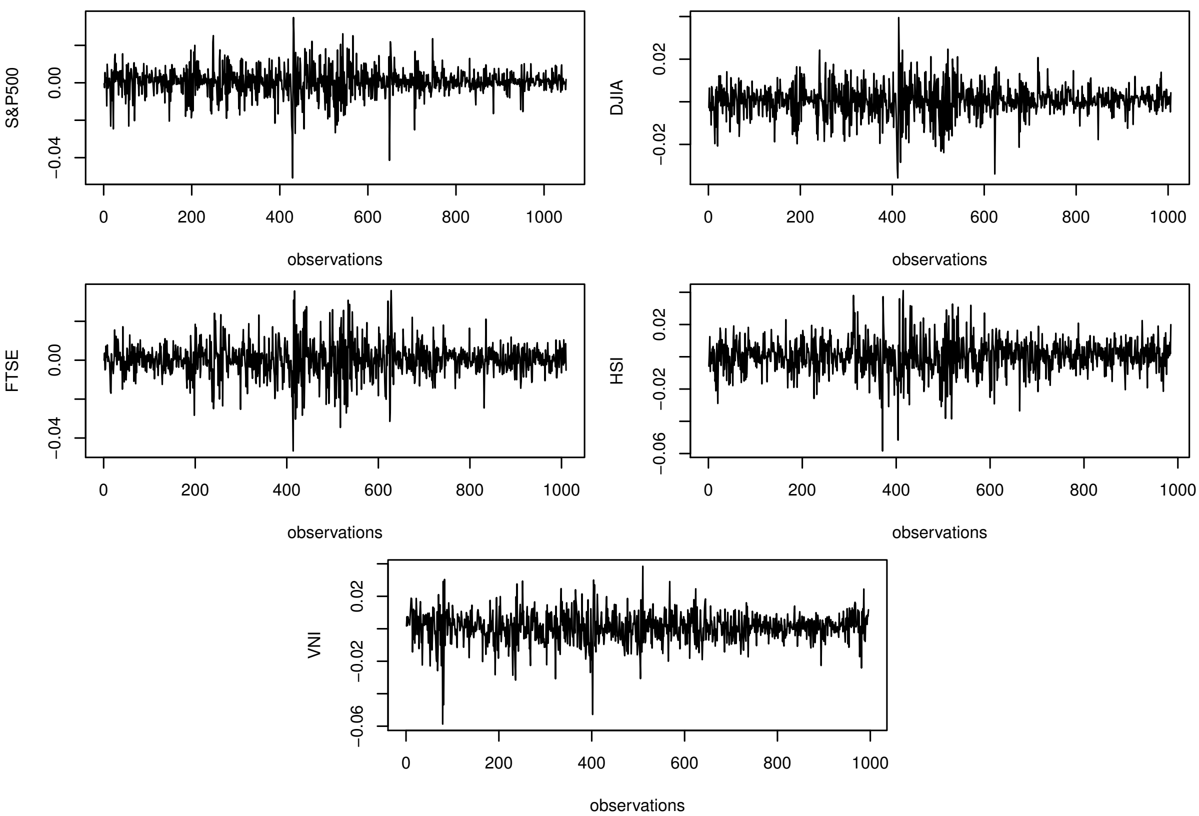

Figure 2. Daily returns of the S&P500 (S&P500), Dow Jones Industrial Average (DJIA), Financial Times Stock Exchange 100 Index (FTSE), Hong Kong Hang Seng Index (HSI) and the Vietnam Sotck Index (VND) stock martket.

where $\widetilde{p} = \min\{m : 1 \le m \le p_n,\ L_m \ge L_h,\ h = 1, 2, ..., p_n\}$, $L_p = Q_p - \pi(p, n, q)$, p_n is an upper bound that tends to infinity with n and

$$\pi(p,n,q) = \begin{cases} p \log n, & \text{if } \max\limits_{1 \le j \le p_n} \left|\widehat{\rho}_j^2\right| \le \sqrt{q \log n} \\ 2p & \text{if } \max\limits_{1 \le j \le p_n} \left|\widehat{\rho}_j^2\right| > \sqrt{q \log n}, \end{cases} \tag{10}$$

where q is some fixed positive number.

In principle, testing the MDH using linear measures of dependence is necessary for the equation (5) to hold but not sufficient. The null hypothesis means that the time series is uncorrelated. It is equivalent to agreement that the time series is not linear dependence. According to Escanciano and Lobato [12]: "These tests are suitable for testing for lack of serial correlation but not necessarily for the MDH, and in fact, they are not consistent against non-martingale difference sequences with zero autocorrelations. These tests are inconsistent because they only employ information contained in the second moments of the process".

Table 1 shows LB_p and M_n test to check whether the returns of five our exchange rates are uncorrelated or not. The table finds that all tested exchange rate returns are not linear dependent (i.e. independent) with an exception of the daily VND for all chosen p (P-values less than 0.05 for all tests).

Table 1. Linear predictability of exchange rates returns

	Statistics (P-value)				
	EUR	GBP(£)	CAN	YEN(¥)	VND
LB_5	3.9656 (0.5544)	8.7828 (0.118)	6.3573 (0.273)	2.4164 (0.789)	40.847 (1.008E-7)
LB_{15}	17.366 (0.2975)	14.796 (0.4662)	17.135 (0.3109)	11.665 (0.7042)	64.344 (4.45E-8)
LB_{25}	26.523 (0.3801)	22.529 (0.605)	23.843 (0.5284)	29.173 (0.2568)	75.829 (5.06E-7)
LB_{50}	55.699 (0.269)	44.39 (0.6971)	48.971 (0.5147)	56.993 (0.2311)	112.18 (1.13E-6)
N_n	0.7937 (0.3729)	0.0009 (0.9756)	0.1553 (0.6934)	0.0017 (0.9668)	6.2554 (0.0124)

The tests for the linear dependence on some stock market indexes are given in Table 2. The results are similar to the exchange rates case with an exception of the FTSE for LB_p test and the VNI for LB_5. Surprisingly, the contradictory results for these stock market indexes obtained by N_n test. These contradictory results may be due to a lack of power of the tests rather than a lack of evidence against the MDH.

Table 2. Linear predictability of stock market returns

	Statistics (P-value)				
	S&P500	DJIA	FTSE	HSI	VNI
LB_5	6.2249 (0.2849)	5.9451 (0.3116)	22.428 (0.0004)	0.99972 (0.9626)	11.119 (0.049)
LB_{15}	8.9002 (0.8827)	15.973 (0.3839)	32.772 (0.005)	9.2918 (0.8618)	24.666 (0.0546)
LB_{25}	22.806 (0.5889)	32.581 (0.1419)	54.576 (0.00055)	16.292 (0.9059)	33.881 (0.1105)
LB_{50}	51.77 (0.4046)	57.742 (0.2109)	89.872 (0.00046)	29.707 (0.99)	64.761(0.0782)
N_n	0.0113 (0.9152)	0.2935 (0.5879)	0.3834 (0.5357)	0.1286 (0.7198)	1.1753 (0.2783)

2.2. Tests based on nonlinear measures of dependence

Similarly to the previous subsection in the case of finite number of lags, the lags (P) are chosen manually. This subsection reviews the use of $w(I_{t-1}) = w_0(\widetilde{Y}_{t,P}, x)$ in (4) where $\widetilde{Y}_{t,P} = (Y_{t-1}, ..., Y_{t-P})'$ and w_0 is a nonlinear function driven by x. The null hypothesis of the MDH is to check how "far" it is from the sample analogue of $E\left[(Y_t - \mu)w_0(\widetilde{Y}_{t,P}, x)\right]$ to zero. This work reviews indicator functions $w_0(\widetilde{Y}_{t,P}, x) = 1(\widetilde{Y}_{t,P} \le x),\ x \in \mathbb{R}$ that were used by Koul and Stute [23] for autoregression check and by Dominguez and Lobato [10] for the MDH test. The later extended Cramer-von Mises (CvM) and Kolmogorov-Smirnov (KS) statistics in Koul and Stute [23] to the multivariate case, that is respectively

$$CvM_{n,P} := \frac{1}{\widehat{\sigma}^2 n^2} \sum_{j=1}^{n} \left[\sum_{t=1}^{n} (Y_t - \overline{Y}) 1(\widetilde{Y}_{t,P} \le \widetilde{Y}_{j,P}) \right]^2, \tag{11}$$

$$KS_{n,P} := \max_{1 \le i \le n} \left| \frac{1}{\widehat{\sigma}\sqrt{n}} \sum_{t=1}^{n} (Y_t - \overline{Y}) 1(\widetilde{Y}_{t,P} \le \widetilde{Y}_{j,P}) \right|, \tag{12}$$

where $\widehat{\sigma}^2 = \frac{1}{n} \sum_{t=1}^{n} (Y_t - \overline{Y})^2$.

There is a notice that the asymptotic null distribution of the test depends on the data generating process in which critical values can not be tabulated. Therefore, it is important to approximate the null distribution by the so-called bootstrap methods. Based on these methods the asymptotic distribution is approximated by replacing $(Y_t - \overline{Y})$ by $(Y_t - \overline{Y})(V_t - \overline{V})$, where $\{V_t\}_{t=1}^n$ is a independent random variables sequence with zero mean and unit variance and also independent of the sequence $\{Y_t\}_{t=1}^n$. Obviously, $\overline{V}$ is the sample mean of $\{V_t\}_{t=1}^n$. This work employs a sequence of i.i.d Bernoulli random variables with $p = P(V_t = 0.5(1-\sqrt{5})) = \dfrac{1+\sqrt{5}}{2\sqrt{5}}$ and $P(V_t = 0.5(1-\sqrt{5})) = 1-p$ for $\{V_t\}$ due to the popular choice in previous literature.

Table 3 and Table 4 report the bootstrap MDH tests on exchange rate and stock market returns using $CvM_{n,P}$ and $KS_{n,P}$ with $P = 1$ and $P = 3$. For the exchange rate returns in Table 3, the results favor the MDH with all of them even with VND/USD exchange rate.

Table 3. Testing the MDH of exchange rates returns: P-values

	EUR	GBP(£)	CAN	YEN(¥)	VND
$CvM_{n,1}$	0.14	0.4433	0.26	0.7233	0.1
$CvM_{n,3}$	0.82	0.69	0.59	0.8	0.2066
$KS_{n,1}$	0.1066	0.6233	0.2333	0.72	0.0866
$KS_{n,3}$	0.4933	0.4466	0.5866	0.87	0.0866

In Table 4, the MDH tests are implemented on five stock market returns. The results support the previous conclusions in linear measure case (see LB_5 in Table 2) that all stock market returns support the MDH except VNI (VN-Index). These results arise the doubts about efficiency of Vietnam stock market compared to others.

Table 4. Testing the MDH of stock market returns: P-values

	S&P500	DJIA	FTSE	HSI	VNI
$CvM_{n,1}$	0.3033	0.11	0.24	0.3633	0.04
$CvM_{n,3}$	0.5466	0.3933	0.2066	0.1866	0.04
$KS_{n,1}$	0.4933	0.1566	0.32	0.5766	0.0233
$KS_{n,3}$	0.73	0.31	0.0966	0.6766	0.0633

The MDH tests based on finite number of lags may miss some dependence structure in the conditional mean at omitted lags because of working on a finite-dimensional information set. Hence, there have been some MDH tests on infinite-dimensional information sets [10, 17]. This paper examined the test of Escanciano and Velasco [13]. This MDH test consists of checking all the pairwise

$$H_0 : \gamma_j = 0 \text{ a.s. } \forall j \geq 1, \tag{13}$$

where $\gamma_j = E[Y_t - \mu | Y_{t-j}]$. Then the measure for the conditional mean dependence in a nonlinear time series framework is given by

$$\gamma_j(x) = E[(Y_t - \mu)e^{ixY_{t-j}}]. \tag{14}$$

The sample counterpart of $\gamma_j(x)$ is

$$\widehat{\gamma}_j(x) = \frac{1}{n-j}\sum_{t=1+j}^{n}(Y_t - \overline{Y}_{n-j})e^{ixY_{t-j}}, \tag{15}$$

where

$$\overline{Y}_{n-j} = \frac{1}{n-j}\sum_{t=1+j}^{n} Y_t. \tag{16}$$

A norm for the Cramer-von Mises test which is called generalized spectral test has the form

$$D_n^2 = \sum_{j=1}^{n-1}(n-j)\frac{1}{(j\pi)^2}\int_{\mathbb{R}} |\widehat{\gamma}_j(x)|^2 \, W(dx), \tag{17}$$

where $W(.)$ is a weighting function satisfying the condition $W : \mathbb{R} \longrightarrow \mathbb{R}^+$ *is non-decreasing, absolutely continuous with respect to Lebesgue measure and with bounded total variation.* In practice, we choose the standard normal cumulative probability distribution function as the weighting function $W(.)$.

Table 5. P-values of generalized spectral test

	EUR	GBP(£)	CAN	YEN(¥)	VND
D_n^2	0.3066	0.72	0.1366	0.5433	0.56
	S&P500	DJIA	FTSE	HSI	VNI
D_n^2	0.9633	0.48	0.8666	0.7266	0.0566

Table 5 reports the P-values of generalized spectral test for MDH on both exchange rates and stock market returns. The results continuously support our previous findings that the VNI returns do not follow a MDS. For the VND/USD exchange rate returns, the nonlinear measure-based MDH tests slightly support MDH whereas linear measure-based MDH tests strongly reject.

3. MULTI-VALUED MARTINGALE DIFFERENCE HYPOTHESIS

This section defines multi-valued martingale difference and examines the multi-valued martingale difference hypothesis tests on our data. Suppose throughout this section that $(\Omega, \mathcal{F}, P)$ is a complete probability space, $\mathcal{F}_0 \subset \mathcal{F}_1 \subset ... \subset \mathcal{F}_n \subset ...$ (with $\mathcal{F}_0 = \{\emptyset, \Omega\}$) is an increasing sequence of sub σ-algebras of $\mathcal{F}$, $\mathbf{E}$ is a Banach space.

Multi-valued random variables and multi-valued martingale were relatively fully introduced in Lee 2013 [26] and the references therein. However, it is difficult to directly define a multi-valued martingale difference sequence from a martingale in the way of single-valued martingale difference due to the lack of subtraction between two sets. Fortunately, it is thanks to the definition of martingale selections that martingale difference selections could be defined and a Castaing representation of a multi-valued martingale difference that consists of martingale difference selections is provided.

3.1. Multi-valued martingale difference

We introduce a definition of multi-valued martingale differences in the form of (2) for a sequence of multi-valued random variables.

Definition 1. (Multi-valued Martingale differences) Let $\{D_n, n \geq 1\}$ be a sequence of multi-valued random variables, $\{D_n, n \geq 1\}$ is said to be a multi-valued Martingale differences if

$$\begin{array}{l}\text{(i)}\;\; D_n \text{ is } \mathcal{F}_n\text{-measurable;}\\ \text{(ii)}\;\; E(D_n|\mathcal{F}_{n-1}) = 0 \text{ for all } n \geq 1.\end{array} \tag{18}$$

Clearly, it is to run into difficulties to work with this definition because the properties of a real single-valued martingale difference are no longer preserved. The following concept relates to the multi-valued and single-valued martingale differences.

Definition 2. (Martingale difference selection) An $\mathbf{E}-$valued martingale $\{(d_n, \mathcal{F}_n), n \geq 1\}$ is said to be a *martingale difference selection* of $\{D_n, n \geq 1\}$ if $d_n \in S^1(D_n)$ for every $n \geq 1$, where

$$S^1(D_n) = \{f \in L^1[\Omega, \mathbf{E}] : f(\omega) \in F_n(\omega) \;\; \text{a.s.}\}. \tag{19}$$

The family of all martingale selections is denoted by $MDs(\{D_n, n \geq 1\})$.

The concept of the projective limit makes it possible to establish the existence of martingale difference selections and to provide a Castaing representation of $\{D_n\}$ that consists of martingale difference selections (see Vu Viet Yen [43]). Note that $proj_{n-1}(\{D_n, n \geq 1\}) = D_{n-1}$ denotes a projection of a sequence $\{D_n, n \geq 1\}$ onto its $(n-1)$th coordinate. For example, $proj_{n-1}(MDs(\{D_n, n \geq 1\}))$ is the family of all martingale difference selections at time $n-1$.

Example. Consider a sequence of asset prices $\{X_t, t \geq 1\}$ and its returns $\{R_t, t \geq 1\}$. One may be more interested in how the trend of the tomorrow return will be than how the value of the tomorrow return will be. For instance, given the state of R_t today is "high", whether the tomorrow R_{t+1} will be "very high", "high", "low", "very low" or "very very low". It means one may want to estimate $E(R_{t+1}|R_t = \text{high})$. From this point of view, it is possible to define a space of sets $\mathbf{E}$ which consists of the elements such as "very very high", "very high", "high", "low", "very low" or "very very low" where each element is represented by an interval of real line $\mathbb{R}$. Now the random variable R_t becomes a set-valued random variable that takes its values in $\mathbf{E}$. The task now is to check whether $\{R_t\}$ follows an MVMD.

3.2. Testing the multi-valued Martingale difference hypothesis

Given a return series $\{R_t, t \geq 1\}$ of an asset price, the following transformation converts $\{R_t, t \geq 1\}$ into multi-valued (or set-valued) random sequence. From the histograms of all our return series show that their values are concentrated in the interval from -0.02 to 0.02 (see Figure 3), we could define elements of $\mathbf{E}$ as follows,

- E_1="normal"$=[-0.004, 0.004]$
- E_2="high"$=[0.004, 0.012]$
- E_3="very high"$=[0.012, 0.02]$
- E_4="very very high"$=[0.02, \max_{t\geq 1} R_t)$
- E_5="low"$=[-0.012, 0.004]$
- E_6="very low"$=[-0.02, -0.012]$
- E_7="very very low"$=(\min_{t\geq 1} R_t, -0.02]$

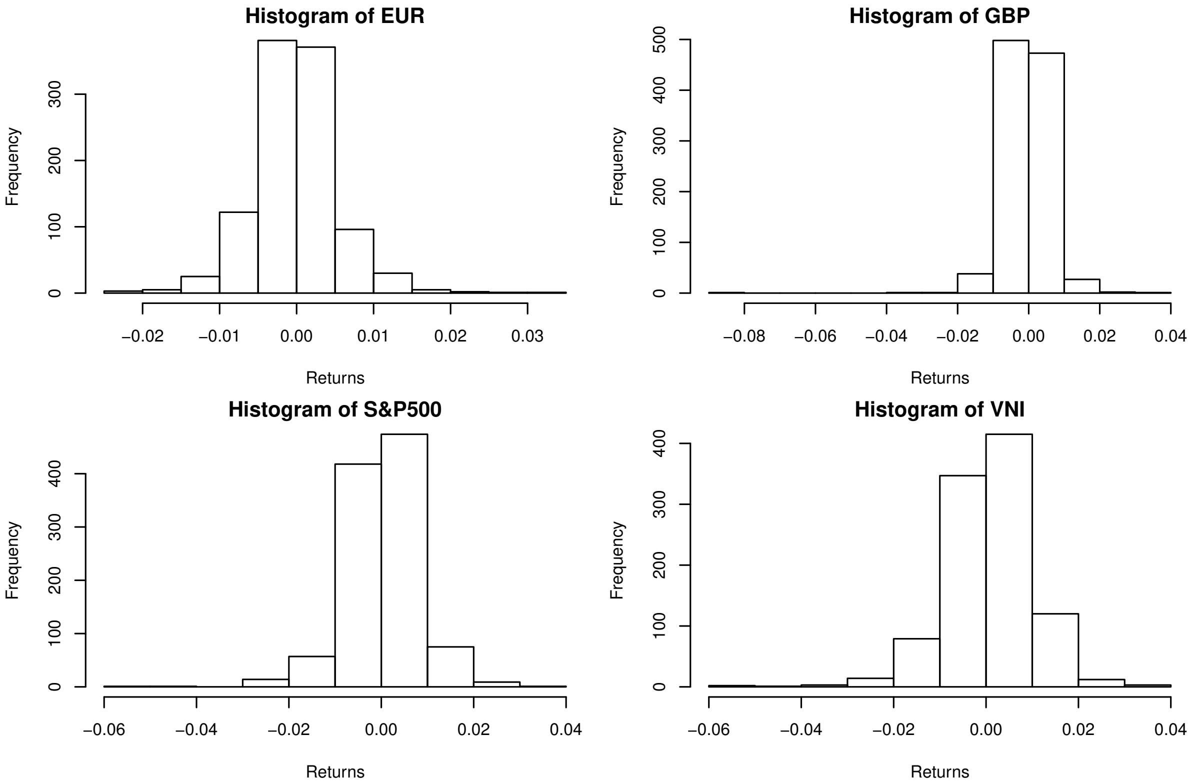

Figure 3. Histograms of some exchange rate and stock market returns

Now $R_t, t \geq 1$ are set-valued (multi-valued) random variables whose values are $E_i, i = 1, ..., 7$. The scenario for this transformation is demonstrated in Figure 4.

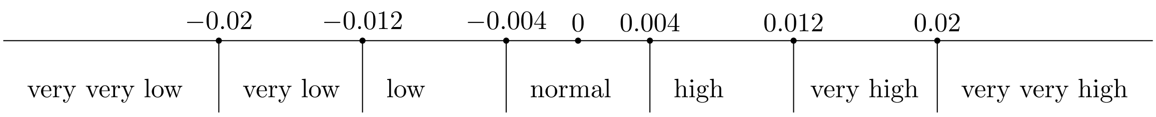

Figure 4. Definitions of set-valued random variables for returns

To test the MVMDH for $\{R_t, t \geq 1\}$, we choose a selection of $\{R_t\}$ by their midpoints denoted by $\{r_t, t \geq 1\}$. Testing the MDH on $\{r_t, t \geq 1\}$ provides the results of MVMDH for $\{R_t, t \geq 1\}$. Figure 5 compares the difference between return time series and return selection time series of EUR/USD and VND/USD exchange rate. One can see that the selection time series of VND/USD is almost "normal" (e.g. no change) whereas EUR/USD's are similar. This means the returns of VND/USD exchange rate are linearly dependent and easy to predict. Others need to test to make conclusions.

Now the MVMDH tests are implemented on our data by testing the MDH on return selections instead of on the returns. The Table 6 reports the results in the case of linear measures for exchange rate data. Compare to Table 1 in Section 2, the results are similar except LB_5 of CAN/USD exchange rate. But there is not enough evidence to reject the MDH for CAN/USD returns.

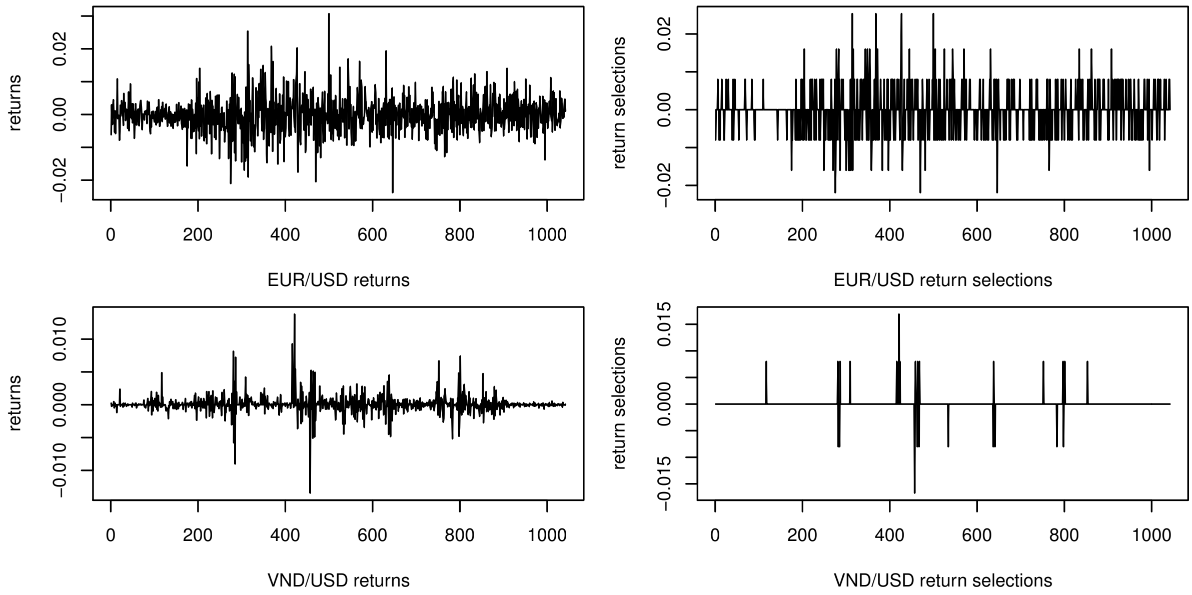

Figure 5. Returns and selection of returns of EUR/USD (above) and VND/USD (below)

Table 6. Linear predictability of multi-valued exchange rates returns

	Statistics (P-value)				
	EUR	GBP(£)	CAN	YEN(¥)	VND
LB_5	1.8263 (0.8726)	9.916 (0.0776)	12.14 (0.0329)	4.1058 (0.5343)	52.589 (4.08E-10)
LB_{15}	8.9807 (0.8785)	22.192 (0.1029)	23.137 (0.0812)	17.263 (0.3034)	71.603 (2.3E-9)
LB_{25}	18.679 (0.8122)	30.467 (0.2073)	34.85 (0.0909)	37.48 (0.0519)	74.821 (7.22E-7)
LB_{50}	40.808 (0.8198)	63.236 (0.0989)	62.298 (0.1137)	67.403 (0.0508)	137.11 (4.8E-10)
N_n	0.0174 (0.8948)	5.8E-5 (0.9938)	0.8821 (0.3476)	0.0161 (0.8988)	6.0418 (0.0139)

Table 7 examines linear predictability of some stock market returns with multi-valued MDH. There are some differences between Table 7 and Table 2. Focus on VNI, Table 2 shows that VNI returns is not linear dependent whereas its selections reject the MDH. This difference implies difficult to predict values of VNI returns but easier to predict its trend. The similar comment is also conformable to FTSE returns.

Table 7. Linear predictability of multi-valued stock market returns

	Statistics (P-value)				
	S&P500	DJIA	FTSE	HSI	VNI
LB_5	3.9887 (0.5511)	5.5736 (0.3499)	32.522 (4.6E-6)	6.0097 (0.3053)	14.432 (0.013)
LB_{15}	7.7672 (0.9328)	17.878 (0.2691)	42.073 (0.0002)	39.033 (0.0006)	29.642 (0.0133)
LB_{25}	21.873 (0.643)	28.37 (0.2911)	63.083 (3.8E-5)	57.949 (0.0002)	38.433 (0.042)
LB_{50}	51.132 (0.429)	48.491 (0.5341)	91.808 (0.0003)	80.924 (0.0036)	69.976 (0.032)
N_n	0.0006 (0.9806)	0.5785 (0.4468)	1.1952 (0.2742)	0.219 (0.6397)	3.6076 (0.0575)

For the nonlinear measures of dependence, the MVMDH tests on exchange rate stock market returns in the case of finite number of lags are given in Table 8 and Table 9. We find that the VND/USD strongly rejects the MVMDH in the linear measure case but it supports MVMDH in the nonlinear measure case. One may understand that VND/USD returns are not nonlinear dependent but linear dependent in the sense of set-valued martingale difference. By contrast, VNI stock market returns reject the MVMDH whatever linear or nonlinear measure case. FTSE returns in Table 4 support the MDH but reject the MVMDH in Table 9. This implies that it is difficult to predict FTSE returns but one can predict its "level of changes" by some suitable nonlinear-based predictors.

Table 8. Nonlinear MVMDH tests of exchange rates returns: P-values

	EUR	GBP(£)	CAN	YEN(¥)	VND
$CvM_{n,1}$	0.3333	0.45	0.33	0.96	0.37
$CvM_{n,3}$	0.52	0.54	0.8633	0.55	0.1266
$KS_{n,1}$	0.5566	0.2133	0.6033	0.85	0.16
$KS_{n,3}$	0.84	0.2533	0.99	0.8066	0.4533

Recall that in equations (11) and (12) both test statistics $CvM_{n,P}$ and $KS_{n,P}$ are based on indicator transformation of the past $w(I_{t-1})$.

Table 9. Nonlinear MVMDH tests of stock market returns: P-values

	S&P500	DJIA	FTSE	HSI	VNI
$CvM_{n,1}$	0.6566	0.3333	0.04	0.22	0
$CvM_{n,3}$	0.6333	0.4533	0.04	0.2	0.0033
$KS_{n,1}$	0.9166	0.63	0.09	0.43	0
$KS_{n,3}$	0.36	0.67	0.0566	0.8566	0.02

Table 10 reports the generalized spectral test for MVMDH on our data in the infinite number of lags case. Note that D_n^2 statistic is based on $w_0(Y_{t-j}, x) = \exp(ixY_{t-j})$ (i.e. exponential dependence of the past $w(I_{t-1})$). The results show that the MVMDH is rejected for all our returns. This asserts that the multi-valued returns of exchange rate or stock market are not exponentially dependent.

Table 10. P-values of generalized spectral test for MVMDH

	EUR	GBP(£)	CAN	YEN(¥)	VND
D_n^2	0.4133	0.9066	0.0733	0.3266	0.8133
	S&P500	DJIA	FTSE	HSI	VNI
D_n^2	0.8666	0.8466	0.5266	0.4733	0.1233

4. CONCLUSIONS

This paper has presented some popular tests for MDH on several exchange rate returns and stock market returns. The tests are included linear measure-based and nonlinear measure-based statistics. These tests are also used to examine the MDH for a multi-valued martingale difference, called MVMDH. The results of MDH and MVMDH tests are recapped in Table 11.

Table 11. Martingale difference hypothesis vs multi-valued Martingale difference hypothesis

MDH-MVMDH										
	Linear measures					Nonlinear measures				
	LB_5	LB_{15}	LB_{25}	LB_{50}	N_n	$CvM_{n,1}$	$CvM_{n,3}$	$KS_{n,1}$	$KS_{n,3}$	D_n^2
EUR	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓
GBP	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓
CAN	✓-✗	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓
YEN	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓
VND	✗-✗	✗-✗	✗-✗	✗-✗	✗-✗	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓
SP500	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓
DJIA	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓
FTSE	✗-✗	✗-✗	✗-✗	✗-✗	✓-✓	✓-✗	✓-✗	✓-✓	✓-✓	✓-✓
HSI	✓-✓	✓-✗	✓-✗	✓-✗	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓	✓-✓
VNI	✗-✗	✓-✗	✓-✗	✓-✗	✓-✗	✗-✗	✗-✗	✗-✗	✓-✗	✓-✓

✓: support H0, ✗: reject H0, left: MDH - right: MVMDH

According to the results, this work provides some following conclusions:

- The MDH tests on some exchange rate returns and stock market returns are in agreement with previous findings (see [16, 18]) that most of them (e.g. EUR, GBP, CAN, YEN, S&P500, DJIA, HSI) support the MDH. Rejecting the MDH makes the confidence in an efficient market where all trades are performed in a "fair game".

- The exceptions of VND, VNI and FTSE are the evidences that the MDH (or predictability) on exchange rate and stock market returns depends on which market is tested. As the MDH is rejected, there exist some kinds of dependence in the series corresponding to the measures of the tests. One may fit several available forecast models as the predictors of the series.

- Interestingly, the tests of MVMDH (multi-valued martingale difference hypothesis) show the same results to MDH for EUR, GBP, CAN, YEN, S&P500, DJIA and HSI. These similarities strengthen their unpredictability even when forecasting the trend of the series. However, the changes from MDH support to MVMDH rejection of HSI and VNI say that it is impossible to predict the values of the return series but possible to predict its trend. This finding helps one to reassess the efficiency of the market or forecast the market tendency.

ACKNOWLEDGMENT

We would like to thank the anonymous reviewers, the editor for very helpful comments and suggestions.

REFERENCES

[1] A. Aswani, "Statistics with set-valued functions: applications to inverse approximate optimization," *Mathematical Programming*, pp. 1–27, 2017.

[2] M. R. Borges, "Efficient market hypothesis in european stock markets," *The European Journal of Finance*, vol. 16, no. 7, pp. 711–726, 2010.

[3] G. E. P. Box and D. A. Pierce, "Distribution of residual autocorrelations in autoregressive-integrated moving average time series models," *Journal of the American Statistical Association*, vol. 65, no. 332, pp. 1509–1526, 1970.

[4] P. J. Brockwell and R. A. Davis, "Time series: Theory and methods," *Springer Series in Statistics, Berlin, New York: Springer, 2nd ed.*, 1991.

[5] C. Castaing, N. Van Quang, and N. T. Thuan, "A new family of convex weakly compact valued random variables in banach space and applications to laws of large numbers," *Statistics & Probability Letters*, vol. 82, no. 1, pp. 84–95, 2012.

[6] W. W. Chen and R. S. Deo, "The variance ratio statistic at large horizons," *Econometric Theory*, vol. 22, no. 2, pp. 206–234, 2006.

[7] I. Choi, "Testing the random walk hypothesis for real exchange rates," *Journal of Applied Econometrics*, vol. 14, no. 3, pp. 293–308, 1999.

[8] K. V. Chow and K. C. Denning, "A simple multiple variance ratio test," *Journal of Econometrics*, vol. 58, no. 3, pp. 385–401, 1993.

[9] R. S. Deo, "Spectral tests of the martingale hypothesis under conditional heteroscedasticity," *Journal of Econometrics*, vol. 99, no. 2, pp. 291–315, 2000.

[10] M. A. Domínguez and I. N. Lobato, "Testing the martingale difference hypothesis," *Econometric Reviews*, vol. 22, no. 4, pp. 351–377, 2003.

[11] S. N. Durlauf, "Spectral based testing of the martingale hypothesis," *Journal of Econometrics*, vol. 50, no. 3, pp. 355–376, 1991.

[12] J. C. Escanciano and I. N. Lobato, "Testing the martingale hypothesis," in *Palgrave handbook of econometrics*. Springer, 2009, pp. 972–1003.

[13] J. C. Escanciano and C. Velasco, "Generalized spectral tests for the martingale difference hypothesis," *Journal of Econometrics*, vol. 134, no. 1, pp. 151–185, 2006.

[14] L. Guan and Y. Wan, "A strong law of large numbers for set-valued negatively dependent random variables," *International Journal of Statistics and Probability*, vol. 5, no. 3, p. 102, 2016.

[15] K. Hamid, M. T. Suleman, S. Z. A. Shah, and R. S. I. Akash, "Testing the weak form of efficient market hypothesis: Empirical evidence from asia-pacific markets," *International Research Journal of Finance and Economics*, no. 58, pp. 121–133, 2010.

[16] Y. Hong and T.-H. Lee, "Inference on predictability of foreign exchange rates via generalized spectrum and nonlinear time series models," *Review of Economics and Statistics*, vol. 85, no. 4, pp. 1048–1062, 2003.

[17] Y. Hong and Y.-J. Lee, "Generalized spectral tests for conditional mean models in time series with conditional heteroscedasticity of unknown form," *The Review of Economic Studies*, vol. 72, no. 2, pp. 499–541, 2005.

[18] D. A. Hsieh, "Testing for nonlinear dependence in daily foreign exchange rates," *Journal of Business*, pp. 339–368, 1989.

[19] D. G. Kendall, *Foundations of a Theory of Random Sets, In Stochastic Geometry.* John Wiley & Sons, 1974.

[20] M. G. Kendall and A. B. Hill, "The analysis of economic time-series-part i: Prices," *Journal of the Royal Statistical Society. Series A (General)*, vol. 116, no. 1, pp. 11–34, 1953.

[21] J. H. Kim, "Wild bootstrapping variance ratio tests," *Economics letters*, vol. 92, no. 1, pp. 38–43, 2006.

[22] M. Kisielewicz, "Martingale representation theorem for set-valued martingales," *Journal of Mathematical Analysis and Applications*, vol. 409, no. 1, pp. 111–118, 2014.

[23] H. L. Koul, W. Stute *et al.*, "Nonparametric model checks for time series," *The Annals of Statistics*, vol. 27, no. 1, pp. 204–236, 1999.

[24] C.-M. Kuan and W.-M. Lee, "A new test of the martingale difference hypothesis," *Studies in Nonlinear Dynamics & Econometrics*, vol. 8, no. 4, 2004.

[25] J.-G. Li and S.-Q. Zheng, "Set-valued stochastic equation with set-valued square integrable martingale," in *ITM Web of Conferences*, vol. 12. EDP Sciences, 2017, p. 03002.

[26] S. Li, Y. Ogura, and V. Kreinovich, *Limit theorems and applications of set-valued and fuzzy set-valued random variables.* Springer Science & Business Media, 2013, vol. 43.

[27] S. Li, Y. Ogura, F. N. Proske, and M. L. Puri, "Central limit theorems for generalized set-valued random variables," *Journal of Mathematical Analysis and Applications*, vol. 285, no. 1, pp. 250–263, 2003.

[28] G. M. Ljung and G. E. Box, "On a measure of lack of fit in time series models," *Biometrika*, vol. 65, no. 2, pp. 297–303, 1978.

[29] A. W. Lo and A. C. MacKinlay, "Stock market prices do not follow random walks: Evidence from a simple specification test," *The Review of Financial Studies*, vol. 1, no. 1, pp. 41–66, 1988.

[30] I. Lobato, J. C. Nankervis, and N. E. Savin, "Testing for autocorrelation using a modified box-pierce q test," *International Economic Review*, vol. 42, no. 1, pp. 187–205, 2001.

[31] I. N. Lobato, J. C. Nankervis, and N. Savin, "Testing for zero autocorrelation in the presence of statistical dependence," *Econometric Theory*, vol. 18, no. 3, pp. 730–743, 2002.

[32] B. G. Malkiel, "Efficient market hypothesis," in *Finance.* Springer, 1989, pp. 127–134.

[33] G. Matheron, *Random Sets and Integral Geometry.* John Wiley and Sons, 1975.

[34] J. L. McCauley, K. E. Bassler, and G. H. Gunaratne, "Martingales, detrending data, and the efficient market hypothesis," *Physica A: Statistical Mechanics and its Applications*, vol. 387, no. 1, pp. 202–216, 2008.

[35] P. Mörters and Y. Peres, *Brownian motion.* Cambridge University Press, 2010, vol. 30.

[36] M. F. Osborne, "Brownian motion in the stock market," *Operations Research*, vol. 7, no. 2, pp. 145–173, 1959.

[37] P. C. Phillips and S. Jin, "Testing the martingale hypothesis," *Journal of Business & Economic Statistics*, vol. 32, no. 4, pp. 537–554, 2014.

[38] M. L. Puri and D. RALESCU, "Integration on fuzzy sets," *Time Series, Fuzzy Analysis and Miscellaneous Topics*, vol. 3, p. 415, 2011.

[39] S. Roberts, "Control chart tests based on geometric moving averages," *Technometrics*, vol. 1, no. 3, pp. 239–250, 1959.

[40] H. J. Skaug and D. Tjøstheim, "A nonparametric test of serial independence based on the empirical distribution function," *Biometrika*, vol. 80, no. 3, pp. 591–602, 1993.

[41] N. Van Quang and N. T. Thuan, "Strong laws of large numbers for adapted arrays of set-valued and fuzzy-valued random variables in banach space," *Fuzzy Sets and Systems*, vol. 209, pp. 14–32, 2012.

[42] J. H. Wright, "Alternative variance-ratio tests using ranks and signs," *Journal of Business & Economic Statistics*, vol. 18, no. 1, pp. 1–9, 2000.

[43] V. V. Yen, "On convergence of multiparameter multivalued martingales," *Acta Mathematica Vietnamica*, vol. 29, pp. 177–184, 2004.

[44] L. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338 – 353, 1965. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S001999586590241X

Received on October 03, 2018
Revised on November 26, 2018